Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 136120, 7 pages
http://dx.doi.org/10.1155/2013/136120

Research Article

Regular Functions with Values in Ternary Number System on the Complex Clifford Analysis

Ji Eun Kim, Su Jin Lim, and Kwang Ho Shon

Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

Correspondence should be addressed to Kwang Ho Shon; khshon@pusan.ac.kr

Received 21 October 2013; Accepted 5 December 2013

Academic Editor: Junesang Choi

We define a new modified basis $\hat{i}$ which is an association of two bases, e_1 and e_2. We give an expression of the form $z = x_0 + \hat{i}\overline{z_0}$, where x_0 is a real number and $\overline{z_0}$ is a complex number on three-dimensional real skew field. And we research the properties of regular functions with values in ternary field and reduced quaternions by Clifford analysis.

1. Introduction

The noncommutative three-dimensional real field $\mathbb{R}^3$ of the hypercomplex numbers is called a ternary number system $\mathbb{T}$. The quaternions are represented by the form $z = \sum_{j=0}^{3} e_j x_j$, where x_j ($j = 0, \dots, 3$) are real numbers on four dimensional real field $\mathbb{R}^4$. Fueter [1] has given a definition of quaternionic functions in $\mathbb{R}^4$ and Deavours [2] and Sudbery [3] have developed theories of quaternionic analysis. Naser [4] investigated some properties of hyperholomorphic functions and Koriyama et al. [5] researched properties of hyperholomorphic functions and holomorphic functions in quaternionic analysis. Nôno [6] obtained several results for regular functions which have a complex number form in quaternion analysis. Cho [7] researched some properties of Euler's formula and De moivre's formula for quaternions. Sangwine and Bihan [8] obtained some results for the quaternionic polar representation with a complex modulus and complex argument inspired by the cayley-dickson form. Fueter [9] obtained some properties of the three variables which are called the Fueter variables and researched the fact that structures lead to the set of all Fueter-regular functions in the general cases of Clifford analysis. By Brackx et al. [10], the theory of Fueter-regularity has been developed and generalized as quaternionic variables for theories of Clifford-valued regular functions.

Lim and Shon [11–13] researched the existence of hyperconjugate harmonic functions of octonion variables, properties of dual quaternion functions, and regularity of functions with values in a noncommutative subalgebra of complex matrix algebras.

We consider that ternary numbers are generated by a new basis $\hat{i}$ and give some properties of regular functions with values in $\mathbb{T}$. Also, we represent the corresponding Euler's formula for the form $z = x_0 + \hat{i}\overline{z_0}$ and investigate calculating rules for regular functions in Clifford analysis. We research new representations of Fueter variables in reduced quaternions with $\hat{i}$ and some characteristics of regularity of functions on the Fueter variable system.

2. Preliminaries

The ternary number system $\mathbb{T}$ is a three dimensional noncommutative and associative real field by three bases e_0, e_1, and e_2 with the following rules:

$$e_1^2 = e_2^2 = -1, \qquad e_1 e_2 = -e_2 e_1, \qquad \overline{e_0} = e_0, \qquad \overline{e_j} = -e_j \quad (j = 1, 2). \tag{1}$$

The element e_0 is the identity of $\mathbb{T}$ and e_1 identifies the imaginary unit $\sqrt{-1}$ in the complex field. We consider an association of two bases e_1 and e_2 as follows:

$$\hat{i} := \frac{ae_1 + be_2}{\sqrt{a^2 + b^2}} = \alpha e_1 + \beta e_2 \quad \text{with } \hat{i}^2 = -1, \tag{2}$$

where $\alpha := a/\sqrt{a^2 + b^2}$, $\beta := b/\sqrt{a^2 + b^2}$, and a, b are real numbers except both zeros.

The number of the skew field $\mathbb{T}$ is

$$\begin{aligned} z &= x_0 + e_1 x_1 + e_2 x_2 \\ &= x_0 + \widehat{i}\overline{z_0}, \end{aligned} \tag{3}$$

where x_j $(j = 0, 1, 2)$ are real variables, $\overline{z_0} = \gamma(x_1 - x_2 e_1 e_2)$, and $\gamma := \alpha + \beta e_1 e_2$.

We define the ternary number system

$$\mathbb{T} := \left\{ z \mid z = x_0 + \widehat{i}\overline{z_0} \right\}. \tag{4}$$

The conjugate number z^* of z in $\mathbb{T}$ is given by the form:

$$z^* = x_0 - \widehat{i}\overline{z_0}. \tag{5}$$

And the norm $|z|$ of z and the inverse z^{-1} of z are given by the following forms:

$$\begin{aligned} |z| &= \sqrt{zz^*} = \sqrt{x_0^2 + \overline{z_0} z_0} = \sqrt{\sum_{j=0}^{2} x_j^2}, \\ z^{-1} &= \frac{z^*}{|z|^2} \quad (z \neq 0), \end{aligned} \tag{6}$$

where $z_0 = \overline{\gamma}(x_1 + x_2 e_1 e_2)$ and $\overline{\gamma} = \alpha - \beta e_1 e_2$.

We define the addition and multiplication of two ternary numbers $z = x_0 + \widehat{i}\overline{z_0}$ and $w = y_0 + \widehat{i}\overline{w_0}$ as follows:

$$\begin{aligned} z + w &= (x_0 + y_0) + \widehat{i}\left(\overline{z_0} + \overline{w_0}\right), \\ zw &= \left(x_0 y_0 - z_0 \overline{w_0}\right) + \widehat{i}\left(x_0 \overline{w_0} + \overline{z_0} y_0\right). \end{aligned} \tag{7}$$

Theorem 1. *Let z be an arbitrary number in $\mathbb{T}$. Then the corresponding Euler formula for z is*

$$e^z = e^{x_0}\left(\cos|z_0| + \frac{z_0}{|z_0|}\widehat{i}\sin|z_0|\right). \tag{8}$$

Moreover, taking logarithms of both sides, one obtains the equation as follows:

$$\ln z = \ln|z| + \frac{z_0}{|z_0|}\widehat{i}\cos^{-1}\left(\frac{x_0}{|z|}\right). \tag{9}$$

Proof. For the number $z = x_0 + \widehat{i}\overline{z_0}$ in $\mathbb{T}$, we get $|\widehat{i}\overline{z_0}| = |\overline{z_0}| = |z_0|$ and $((z_0/|z_0|)\widehat{i})^2 = -1$. Then,

$$\begin{aligned} e^z &= e^{x_0 + \widehat{i}\overline{z_0}} = e^{x_0} e^{(\widehat{i}\overline{z_0}/|\widehat{i}\overline{z_0}|)|\widehat{i}\overline{z_0}|} \\ &= e^{x_0}\left(\cos|z_0| + \frac{z_0}{|z_0|}\widehat{i}\sin|z_0|\right). \end{aligned} \tag{10}$$

From

$$\begin{aligned} z &= |z|\left(\frac{x_0}{|z|} + \frac{z_0}{|z_0|}\widehat{i}\frac{|z_0|}{|z|}\right) \\ &= |z|\left\{\cos\left(\cos^{-1}\left(\frac{x_0}{|z|}\right)\right)\right. \\ &\quad \left. + \frac{z_0}{|z_0|}\widehat{i}\sin\left(\cos^{-1}\left(\frac{x_0}{|z|}\right)\right)\right\}, \end{aligned} \tag{11}$$

we have

$$\ln z = \ln|z| + \frac{z_0}{|z_0|}\widehat{i}\cos^{-1}\left(\frac{x_0}{|z|}\right). \tag{12}$$

We consider the following differential operators:

$$\begin{aligned} D &:= \frac{1}{2}\sum_{j=0}^{2}\overline{e_j}\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \widehat{i}\frac{\partial}{\partial z_0}\right), \\ D^* &= \frac{1}{2}\sum_{j=0}^{2} e_j\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \widehat{i}\frac{\partial}{\partial z_0}\right), \end{aligned} \tag{13}$$

where $\partial/\partial z_0 = \gamma(\partial/\partial x_1 - e_1 e_2(\partial/\partial x_2))$ and $\partial/\partial\overline{z_0} = \overline{\gamma}(\partial/\partial x_1 + e_1 e_2(\partial/\partial x_2))$. Then the Laplacian operator is

$$4\Delta := DD^* = D^* D = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z_0 \partial\overline{z_0}} = \sum_{j=0}^{2}\frac{\partial^2}{\partial x_j^2}. \tag{14}$$

Let Ω be an open set in $\mathbb{R}^3$. The function $f(z)$ that is defined by the following form in Ω with values in $\mathbb{T}$:

$$f : \Omega \longrightarrow \mathbb{T} \tag{15}$$

satisfies

$$z = \left(x_0, \overline{z_0}\right) \in \Omega \longmapsto f(z) = u_0\left(x_0, \overline{z_0}\right) + \widehat{i}\overline{f_0}\left(x_0, \overline{z_0}\right) \in \mathbb{T}, \tag{16}$$

where u_j $(j = 0, 1, 2)$ are real-valued functions and

$$f_0 = \overline{\gamma}\left(u_1 + u_2 e_1 e_2\right), \qquad \overline{f_0} = \gamma\left(u_1 - u_2 e_1 e_2\right) \tag{17}$$

are complex-valued functions with values in $\mathbb{T}$. □

Remark 2. The operators D and D^* act for the function $f(z)$ on $\mathbb{T}$ as follows:

$$\begin{aligned} Df &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ D^* f &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ fD &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial f_0}{\partial z_0}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ fD^* &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}. \end{aligned} \tag{18}$$

3. Properties of Regular Functions with Values in $\mathbb{T}$

Definition 3. Let Ω be an open set in $\mathbb{R}^3$. A function $f(z) = u_0(x_0, \overline{z_0}) + \widehat{i}\overline{f_0}(x_0, \overline{z_0})$ is said to be L(R)-regular in Ω, if the following two conditions are satisfied:

(i) u_0 and f_0 are continuously differential functions on Ω;

(ii) $D^*f(z) = 0$ ($f(z)D^* = 0$) on Ω.

Remark 4. The left equation (ii) of Definition 3 is equivalent to the following:

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial \overline{f_0}}{\partial \overline{z_0}}, \qquad \frac{\partial \overline{f_0}}{\partial x_0} = -\frac{\partial u_0}{\partial z_0}. \tag{19}$$

The equations in (19) are called the corresponding Cauchy-Riemann system for $f(z)$ in $\mathbb{T}$. The right equation (ii) of Definition 3 is equivalent to (19). When the function $f(z) = u_0(x_0, \overline{z_0}) + \widehat{i}\,\overline{f_0}(x_0, \overline{z_0})$ is a L-regular function on $\Omega \subset \mathbb{R}^3$, simply we say that $f(z)$ is a regular function on $\Omega \subset \mathbb{R}^3$. In this case, we often say that $f(z)$ is a biregular function on $\Omega \subset \mathbb{R}^3$.

Remark 5. Let Ω be an open set in $\mathbb{R}^3$ and let $f(z)$ be a regular function on Ω. Then we can replace the corresponding Cauchy-Riemann system in $\mathbb{R}^3$ as follows:

$$\begin{aligned} \frac{\partial u_0}{\partial x_0} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}, \\ \frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \qquad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \end{aligned} \tag{20}$$

where u_j $(j = 0, 1, 2)$ are real-valued functions.

Theorem 6. *Let Ω be an open set in $\mathbb{R}^3$ and let f be a regular function on Ω. Then the derivative f' of f defined by Df is*

$$f' = \frac{\partial f}{\partial x_0} = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{21}$$

on Ω.

Proof. By the definition of regular function with values in $\mathbb{T}$, we have

$$\begin{aligned} Df &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \widehat{i}\left(\frac{\partial \overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\} \\ &= \frac{\partial u_0}{\partial x_0} + \widehat{i}\frac{\partial \overline{f_0}}{\partial x_0} = \frac{\partial f}{\partial x_0} \end{aligned} \tag{22}$$

on Ω. And

$$Df = \frac{\partial \overline{f_0}}{\partial \overline{z_0}} - \widehat{i}\frac{\partial u_0}{\partial z_0} = -\widehat{i}\left(\frac{\partial}{\partial z_0}\widehat{i}\,\overline{f_0} + \frac{\partial u_0}{\partial z_0}\right) = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{23}$$

on Ω. □

Theorem 7. *Let Ω be an open set in $\mathbb{R}^3$ and let $f = u_0 + \widehat{i}\,\overline{f_0}$ be a function with values in $\mathbb{T}$. Suppose that $\partial f/\partial x_0$ and $\partial f/\partial z_0$ exist and are continuous on Ω. If*

$$\frac{\partial f}{\partial x_0} = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{24}$$

on Ω, then f is regular on Ω.

Proof. Since $\partial f/\partial x_0 = -\widehat{i}(\partial f/\partial z_0)$, we have

$$\frac{\partial f}{\partial x_0} = \frac{\partial u_0}{\partial x_0} + \widehat{i}\frac{\partial \overline{f_0}}{\partial x_0}. \tag{25}$$

Hence, we have $D^* f = 0$ and then f is regular on Ω. □

Definition 8. Let Ω be an open set in $\mathbb{R}^3$. A function $f = u_0 + \widehat{i}\,\overline{f_0}$ is said to be harmonic on Ω if all its components u_0 and $\overline{f_0}$ of f are harmonic on Ω.

Proposition 9. *Let Ω be an open set in $\mathbb{R}^3$. If the function f is regular on Ω, then f is harmonic on Ω.*

Proof. Since f is regular function on Ω, we have

$$\begin{aligned} DD^*\overline{f_0} = \frac{1}{4}&\left\{\left(\frac{\partial}{\partial x_0}\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial}{\partial \overline{z_0}}\frac{\partial \overline{f_0}}{\partial z_0}\right)\right. \\ &\left.+\widehat{i}\left(\frac{\partial}{\partial x_0}\frac{\partial \overline{f_0}}{\partial z_0} - \frac{\partial}{\partial z_0}\frac{\partial \overline{f_0}}{\partial x_0}\right)\right\} = 0. \end{aligned} \tag{26}$$

Similarly, we can prove that $DD^*u_0 = 0$. So, we obtain the result. □

Proposition 10. *Let Ω be an open set in $\mathbb{R}^3$ and let $f = u_0 + \widehat{i}\,\overline{f_0}$ and $g = v_0 + \widehat{i}\,\overline{g_0}$ be regular functions on Ω. Then the following properties hold:*

(i) *$f\alpha$ is regular on Ω, if α is any ternary constant;*

(ii) *αf is not regular on Ω, if α is any ternary constant;*

(iii) *$f \pm g$ is regular on Ω;*

(iv) *fg is not regular on Ω. Moreover, if g is a real-valued function, then fg is regular on Ω.*

Proof. It is sufficient to show the second condition of Definition 3.

(i) Let α be a ternary constant with $\alpha = a_0 + \widehat{i}\,\overline{\alpha_0}$, where

$$\alpha_0 = \frac{c_1a_1 + c_2a_2}{\sqrt{c_1^2 + c_2^2}} + \frac{c_2a_1 - c_1a_2}{\sqrt{c_1^2 + c_2^2}}e_1e_2 \tag{27}$$

and a_0, a_1, a_2, c_1, and c_2 are real numbers. Then the equation

$$\begin{aligned} D^*(f\alpha) &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \widehat{i}\frac{\partial}{\partial z_0}\right) \\ &\quad\times\left\{\left(u_0a_0 - f_0\overline{\alpha_0}\right) + \widehat{i}\left(u_0\overline{\alpha_0} + \overline{f_0}a_0\right)\right\} \\ &= \frac{1}{2}\left(\left(\frac{\partial u_0}{\partial x_0}a_0 - \frac{\partial f_0}{\partial x_0}\overline{\alpha_0} - \frac{\partial u_0}{\partial \overline{z_0}}\overline{\alpha_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}a_0\right)\right. \\ &\quad\left.+\widehat{i}\left(\frac{\partial u_0}{\partial x_0}\overline{\alpha_0} + \frac{\partial \overline{f_0}}{\partial x_0}a_0 + \frac{\partial u_0}{\partial z_0}a_0 - \frac{\partial f_0}{\partial z_0}\overline{\alpha_0}\right)\right) \\ &= 0. \end{aligned} \tag{28}$$

Hence, $f\alpha$ is regular on Ω.

(ii) Since

$$D^{*}(\alpha f)=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+\hat{i}\frac{\partial}{\partial z_0}\right)\times\left\{\left(a_0u_0-\alpha_0\overline{f_0}\right)+\hat{i}\left(a_0\overline{f_0}+\overline{\alpha_0}u_0\right)\right\}$$
$$=\frac{1}{2}\left(\left(a_0\frac{\partial u_0}{\partial x_0}-\alpha_0\frac{\partial\overline{f_0}}{\partial x_0}-\overline{\alpha_0}\frac{\partial f_0}{\partial x_0}-\frac{\partial u_0}{\partial z_0}\alpha_0\right)+\hat{i}\left(a_0\frac{\partial\overline{f_0}}{\partial x_0}+\overline{\alpha_0}\frac{\partial u_0}{\partial x_0}+a_0\frac{\partial u_0}{\partial z_0}-\alpha_0\frac{\partial\overline{f_0}}{\partial z_0}\right)\right) \quad (29)$$

is not zero, αf is not always regular on Ω.

(iii) Since

$$D^{*}(f\pm g)=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+\hat{i}\frac{\partial}{\partial z_0}\right)\left\{\left(u_0\pm v_0\right)+\hat{i}\left(\overline{f_0}\pm\overline{g_0}\right)\right\}$$
$$=\frac{1}{2}\left(\left(\frac{\partial u_0}{\partial x_0}\pm\frac{\partial v_0}{\partial x_0}-\frac{\partial\overline{f_0}}{\partial\overline{z_0}}\mp\frac{\partial\overline{g_0}}{\partial\overline{z_0}}\right)+\hat{i}\left(\frac{\partial u_0}{\partial z_0}\pm\frac{\partial v_0}{\partial z_0}+\frac{\partial\overline{f_0}}{\partial x_0}a_0\pm\frac{\partial\overline{g_0}}{\partial x_0}\right)\right)=0, \quad (30)$$

$f\pm g$ is regular on Ω.

(iv) Since

$$D^{*}(fg)=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+\hat{i}\frac{\partial}{\partial z_0}\right)\times\left\{\left(u_0v_0-f_0\overline{g_0}\right)+\hat{i}\left(u_0\overline{g_0}+\overline{f_0}v_0\right)\right\}$$
$$=\frac{1}{2}\left(\left(\frac{\partial u_0}{\partial x_0}-\frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right)v_0+u_0\left(\frac{\partial v_0}{\partial x_0}-\frac{\partial\overline{g_0}}{\partial\overline{z_0}}\right)-\left(\frac{\partial f_0}{\partial x_0}+\frac{\partial u_0}{\partial\overline{z_0}}\right)\overline{g_0}-\left(f_0\frac{\partial\overline{g_0}}{\partial x_0}+\overline{f_0}\frac{\partial v_0}{\partial\overline{z_0}}\right)\right.$$
$$+\hat{i}\left\{\left(\frac{\partial u_0}{\partial x_0}-\frac{\partial f_0}{\partial z_0}\right)\overline{g_0}+u_0\left(\frac{\partial\overline{g_0}}{\partial x_0}+\frac{\partial v_0}{\partial z_0}\right)+\left(\frac{\partial\overline{f_0}}{\partial x_0}+\frac{\partial u_0}{\partial z_0}\right)v_0\left.+\left(\overline{f_0}\frac{\partial v_0}{\partial x_0}-f_0\frac{\partial\overline{g_0}}{\partial z_0}\right)\right\}\right)$$
$$=\frac{1}{2}\left(-\left(f_0\frac{\partial\overline{g_0}}{\partial x_0}+\overline{f_0}\frac{\partial v_0}{\partial\overline{z_0}}\right)+\hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0}-f_0\frac{\partial\overline{g_0}}{\partial z_0}\right)\right) \quad (31)$$

is not zero, fg is not always regular on Ω. □

Theorem 11. *Let Ω be an open set in $\mathbb{R}^3$ and let f and g be regular functions on Ω. Then we have the following equations:*

$$2D^{*}(fg)=(D^{*}f)g+f\frac{\partial g}{\partial x_0}+\hat{i}\left(u_0\frac{\partial g}{\partial z_0}+\hat{i}\overline{f_0}\frac{\partial g}{\partial\overline{z_0}}\right). \quad (32)$$

$$2D(fg)=(Df)g+f\frac{\partial g}{\partial x_0}-\hat{i}\left(u_0\frac{\partial g}{\partial z_0}+\hat{i}\overline{f_0}\frac{\partial g}{\partial\overline{z_0}}\right). \quad (33)$$

Proof. From the proof of Proposition 10, we have the following equations:

$$2D^{*}(fg)=\left\{\left(\frac{\partial u_0}{\partial x_0}-\frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right)+\hat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0}+\frac{\partial u_0}{\partial z_0}\right)\right\}\left(v_0+\hat{i}\overline{g_0}\right)-\left(f_0\frac{\partial\overline{g_0}}{\partial x_0}+\overline{f_0}\frac{\partial v_0}{\partial\overline{z_0}}\right)+\hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0}-f_0\frac{\partial\overline{g_0}}{\partial z_0}\right)$$
$$=\left\{\left(\frac{\partial u_0}{\partial x_0}-\frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right)+\hat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0}+\frac{\partial u_0}{\partial z_0}\right)\right\}\left(v_0+\hat{i}\overline{g_0}\right)+u_0\left(\frac{\partial v_0}{\partial x_0}-\frac{\partial\overline{g_0}}{\partial\overline{z_0}}\right)-\left(f_0\frac{\partial\overline{g_0}}{\partial x_0}+\overline{f_0}\frac{\partial v_0}{\partial\overline{z_0}}\right)$$
$$+\hat{i}u_0\left(\frac{\partial\overline{g_0}}{\partial x_0}+\frac{\partial v_0}{\partial z_0}\right)+\hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0}-f_0\frac{\partial\overline{g_0}}{\partial z_0}\right)$$
$$=(D^{*}f)g+f\frac{\partial g}{\partial x_0}+\hat{i}\left(u_0\frac{\partial g}{\partial z_0}+\hat{i}\overline{f_0}\frac{\partial g}{\partial\overline{z_0}}\right). \quad (34)$$

Similarly, we can prove (33).

We let

$$k=e_1e_2\frac{1}{2}dz_0\wedge d\overline{z_0}+e_2\alpha dx_0\wedge d\overline{z_0}-e_1\beta dx_0\wedge d\overline{z_0}. \quad (35)$$

□

Theorem 12. *Let Ω be an open set in $\mathbb{R}^3$ and U be any domain in Ω with smooth boundary bU such that $U\subset\Omega$. If $f=u_0+\hat{i}\overline{f_0}$ is a regular function on Ω, then*

$$\int_{bU}kf=0, \quad (36)$$

where kf is the ternary product of the form k on the function $f(z)$.

Proof. Since the function $f=u_0+e_1\alpha\overline{f_0}+e_2\beta\overline{f_0}$ exists, we have

$$kf=\left(e_1e_2\frac{1}{2}u_0-e_2\frac{1}{2}\alpha\overline{f_0}+e_1\frac{1}{2}\beta\overline{f_0}\right)dz_0\wedge d\overline{z_0}+\left(e_2\alpha u_0-e_1\beta u_0\right)dx_0\wedge d\overline{z_0}+\left(-e_1e_2\alpha^2\overline{f_0}-e_1e_2\beta^2\overline{f_0}\right)dx_0\wedge dz_0. \quad (37)$$

Then

$$
\begin{aligned}
d(kf) = {} & e_1e_2\left(\frac{\partial u_0}{\partial x_0} - \alpha^2\frac{\partial \overline{f_0}}{\partial \overline{z_0}} - \beta^2\frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right)dV \\
& + e_2\left(-\alpha\frac{\partial \overline{f_0}}{\partial x_0} - \alpha\frac{\partial u_0}{\partial z_0}\right)dV \\
& + e_1\left(\beta\frac{\partial \overline{f_0}}{\partial x_0} + \beta\frac{\partial u_0}{\partial z_0}\right)dV \\
& + \left(-\alpha\beta\frac{\partial \overline{f_0}}{\partial \overline{z_0}} + \alpha\beta\frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right)dV \\
= {} & \left\{e_1e_2\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) - e_2\alpha\left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right. \\
& \left. + e_1\beta\left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\} dV,
\end{aligned}
\tag{38}
$$

where $dV = dx_0 \wedge dz_0 \wedge d\overline{z_0}$ in U, and by the corresponding Cauchy-Riemann system for $f(z)$ in $\mathbb{T}$, $d(kf) = 0$. By Stokes theorem, we obtain the result. □

Remark 13. Since

$$
\left(\widehat{i}\overline{z_0}\right)^k = \begin{cases} (-1)^{k/2}(|z_0|)^k, & k\text{: even} \\ (-1)^{[k/2]}\widehat{i}(|z_0|)^{k-1}\overline{z_0}, & k\text{: odd,} \end{cases} \tag{39}
$$

we have

$$
z^n = \sum_{k=0}^{n} \alpha(k)\, x_0^{n-k} |z_0|^{[k/2]} \overline{z_0}^{\delta_k}, \tag{40}
$$

where

$$
\alpha(k) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k/2}, & k\text{: even} \\ \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{[k/2]}\widehat{i}, & k\text{: odd,} \end{cases}
\qquad
\delta_k = \begin{cases} 0, & k\text{: even} \\ 1, & k\text{: odd.} \end{cases} \tag{41}
$$

And $[k/2]$ is the greatest integer that is less than or equal to $k/2$.

Theorem 14. *Let f be a homogeneous polynomial of degree n with respect to the variables x_0 and $\overline{z_0}$. If f is regular on Ω, then*

$$
f(z) = \frac{1}{n!}\frac{\partial^n f(z)}{\partial x_0^n} z^n, \tag{42}
$$

$$
f(z) = \left(-\widehat{i}\right)^n \frac{1}{n!}\frac{\partial^n f(z)}{\partial z_0^{n-r}\partial \overline{z_0}^r} z^n, \tag{43}
$$

where r is a nonnegative integer.

Proof. Since $f(z)$ is a homogeneous polynomial, then

$$
f(z) = \frac{1}{n}\frac{\partial f(z)}{\partial x_0} z. \tag{44}
$$

Also, since $\partial f(z)/\partial x_0$ is a homogeneous polynomial of degree $n-1$, we have

$$
\frac{\partial f(z)}{\partial x_0} = \frac{1}{n-1}\frac{\partial^2 f(z)}{\partial x_0^2} z. \tag{45}
$$

Then we have

$$
f(z) = \frac{1}{n(n-1)}\frac{\partial^2 f(z)}{\partial x_0^2} z^2. \tag{46}
$$

Continuing this process, we can get the result (42). Similarly, we obtain the result (43). □

4. Properties of Regular Functions with Values in $\mathbb{T}(\mathbb{C})$

We define the number system

$$
\mathbb{T}(\mathbb{C}) = \left\{z \mid z = \widehat{i}\gamma\left(z_1 - e_1e_2z_2\right)\right\}, \tag{47}
$$

where $z_1 = x_1 - (1/2)e_1x_0$ and $z_2 = x_2 - (1/2)e_2x_0$.

The non-commutative multiplication of two numbers $z = \widehat{i}\gamma(z_1 - e_1e_2z_2)$ and $w = \widehat{i}\gamma(w_1 - e_1e_2w_2)$ is defined by

$$
\begin{aligned}
zw &= -\left\{(z_1w_1 + z_2w_2) + e_1e_2\left(\overline{z_2}w_1 - \overline{z_1}w_2\right)\right\}, \\
wz &= -\left\{(w_1z_1 + w_2z_2) + e_1e_2\left(\overline{w_2}z_1 - \overline{w_1}z_2\right)\right\}.
\end{aligned} \tag{48}
$$

The conjugate number z^* of z in $\mathbb{T}(\mathbb{C})$ is given by the following:

$$
z^* = -\widehat{i}\gamma\left(\overline{z_1} - e_1e_2\overline{z_2}\right). \tag{49}
$$

And the norm $|z|$ of z and the inverse z^{-1} of z are given by the following forms:

$$
\begin{aligned}
|z| &= \sqrt{zz^*} = \sqrt{z^*z} \\
&= \sqrt{\left(z_1\overline{z_1} + z_2\overline{z_2}\right) + e_1e_2\left(\overline{z_2z_1} - \overline{z_1z_2}\right)} \\
&= \sqrt{\sum_{j=0}^{2} x_j^2}, \\
z^{-1} &= \frac{z^*}{|z|^2} \quad (z \neq 0).
\end{aligned} \tag{50}
$$

We consider the following differential operators:

$$
D = -\frac{1}{2}\widehat{i}\gamma\left(D_{z_1} - e_1e_2D_{z_2}\right), \qquad D^* = \frac{1}{2}\widehat{i}\gamma\left(D_{\overline{z_1}} - e_1e_2D_{\overline{z_2}}\right), \tag{51}
$$

where

$$
D_{z_1} = \frac{1}{2}e_1\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}, \qquad D_{z_2} = \frac{1}{2}e_2\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2}. \tag{52}
$$

Then the Laplacian operator is

$$4\Delta := DD^{*} = D^{*}D = \sum_{j=0}^{2} \frac{\partial^2}{\partial x_j^2}. \tag{53}$$

Let G be an open set in $\mathbb{C}^2$. The function $f(z)$ that is defined by the following form in G with values in $\mathbb{T}(\mathbb{C})$:

$$f : G \rightarrow \mathbb{T}(\mathbb{C}) \tag{54}$$

satisfies

$$\begin{aligned} z = (z_1, z_2) \in G \longmapsto f(z) &= f(z_1, z_2) \\ &= \hat{i}\gamma (f_1(z_1, z_2) - e_1 e_2 f_2(z_1, z_2)), \end{aligned} \tag{55}$$

where $f_1 = u_1 - (1/2)e_1 u_0$ and $f_2 = u_2 - (1/2)e_2 u_0$ are complex-valued functions with values in $\mathbb{T}(\mathbb{C})$ and u_j $(j = 0, 1, 2)$ are real-valued functions.

Remark 15. The operators D and D^{*} act for a function $f(z)$ on $\mathbb{T}(\mathbb{C})$ as follows:

$$\begin{aligned} Df &= -\hat{i}^2 \left\{ \left(D_{z_1} f_1 + D_{z_2} f_2\right) + e_1 e_2 \left(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2\right) \right\}, \\ D^{*} f &= \hat{i}^2 \left\{ \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2\right) + e_1 e_2 \left(D_{z_2} f_1 - D_{z_1} f_2\right) \right\}. \end{aligned} \tag{56}$$

We define a commutative multiplication of two numbers $z = \hat{i}\gamma(z_1 - e_1 e_2 z_2)$ and $w = \hat{i}\gamma(w_1 - e_1 e_2 w_2)$ by

$$\begin{aligned} z \odot w = w \odot z &= \frac{1}{2}(zw + wz) \\ &= \frac{1}{2}\hat{i}^2 \{ (z_1 w_1 + z_2 w_2 + w_1 z_1 + w_2 z_2) \\ &\quad + e_1 e_2 (\overline{z_2} w_1 - \overline{z_1} w_2 \\ &\quad + \overline{w_2} z_1 - \overline{w_1} z_2) \}. \end{aligned} \tag{57}$$

Remark 16. The operators D and D^{*} act for a function $f(z)$ on $\mathbb{T}(\mathbb{C})$ as follows:

$$\begin{aligned} D \odot f &= \frac{1}{2}(Df + fD) \\ &= \Big\{ \left(D_{z_1} f_1 + D_{z_2} f_2\right) \\ &\quad + \frac{1}{2} e_1 e_2 \Big(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2 \\ &\quad + \overline{f_2} D_{z_1} - \overline{f_1} D_{z_2}\Big) \Big\} \\ &= \Big\{ \Big(D_{z_1} f_1 + D_{z_2} f_2 + \frac{1}{2}\frac{\partial u_0}{\partial x_0}\Big) \\ &\quad + \frac{1}{2} e_1 e_2 \Big(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2 \\ &\quad + D_{z_1} \overline{f_2} - D_{z_2} \overline{f_1}\Big) \Big\}, \end{aligned}$$

$$\begin{aligned} D^{*} \odot f &= \frac{1}{2}(D^{*} f + f D^{*}) \\ &= -\Big\{ \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2\right) \\ &\quad + \frac{1}{2} e_1 e_2 \Big(D_{z_2} f_1 - D_{z_1} f_2 \\ &\quad + \overline{f_2} D_{\overline{z_1}} - \overline{f_1} D_{\overline{z_2}}\Big) \Big\} \\ &= -\Big\{ \Big(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 - \frac{1}{2}\frac{\partial u_0}{\partial x_0}\Big) \\ &\quad + \frac{1}{2} e_1 e_2 \Big(D_{z_2} f_1 - D_{z_1} f_2 \\ &\quad + D_{\overline{z_1}} \overline{f_2} - D_{\overline{z_2}} \overline{f_1}\Big) \Big\}. \end{aligned} \tag{58}$$

Definition 17. Let G be a domain in $\mathbb{C}^2$. A function $f = \hat{i}\gamma(f_1 - e_1 e_2 f_2)$ is said to be dot-regular in G if the following two conditions are satisfied:

(i) f_1 and f_2 are differential functions in G,

(ii) $D^{*} \odot f = 0$ in G.

Remark 18. The above equation (ii) of Definition 17 is equivalent as follows:

$$\begin{aligned} D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 &= \frac{1}{2}\frac{\partial u_0}{\partial x_0}, \\ D_{z_2} f_1 - D_{z_1} f_2 &= D_{\overline{z_2}} \overline{f_1} - D_{\overline{z_1}} \overline{f_2}. \end{aligned} \tag{59}$$

Theorem 19. *Let G be an open set in $\mathbb{C}^2$ and let f be a dot-regular function on G. Then the derivative f' of f defined by $D \odot f$ is*

$$\begin{aligned} f' &= 2\hat{i}\gamma \left(D_{\overline{z_1}} - D_{z_1}\right) f = 2e_1 \left(D_{\overline{z_1}} - D_{z_1}\right) f, \\ f' &= -2\hat{i}\gamma \left(D_{z_2} - D_{\overline{z_2}}\right) f = 2e_2 \left(D_{\overline{z_2}} - D_{z_2}\right) f. \end{aligned} \tag{60}$$

Proof. By the definition of a dot-regular function with values in $\mathbb{T}(\mathbb{C})$, we have

$$\begin{aligned} D \odot f &= \Big(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 + e_1 \frac{\partial u_1}{\partial x_0} + e_2 \frac{\partial u_2}{\partial x_0} + \frac{3}{2}\frac{\partial u_0}{\partial x_0} \Big) \\ &\quad + \frac{1}{2} e_1 e_2 \Big(D_{z_2} f_1 - D_{z_1} f_2 + D_{\overline{z_1}} \overline{f_2} \\ &\quad - D_{\overline{z_2}} \overline{f_1} - 2e_2 \frac{\partial u_1}{\partial x_0} + 2e_1 \frac{\partial u_2}{\partial x_0} \Big) \\ &= 2\hat{i}\gamma \left(D_{\overline{z_1}} - D_{z_1}\right) f \end{aligned} \tag{61}$$

on G. And, similarly, we have

$$D \odot f = -2\hat{i}\gamma \left(D_{z_2} - D_{\overline{z_2}}\right) f \tag{62}$$

on G. □

Acknowledgment

Kwang Ho Shon was supported by the Research Fund Program of Research Institute for Basic Sciences, Pusan National University, Korea, 2013, Project no. RIBS-PNU-2013-101.

References

[1] R. Fueter, "Die funktionentheorie der differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen variablen," *Commentarii Mathematici Helvetici*, vol. 7, no. 1, pp. 307–330, 1935.

[2] C. A. Deavours, "The quaternion calculus," *The American Mathematical Monthly*, vol. 80, pp. 995–1008, 1973.

[3] A. Sudbery, "Quaternionic analysis," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 85, no. 2, pp. 199–224, 1979.

[4] M. Naser, "Hyperholomorphic functions," *Silberian Mathematical Journal*, vol. 12, pp. 959–968, 1971.

[5] H. Koriyama, H. Mae, and K. Nôno, "Hyperholomorphic functions and holomorphic functions in quaternionic analysis," *Bulletin of Fukuoka University of Education*, vol. 60, pp. 1–9, 2011.

[6] K. Nôno, "Hyperholomorphic functions of a quaternion variable," *Bulletin of Fukuoka University of Education*, vol. 32, pp. 21–37, 1983.

[7] E. Cho, "De moivre's formula for quaternions," *Applied Mathematics Letters*, vol. 11, no. 6, pp. 33–35, 1998.

[8] S. J. Sangwine and N. L. Bihan, "Quaternion polar representation with a complex modulus and complex argument inspired by the cayley-dickson form," *Advances in Applied Clifford Algebras*, vol. 20, no. 1, pp. 111–120, 2010.

[9] R. Fueter, "Die theorie der regulären funktionen einer quaternionenvariablen," in *Comptès Rendus du Congrès International des Mathèmaticiens, Oslo 1936*, vol. 1, pp. 75–91, 1935.

[10] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, vol. 76 of *Research Notes in Mathematics*, 1982.

[11] S. J. Lim and K. H. Shon, "Hyperholomorphic fucntions and hyperconjugate harmonic functions of octonion variables," *Journal of Inequalities and Applications*, vol. 77, pp. 1–8, 2013.

[12] S. J. Lim and K. H. Shon, "Dual quaternion functions and its applications," *Journal of Applied Mathematics*, vol. 2013, Article ID 583813, 6 pages, 2013.

[13] S. J. Lim and K. H. Shon, "Regularity of functions with values in a non-commutativealgebra of complex matrix algebras," Preprint.