

Research Article

Exponential Stability of Impulsive Delayed Reaction-Diffusion Cellular Neural Networks via Poincaré Integral Inequality

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Received 15 November 2012; Accepted 8 February 2013

Academic Editor: Qi Luo

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This work is devoted to the stability study of impulsive cellular neural networks with time-varying delays and reaction-diffusion terms. By means of new Poincaré integral inequality and Gronwall-Bellman-type impulsive integral inequality, we summarize some novel and concise sufficient conditions ensuring the global exponential stability of equilibrium point. The provided stability criteria are applicable to Dirichlet boundary condition and show that not only the reaction-diffusion coefficients but also the regional features including the boundary and dimension of spatial variable can influence the stability. Two examples are finally illustrated to demonstrate the effectiveness of our obtained results.

1. Introduction

Cellular neural networks (CNNs), proposed by Chua and Yang in 1988 [1, 2], have been the focus of a number of investigations due to their potential applications in various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision [3–7]. As the switching speed of neurons and amplifiers is finite in the implementation of neural networks, time delays are inevitable and therefore a type of more effective models is afterwards introduced, called delayed cellular neural networks (DCNNs). Actually, DCNNs have been found to be helpful in solving some dynamic image processing and pattern recognition problems.

As we all know, all the applications of CNNs and DCNNs depend heavily on the dynamic behaviors such as stability, convergence, and oscillatory [8, 9], wherein stability analysis is a major concern in the designs and applications. Correspondingly, the stability of CNNs and DCNNs is a subject of current interest and considerable theoretical efforts have been put into this topic with many good results reported (see, e.g., [10–13]).

With reference to neural networks, however, it is noteworthy that the state of electronic networks is often subject to instantaneous perturbations which may be caused by a

switching phenomenon, frequency change, or other sudden noise. On this account, neural networks will experience abrupt change at certain instants, exhibiting impulse effects [14, 15]. For instance, according to Arbib [16] and Haykin [17], when a stimulus from the body or the external environment is received by receptors, the electrical impulses will be conveyed to the neural net and impulse effects arise naturally in the net. In view of this discovery, many scientists have shown growing interests in the influence that the impulses may have on CNNs or DCNNs with a result that a large number of relevant results have been achieved (see, e.g., [18–24]).

Besides impulsive effects, diffusing effects are also non-ignorable in reality since the diffusion is unavoidable when the electrons are moving in asymmetric electromagnetic fields. Therefore, the model of impulsive delayed reaction-diffusion neural networks appears as a natural description of the observed evolution phenomena of several real world problems. This one acknowledgement poses a new challenge to the stability research of neural networks.

So far, there have been some theoretical achievements [25–33] on the stability of impulsive delayed reaction-diffusion neural networks. Previously, authors of [27–32] studied the stability of impulsive delayed reaction-diffusion neural networks and put forward several stability criteria by impulsive differential inequality and Green formula, wherein

the reaction-diffusion term is evaluated to be less than zero by means of Green formula and thereby the presented stability criteria are shown to be wholly independent of diffusion. According to this result, we fail to see the influence of diffusion on stability.

Recently, it is encouraging that, for impulsive delayed reaction-diffusion neural network, some new stability criteria involving diffusion are obtained in [25, 26, 33–36]. Meanwhile the estimation of reaction-diffusion term is not merely less than zero, instead a more accurate one is given; that is, the reaction-diffusion term is verified to be less than a negative definite term by using some inequalities together with Green formula. It is thereby testified that the diffusion does contribute to the stability of impulsive neural networks.

In [25], the authors quoted the following inequality to deal with the reaction-diffusion terms:

$$\int_{\Omega^*} \left| \frac{\partial v(x)}{\partial x_j} \right|^2 dx \geq \frac{1}{l_j^2} \int_{\Omega^*} v^2(x) dx, \quad (1)$$

where Ω^* is a cube $|x_j| < l_j$ ($j = 1, 2, \dots, m$) and $v(x)$ is a real-valued function belonging to $C_0^1(\Omega^*)$. We can easily derive from this inequality that

$$\int_{\Omega^*} |\nabla v|^2 dx \geq \left(\int_{\Omega^*} v^2(x) dx \right) \left(\sum_{j=1}^m \frac{1}{l_j^2} \right). \quad (2)$$

For better exploring the influence of diffusion on stability, we wonder if we can get a more accurate estimate of reaction-diffusion term. Fortunately, we find the following new Poincaré integral inequality supporting this idea:

$$\int_{\mathcal{S}} |\nabla v(x)|^2 dx \geq \frac{4n}{B^2} \int_{\mathcal{S}} v^2(x) dx. \quad (3)$$

One can refer to Lemma 3 in Section 2 for the details of this inequality.

On the other hand, it is well known that the theory of differential and integral inequalities plays an important role in the qualitative and quantitative study of solution to differential equations. Up till now, there have been many applications of impulsive differential inequalities to impulsive dynamic systems, followed by lots of stability criteria provided. However, these stability criteria appear a bit complicated and we wonder if we can deduce relatively concise stability criteria by using impulsive integral inequalities.

Motivated by these, we attempt to, for impulsive delayed neural networks, employ new Poincaré integral inequality to further investigate the influence of diffusion on the stability and combine Gronwall-Bellman-type impulsive integral inequality so as to provide some new and concise stability criteria. The rest of this paper is organized as follows. In Section 2, the model of impulsive cellular neural networks with time-varying delays and reaction-diffusion terms as well as Dirichlet boundary condition is outlined; in addition, some facts and lemmas are introduced for later reference. In Section 3, we provide a new estimate on the reaction-diffusion term by the agency of new Poincaré integral inequality and then discuss the global exponential stability

of equilibrium point by utilizing Gronwall-Bellman-type impulsive integral inequality with a result of some novel and concise stability criteria presented. To conclude, two illustrative examples are given in Section 4 to verify the effectiveness of our obtained results.

2. Preliminaries

Let $R_+ = [0, \infty)$ and $t_0 \in R_+$. Let R^n denote the n -dimensional Euclidean space, and let $\Omega = \prod_{i=1}^m [d_i, k_i]$ be a fixed rectangular region in R^m and $M := \max\{k_i - d_i : i = 1, \dots, m\}$. As usual, denote

$$C_0^1(\Omega) = \left\{ v \mid v \text{ and } D_j v = \frac{\partial v}{\partial x_j} \text{ are continuous on } \Omega, \right. \\ \left. v|_{\partial\Omega} = 0, 1 \leq j \leq m \right\}. \quad (4)$$

Consider the following impulsive cellular neural network with time-varying delays and reaction-diffusion terms:

$$\frac{\partial u_i(t, x)}{\partial t} = \sum_{s=1}^m \frac{\partial}{\partial x_s} \left(D_{is} \frac{\partial u_i(t, x)}{\partial x_s} \right) - a_i u_i(t, x) \\ + \sum_{j=1}^n b_{ij} f_j(u_j(t, x)) + \sum_{j=1}^n c_{ij} f_j(u_j(t - \tau_j(t), x)) \\ t \geq t_0, t \neq t_k, x \in \Omega, \\ i = 1, 2, \dots, n, k = 1, 2, \dots, \quad (5)$$

$$u_i(t_k + 0, x) = u_i(t_k, x) + P_{ik}(u_i(t_k, x)), \\ x \in \Omega, i = 1, 2, \dots, n, k = 1, 2, \dots, \quad (6)$$

where n corresponds to the numbers of units in a neural network; $x = (x_1, \dots, x_m)^T \in \Omega$, $u_i(t, x)$ denotes the state of the i th neuron at time t and in space x ; $D_{is} = \text{const} > 0$ represents transmission diffusion of the i th unit; activation function $f_j(u_j(t, x))$ stands for the output of the j th unit at time t and in space x ; b_{ij} , c_{ij} , and a_i are constants: b_{ij} indicates the connection strength of the j th unit on the i th unit at time t and in space x , c_{ij} denotes the connection weight of the j th unit on the i th unit at time $t - \tau_j(t)$ and in space x , where $\tau_j(t)$ corresponds to the transmission delay along the axon of the j th unit, satisfying $0 \leq \tau_j(t) \leq \tau$ ($\tau = \text{const}$) and $\tau_j(t) < (1 - (1/h))$ ($h > 0$), and $a_i > 0$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t and in space x . The fixed moments t_k ($k = 1, 2, \dots$) are called impulsive moments meeting $0 \leq t_0 < t_1 < t_2 < \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$; $u_i(t_k + 0, x)$ and $u_i(t_k - 0, x)$ represent the right-hand and left-hand limit of $u_i(t, x)$ at time t_k and in space x , respectively. $P_{ik}(u_i(t_k, x))$ stands for the abrupt change of $u_i(t, x)$ at the impulsive moment t_k and in space x .

Denote by $u(t, x) = u(t, x; t_0, \varphi)$, $u \in R^n$, the solution of system (5)-(6), satisfying the initial condition

$$u(s, x; t_0, \varphi) = \varphi(s, x), \quad t_0 - \tau \leq s \leq t_0, \quad x \in \Omega, \quad (7)$$

and Dirichlet boundary condition

$$u(t, x; t_0, \varphi) = 0, \quad t \geq t_0, \quad x \in \partial\Omega, \quad (8)$$

where the vector-valued function $\varphi(s, x) = (\varphi_1(s, x), \dots, \varphi_n(s, x))^T$ is such that $\int_{\Omega} \sum_{i=1}^n \varphi_i^2(s, x) dx$ is bounded on $[t_0 - \tau, t_0]$.

The solution $u(t, x) = u(t, x; t_0, \varphi) = (u_1(t, x; t_0, \varphi), \dots, u_n(t, x; t_0, \varphi))^T$ of problem (5)-(8) is, for the time variable t , a piecewise continuous function with the first kind discontinuity at the points t_k ($k = 1, 2, \dots$), where it is continuous from the left; that is, the following relations are true:

$$\begin{aligned} u_i(t_k - 0, x) &= u_i(t_k, x), \\ u_i(t_k + 0, x) &= u_i(t_k, x) + P_{ik}(u_i(t_k, x)). \end{aligned} \quad (9)$$

Throughout this paper, the norm of $u(t, x; t_0, \varphi)$ is defined by

$$\|u(t, x; t_0, \varphi)\|_{\Omega}^2 = \sum_{i=1}^n \int_{\Omega} u_i^2(t, x; t_0, \varphi) dx. \quad (10)$$

Before proceeding, we introduce two hypotheses as follows:

- (H1) $f_i(\bullet) : R \rightarrow R$ satisfies $f_i(0) = 0$, and there exists a constant $l_i > 0$ such that $|f_i(y_1) - f_i(y_2)| \leq l_i |y_1 - y_2|$ for all $y_1, y_2 \in R$ and $i = 1, 2, \dots, n$.
- (H2) $P_{ik}(\bullet) : R \rightarrow R$ is continuous and $P_{ik}(0) = 0$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$.

According to (H1) and (H2), it is easy to see that problem (5)-(8) admits an equilibrium point $u = 0$.

Definition 1 (see [25]). The equilibrium point $u = 0$ of problem (5)-(8) is said to be globally exponentially stable if there exist constants $\kappa > 0$ and $\omega \geq 1$ such that

$$\|u(t, x; t_0, \varphi)\|_{\Omega} \leq \omega \|\varphi\|_{\Omega} e^{-\kappa(t-t_0)}, \quad t \geq t_0, \quad (11)$$

where $\|\varphi\|_{\Omega}^2 = \sup_{t_0-\tau \leq s \leq t_0} \sum_{i=1}^n \int_{\Omega} \varphi_i^2(s, x) dx$.

Lemma 2 (see [37] Gronwall-Bellman-type Impulsive Integral Inequality). Assume that

- (A1) the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \dots$, with $\lim_{k \rightarrow \infty} t_k = \infty$,
- (A2) $q \in PC^1[R_+, R]$ and $q(t)$ is left-continuous at t_k , $k = 1, 2, \dots$,
- (A3) $p \in C[R_+, R_+]$ and for $k = 1, 2, \dots$,

$$q(t) \leq c + \int_{t_0}^t p(s) q(s) ds + \sum_{t_0 < t_k < t} \eta_k q(t_k), \quad t \geq t_0, \quad (12)$$

where $\eta_k \geq 0$ and $c = \text{const}$. Then,

$$q(t) \leq c \prod_{t_0 < t_k < t} (1 + \eta_k) \exp\left(\int_{t_0}^t p(s) ds\right), \quad t \geq t_0. \quad (13)$$

Lemma 3 (see [38] Poincaré integral inequality). Let $\mathcal{S} = \prod_{i=1}^n [a_i, b_i]$ be a fixed rectangular region in R^n and $B := \max\{b_i - a_i : i = 1, \dots, n\}$. For any $v(x) \in C_0^1(\mathcal{S})$,

$$\int_{\mathcal{S}} v^2(x) dx \leq \frac{B^2}{4n} \int_{\mathcal{S}} |\nabla v(x)|^2 dx. \quad (14)$$

Remark 4. According to Lemma 2.1 in [25], we know if \mathcal{S} is a cube $|x_j| < l_j$ ($j = 1, 2, \dots, m$) and $v(x)$ is a real-valued function belonging to $C_0^1(\mathcal{S})$, then

$$\int_{\mathcal{S}} \left| \frac{\partial v(x)}{\partial x_j} \right|^2 dx \geq \frac{1}{l_j^2} \int_{\mathcal{S}} v^2(x) dx, \quad (15)$$

which yields

$$\int_{\mathcal{S}} |\nabla v|^2 dx \geq \left(\int_{\mathcal{S}} v^2(x) dx \right) \left(\sum_{j=1}^m \frac{1}{l_j^2} \right). \quad (16)$$

Through the simple example as follows, we can find that in some cases the estimate $\int_{\mathcal{S}} |\nabla v(x)|^2 dx \geq (4n/B^2) \int_{\mathcal{S}} v^2(x) dx$ shown in Lemma 3 can do better. Let $\mathcal{S} = [0, 1] \times [0, 2]$, we derive from Lemma 2.1 in [25] that

$$\int_{\mathcal{S}} |\nabla v|^2 dx \geq \left(\int_{\mathcal{S}} v^2(x) dx \right) \left(\sum_{j=1}^m \frac{1}{l_j^2} \right) = \frac{5}{4} \int_{\mathcal{S}} v^2(x) dx, \quad (17)$$

whereas the application of Lemma 3 of this paper will give

$$\int_{\mathcal{S}} |\nabla v(x)|^2 dx \geq \frac{4n}{B^2} \int_{\mathcal{S}} v^2(x) dx = 2 \int_{\mathcal{S}} v^2(x) dx, \quad (18)$$

which is obviously superior to $\int_{\mathcal{S}} |\nabla v|^2 dx \geq (5/4) (\int_{\mathcal{S}} v^2(x) dx)$.

3. Main Results

Theorem 5. Provided that one has the following:

- (1) let $\underline{D} = \min\{D_{is} : i = 1, \dots, n; s = 1, \dots, m\} > 0$ and denote $8m\underline{D}/M^2 = \chi$;
- (2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik} u_i(t_k, x)$, $0 \leq \theta_{ik} \leq 2$;
- (3) there exists a constant $\gamma > 0$ satisfying $\gamma + \lambda + hpe^{\gamma\tau} > 0$ as well as $\lambda + hpe^{\gamma\tau} < 0$, where $\lambda = \max_{i=1, \dots, n} (-\chi - 2a_i + \sum_{j=1}^n (b_{ij}^2 + c_{ij}^2)) + \rho$, $\rho = n \max_{i=1, \dots, n} (l_i^2)$;

then, the equilibrium point $u = 0$ of problem (5)-(8) is globally exponentially stable with convergence rate $-(\lambda + hpe^{\gamma\tau})/2$.

Proof. Multiplying both sides of (5) by $u_i(t, x)$ and integrating with respect to spatial variable x on Ω , we get

$$\begin{aligned} & \frac{d \left(\int_{\Omega} u_i^2(t, x) dx \right)}{dt} \\ &= 2 \sum_{s=1}^m \int_{\Omega} u_i(t, x) \frac{\partial}{\partial x_s} \left(D_{is} \frac{\partial u_i(t, x)}{\partial x_s} \right) dx \\ & \quad - 2a_i \int_{\Omega} u_i^2(t, x) dx \\ & \quad + 2 \sum_{j=1}^n b_{ij} \int_{\Omega} u_i(t, x) f_j(u_j(t, x)) dx \\ & \quad + 2 \sum_{j=1}^n c_{ij} \int_{\Omega} u_i(t, x) f_j(u_j(t - \tau_j(t), x)) dx \\ & \quad t \geq t_0, \quad t \neq t_k, \quad i = 1, \dots, n, \quad k = 1, 2, \dots \end{aligned} \quad (19)$$

Regarding the right-hand part of (19), the first term becomes by using Green formula, Dirichlet boundary condition, Lemma 3, and condition (1) of Theorem 5

$$\begin{aligned} & 2 \sum_{s=1}^m \int_{\Omega} u_i(t, x) \frac{\partial}{\partial x_s} \left(D_{is} \frac{\partial u_i(t, x)}{\partial x_s} \right) dx \\ &= -2 \sum_{s=1}^m \int_{\Omega} D_{is} \left(\frac{\partial u_i(t, x)}{\partial x_s} \right)^2 dx \\ &\leq \frac{-8mD}{M^2} \int_{\Omega} u_i^2(t, x) dx \triangleq -\chi \int_{\Omega} u_i^2(t, x) dx. \end{aligned} \quad (20)$$

Moreover, From (H1), we have

$$\begin{aligned} & 2 \sum_{j=1}^n b_{ij} \int_{\Omega} u_i(t, x) f_j(u_j(t, x)) dx \\ &\leq 2 \sum_{j=1}^n |b_{ij}| \int_{\Omega} |u_i(t, x)| |f_j(u_j(t, x))| dx \\ &\leq 2 \sum_{j=1}^n \int_{\Omega} l_j |b_{ij}| |u_i(t, x)| |u_j(t, x)| dx \\ &\leq \sum_{j=1}^n \int_{\Omega} (b_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t, x)) dx, \\ & 2 \sum_{j=1}^n c_{ij} \int_{\Omega} u_i(t, x) f_j(u_j(t - \tau_j(t), x)) dx \\ &\leq 2 \sum_{j=1}^n |c_{ij}| \int_{\Omega} |u_i(t, x)| |f_j(u_j(t - \tau_j(t), x))| dx \\ &\leq 2 \sum_{j=1}^n \int_{\Omega} l_j |c_{ij}| |u_i(t, x)| |u_j(t - \tau_j(t), x)| dx \\ &\leq \sum_{j=1}^n \int_{\Omega} (c_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t - \tau_j(t), x)) dx. \end{aligned} \quad (21)$$

(22)

Consequently, substituting (20)–(22) into (19) produces

$$\begin{aligned} & \frac{d \left(\int_{\Omega} u_i^2(t, x) dx \right)}{dt} \\ &\leq -\chi \int_{\Omega} u_i^2(t, x) dx - 2a_i \int_{\Omega} u_i^2(t, x) dx \\ & \quad + \sum_{j=1}^n \int_{\Omega} (b_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t, x)) dx \\ & \quad + \sum_{j=1}^n \int_{\Omega} (c_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t - \tau_j(t), x)) dx \end{aligned} \quad (23)$$

for $t \geq t_0, t \neq t_k, i = 1, \dots, n, k = 1, 2, \dots$

Define a Lyapunov function $V_i(t)$ as $V_i(t) = \int_{\Omega} u_i^2(t, x) dx$. It is easy to find that $V_i(t)$ is a piecewise continuous function with the first kind discontinuous points t_k ($k = 1, 2, \dots$), where it is continuous from the left, that is, $V_i(t_k - 0) = V_i(t_k)$ ($k = 1, 2, \dots$). In addition, we also see

$$V_i(t_k + 0) \leq V_i(t_k), \quad k = 0, 1, 2, \dots, \quad (24)$$

as $V_i(t_0 + 0) \leq V_i(t_0)$ and the following estimate derived from condition (2) of Theorem 5:

$$\begin{aligned} u_i^2(t_k + 0, x) &= (-\theta_{ik} u_i(t_k, x) + u_i(t_k, x))^2 \\ &= (1 - \theta_{ik})^2 u_i^2(t_k, x) \leq u_i^2(t_k, x), \end{aligned} \quad (25)$$

$k = 1, 2, \dots$

Put $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \dots$. It then results from (23) that

$$\begin{aligned} & \frac{dV_i(t)}{dt} \leq -\chi \int_{\Omega} u_i^2(t, x) dx - 2a_i \int_{\Omega} u_i^2(t, x) dx \\ & \quad + \sum_{j=1}^n \int_{\Omega} (b_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t, x)) dx \\ & \quad + \sum_{j=1}^n \int_{\Omega} (c_{ij}^2 u_i^2(t, x) + l_j^2 u_j^2(t - \tau_j(t), x)) dx \\ &\leq \left(-\chi - 2a_i + \sum_{j=1}^n b_{ij}^2 + \sum_{j=1}^n c_{ij}^2 \right) V_i(t) \\ & \quad + \max_{i=1, \dots, n} (l_i^2) \sum_{j=1}^n V_j(t) \\ & \quad + \max_{i=1, \dots, n} (l_i^2) \sum_{j=1}^n V_j(t - \tau_j(t)) \\ & \quad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots \end{aligned} \quad (26)$$

Choose $V(t)$ of the form $V(t) = \sum_{i=1}^n V_i(t)$. From (26), one reads

$$\frac{dV(t)}{dt} \leq \lambda V(t) + \rho \sum_{j=1}^n V_j(t - \tau_j(t)), \quad (27)$$

$$t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots,$$

where $\lambda = \max_{i=1, \dots, n} (-\chi - 2a_i + \sum_{j=1}^n (b_{ij}^2 + c_{ij}^2)) + \rho$ and $\rho = n \max_{i=1, \dots, n} (l_i^2)$.

Now construct $V^*(t) = e^{\gamma(t-t_0)} V(t)$ again, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$. Evidently, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k ($k = 1, 2, \dots$), where it is continuous from the left, that is, $V^*(t_k - 0) = V^*(t_k)$ ($k = 1, 2, \dots$). Moreover, at $t = t_k$ ($k = 0, 1, 2, \dots$), we find by use of (24)

$$V^*(t_k + 0) \leq V^*(t_k), \quad k = 0, 1, 2, \dots \quad (28)$$

Set $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \dots$. By virtue of (27), one has

$$\begin{aligned} \frac{dV^*(t)}{dt} &= \gamma e^{\gamma(t-t_0)} V(t) + e^{\gamma(t-t_0)} \frac{dV(t)}{dt} \\ &\leq \gamma e^{\gamma(t-t_0)} V(t) \\ &\quad + \left(\lambda V(t) + \rho \sum_{j=1}^n V_j(t - \tau_j(t)) \right) e^{\gamma(t-t_0)} \\ &= (\gamma + \lambda) V^*(t) + \rho e^{\gamma(t-t_0)} \sum_{j=1}^n V_j(t - \tau_j(t)) \\ &\quad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots \end{aligned} \quad (29)$$

Choose small enough $\varepsilon > 0$. Integrating (29) from $t_k + \varepsilon$ to t gives

$$\begin{aligned} V^*(t) &\leq V^*(t_k + \varepsilon) + (\gamma + \lambda) \int_{t_k + \varepsilon}^t V^*(s) ds \\ &\quad + \int_{t_k + \varepsilon}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \\ &\quad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, \end{aligned} \quad (30)$$

which yields, after letting $\varepsilon \rightarrow 0$ in (30),

$$\begin{aligned} V^*(t) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ &\quad + \int_{t_k}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \\ &\quad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots \end{aligned} \quad (31)$$

Next we will estimate the value of $V^*(t)$ at $t = t_{k+1}$, $k = 0, 1, 2, \dots$. For small enough $\varepsilon > 0$, we put $t = t_{k+1} - \varepsilon$. An application of (31) leads to, for $k = 0, 1, 2, \dots$,

$$\begin{aligned} V^*(t_{k+1} - \varepsilon) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1} - \varepsilon} V^*(s) ds \\ &\quad + \int_{t_k}^{t_{k+1} - \varepsilon} \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds. \end{aligned} \quad (32)$$

If we let $\varepsilon \rightarrow 0$ in (32), there results

$$\begin{aligned} V^*(t_{k+1} - 0) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1}} V^*(s) ds \\ &\quad + \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds, \\ &\quad k = 0, 1, 2, \dots \end{aligned} \quad (33)$$

Note that $V^*(t_{k+1} - 0) = V^*(t_{k+1})$ is applicable for $k = 0, 1, 2, \dots$. Thus,

$$\begin{aligned} V^*(t_{k+1}) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1}} V^*(s) ds \\ &\quad + \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \end{aligned} \quad (34)$$

holds for $k = 0, 1, 2, \dots$. By synthesizing (31) and (34), we then arrive at

$$\begin{aligned} V^*(t) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ &\quad + \int_{t_k}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \\ &\quad t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots \end{aligned} \quad (35)$$

This, together with (28), results in

$$\begin{aligned} V^*(t) &\leq V^*(t_k) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ &\quad + \int_{t_k}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \end{aligned} \quad (36)$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \dots$

Recalling assumptions that $0 \leq \tau_j(t) \leq \tau$ and $\dot{\tau}_j(t) < (1 - (1/h))(h > 0)$, we obtain

$$\begin{aligned} &\int_{t_k}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s)) ds \\ &= \sum_{j=1}^n \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} \rho e^{\gamma(\theta + \tau_j(s) - t_0)} V_j(\theta) \frac{1}{1 - \dot{\tau}_j(s)} d\theta \\ &\leq h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta. \end{aligned} \quad (37)$$

Hence,

$$\begin{aligned} V^*(t) &\leq V^*(t_k) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(s-t_0)} V_j(s) ds \\ &\quad t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots \end{aligned} \quad (38)$$

By induction argument, we reach

$$\begin{aligned} V^*(t_k) &\leq V^*(t_{k-1}) + (\gamma + \lambda) \int_{t_{k-1}}^{t_k} V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(s-t_0)} V_j(s) ds, \\ &\quad \vdots \\ V^*(t_2) &\leq V^*(t_1) + (\gamma + \lambda) \int_{t_1}^{t_2} V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(s-t_0)} V_j(s) ds, \\ V^*(t_1) &\leq V^*(t_0) + (\gamma + \lambda) \int_{t_0}^{t_1} V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(s-t_0)} V_j(s) ds. \end{aligned} \quad (39)$$

Therefore,

$$\begin{aligned} V^*(t) &\leq V^*(t_0) + (\gamma + \lambda) \int_{t_0}^t V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(s-t_0)} V_j(s) ds \\ &\leq V^*(t_0) + (\gamma + \lambda) \int_{t_0}^t V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^t e^{\gamma(s-t_0)} V_j(s) ds \\ &= V^*(t_0) + (\gamma + \lambda + h\rho e^{\gamma\tau}) \int_{t_0}^t V^*(s) ds \\ &\quad + h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)} V_j(s) ds \\ &\quad t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots \end{aligned} \quad (40)$$

Since

$$\begin{aligned} &h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)} V_j(s) ds \\ &\leq h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau}^{t_0} V_j(s) ds \\ &= h\rho e^{\gamma\tau} \int_{t_0-\tau}^{t_0} \left(\sum_{j=1}^n \int_{\Omega} \varphi_j^2(s, x) dx \right) ds \\ &\leq \tau h\rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2, \end{aligned} \quad (41)$$

we claim

$$\begin{aligned} V^*(t) &\leq V^*(t_0) + \tau h\rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2 \\ &\quad + (\gamma + \lambda + h\rho e^{\gamma\tau}) \int_{t_0}^t V^*(s) ds \\ &\quad t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots \end{aligned} \quad (42)$$

According to Lemma 2, we know

$$\begin{aligned} V^*(t) &\leq \left(V^*(t_0) + \tau h\rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2 \right) \\ &\quad \times \exp \{ (\gamma + \lambda + h\rho e^{\gamma\tau}) (t - t_0) \}, \quad t \geq t_0 \end{aligned} \quad (43)$$

which reduces to

$$\begin{aligned} &\|u(t, x; t_0, \varphi)\|_{\Omega} \\ &\leq \sqrt{1 + \tau h\rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2} \exp \left\{ \left(\frac{\lambda + h\rho e^{\gamma\tau}}{2} \right) (t - t_0) \right\}, \\ &\quad t \geq t_0. \end{aligned} \quad (44)$$

This completes the proof. \square

Remark 6. According to Theorem 5, we see that the diffusion can really influence the stability of equilibrium point $u = 0$ of problem (5)–(8), wherein the factors embrace not only the reaction-diffusion coefficients but also the regional features including the dimension and boundary of spatial variable. Owing to the employ of new Poincaré integral inequality, in this paper, the estimation of reaction-diffusion terms is superior to that in [25] in some cases, and this will be helpful to further know the influence of diffusion on stability. What is more, from condition (1) of Theorem 5, we also see that the dimension of spatial variable has an impact on the stability while this is not mentioned in [25].

Remark 7. Among the three conditions of Theorem 5, condition (3) is critical and therefore we must ensure the existence of constant $\gamma > 0$. Fortunately, it is not difficult to find that there must exist a constant $\gamma > 0$ satisfying condition (3) if $\lambda < -h\rho$ which is easily checked.

Theorem 8. *Providing that one has the following:*

- (1) let $\underline{D} = \min\{D_{is} : i = 1, \dots, n; s = 1, \dots, m\} > 0$ and denote $8m\underline{D}/M^2 = \chi$;
- (2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik}u_i(t_k, x)$, $1 - \sqrt{1 + \alpha} \leq \theta_{ik} \leq 1 + \sqrt{1 + \alpha}$, $\alpha \geq 0$;
- (3) $\inf_{k=1,2,\dots} (t_k - t_{k-1}) \geq \mu$;
- (4) there exists a constant $\gamma > 0$ which satisfies $\gamma + \lambda + hpe^{\gamma\tau} > 0$ and $\lambda + hpe^{\gamma\tau} + (\ln(1 + \alpha)/\mu) < 0$, where $\lambda = \max_{i=1,\dots,n} (-\chi - 2a_i + \sum_{j=1}^n (b_{ij}^2 + c_{ij}^2)) + \rho$ and $\rho = n \max_{i=1,\dots,n} (l_i^2)$;

then, the equilibrium point $u = 0$ of problem (5)–(8) is globally exponentially stable with convergence rate $-(1/2)(\lambda + hpe^{\gamma\tau} + (\ln(1 + \alpha)/\mu))$.

Proof. Define Lyapunov function V of the form $V(t) = \sum_{i=1}^n V_i(t)$, where $V_i(t) = \int_{\Omega} u_i^2(t, x) dx$. Obviously, $V(t)$ is a piecewise continuous function with the first kind discontinuous points t_k , $k = 1, 2, \dots$, where it is continuous from the left, that is, $V(t_k - 0) = V(t_k)$ ($k = 1, 2, \dots$). Furthermore, when $t = t_k$ ($k = 0, 1, 2, \dots$), it follows from condition (2) of Theorem 8 that

$$\begin{aligned} u_i^2(t_k + 0, x) - u_i^2(t_k, x) \\ = (1 - \theta_{ik})^2 u_i^2(t_k, x) - u_i^2(t_k, x) \leq \alpha u_i^2(t_k, x). \end{aligned} \quad (45)$$

Thereby,

$$V(t_k + 0) \leq \alpha V(t_k) + V(t_k), \quad k = 0, 1, 2, \dots \quad (46)$$

Construct another Lyapunov function $V^*(t) = e^{\gamma(t-t_0)} \times V(t)$, where $\gamma > 0$ satisfies $\gamma + \lambda + hpe^{\gamma\tau} > 0$ and $\lambda + hpe^{\gamma\tau} + (\ln(1 + \alpha)/\mu) < 0$. Then, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k , $k = 1, 2, \dots$, where it is continuous from the left, and for $t = t_k$ ($k = 0, 1, 2, \dots$), it results from (46) that

$$V^*(t_k + 0) \leq \alpha V^*(t_k) + V^*(t_k), \quad k = 0, 1, 2, \dots \quad (47)$$

Set $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \dots$. Following the same procedure as in Theorem 5, we get

$$\begin{aligned} V^*(t) &\leq V^*(t_k + 0) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ &\quad + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta \end{aligned} \quad (48)$$

$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots$

The relations (47) and (48) yield

$$\begin{aligned} V^*(t) - V^*(t_k) \\ \leq \alpha V^*(t_k) + (\gamma + \lambda) \int_{t_k}^t V^*(s) ds \\ + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta \end{aligned} \quad (49)$$

$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots$

By induction argument, we reach

$$\begin{aligned} V^*(t_k) - V^*(t_{k-1}) \\ \leq \alpha V^*(t_{k-1}) + (\gamma + \lambda) \int_{t_{k-1}}^{t_k} V^*(s) ds \\ + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_{k-1} - \tau_j(t_{k-1})}^{t_k - \tau_j(t_k)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta, \\ \vdots \\ V^*(t_2) - V^*(t_1) \\ \leq \alpha V^*(t_1) + (\gamma + \lambda) \int_{t_1}^{t_2} V^*(s) ds \\ + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_1 - \tau_j(t_1)}^{t_2 - \tau_j(t_2)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta, \\ V^*(t_1) - V^*(t_0) \\ \leq \alpha V^*(t_0) + (\gamma + \lambda) \int_{t_0}^{t_1} V^*(s) ds \\ + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t_1 - \tau_j(t_1)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta. \end{aligned} \quad (50)$$

Hence,

$$\begin{aligned} V^*(t) - V^*(t_0) \\ \leq \alpha V^*(t_0) + (\gamma + \lambda) \int_{t_0}^t V^*(s) ds + hpe^{\gamma\tau} \\ \times \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta + \alpha \sum_{t_0 < t_k < t} V(t_k) \\ \leq \alpha V^*(t_0) + (\gamma + \lambda + hpe^{\gamma\tau}) \int_{t_0}^t V^*(s) ds \\ + hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t_0} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta + \alpha \sum_{t_0 < t_k < t} V(t_k) \\ t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots \end{aligned} \quad (51)$$

Introducing $hpe^{\gamma\tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t_0} e^{\gamma(\theta - t_0)} V_j(\theta) d\theta \leq \tau hpe^{\gamma\tau} \times \|\varphi\|_{\Omega}^2$ as shown in the proof of Theorem 5 into (51), (51) becomes

$$\begin{aligned} V^*(t) - V^*(t_0) \\ \leq \alpha V^*(t_0) + \tau hpe^{\gamma\tau} \|\varphi\|_{\Omega}^2 \end{aligned}$$

$$\begin{aligned}
& + (\gamma + \lambda + h\rho e^{\gamma\tau}) \int_{t_0}^t V^*(s) ds + \alpha \sum_{t_0 < t_k < t} V(t_k) \\
& t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \dots
\end{aligned} \quad (52)$$

It then results from Lemma 2 that, for $t \geq t_0$,

$$\begin{aligned}
V^*(t) & \leq \left((\alpha + 1) V^*(t_0) + \tau h \rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2 \right) \\
& \times \prod_{t_0 < t_k < t} (1 + \alpha) \exp((\gamma + \lambda + h\rho e^{\gamma\tau})(t - t_0)) \\
& = \left((\alpha + 1) V^*(t_0) + \tau h \rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2 \right) \\
& \times (1 + \alpha)^k \exp((\gamma + \lambda + h\rho e^{\gamma\tau})(t - t_0)).
\end{aligned} \quad (53)$$

On the other hand, since $\inf_{k=1,2,\dots} (t_k - t_{k-1}) \geq \mu$, one has $k \leq (t_k - t_0)/\mu$. Thereby,

$$\begin{aligned}
(1 + \alpha)^k & \leq \exp \left\{ \frac{\ln(1 + \alpha)}{\mu} (t_k - t_0) \right\} \\
& \leq \exp \left\{ \frac{\ln(1 + \alpha)}{\mu} (t - t_0) \right\}
\end{aligned} \quad (54)$$

and (53) can be rewritten as

$$\begin{aligned}
V^*(t) & \leq \left((\alpha + 1) V^*(t_0) + \tau h \rho e^{\gamma\tau} \|\varphi\|_{\Omega}^2 \right) \\
& \times \exp \left(\left(\gamma + \lambda + h\rho e^{\gamma\tau} + \frac{\ln(1 + \alpha)}{\mu} \right) (t - t_0) \right)
\end{aligned} \quad (55)$$

which implies

$$\begin{aligned}
& \|u(t, x; t_0, \varphi)\|_{\Omega} \\
& \leq \sqrt{(\alpha + 1 + \tau h \rho e^{\gamma\tau})} \|\varphi\|_{\Omega} \\
& \times \exp \left(\frac{1}{2} \left(\lambda + h\rho e^{\gamma\tau} + \frac{\ln(1 + \alpha)}{\mu} \right) (t - t_0) \right), \\
& t \geq t_0.
\end{aligned} \quad (56)$$

The proof is completed. \square

As

$$\begin{aligned}
& 2 \sum_{j=1}^n b_{ij} \int_{\Omega} u_i(t, x) f(u_j(t, x)) dx \\
& \leq \sum_{j=1}^n \int_{\Omega} \left(\varepsilon_1 b_{ij}^2 u_i^2(t, x) + \frac{l_j^2}{\varepsilon_1} u_j^2(t, x) \right) dx, \\
& 2 \sum_{j=1}^n c_{ij} \int_{\Omega} u_i(t, x) f(u_j(t - \tau_j(t), x)) dx \\
& \leq \sum_{j=1}^n \int_{\Omega} \left(\varepsilon_2 c_{ij}^2 u_i^2(t, x) + \frac{l_j^2}{\varepsilon_2} u_j^2(t - \tau_j(t), x) \right) dx
\end{aligned} \quad (57)$$

hold for any $\varepsilon_1, \varepsilon_2 > 0$. In the sequel, analogous to the proofs of Theorems 5 and 8 we arrive at the following.

Theorem 9. *Provided that one has the following:*

- (1) let $\underline{D} = \min\{D_{is} : i = 1, \dots, n; s = 1, \dots, m\} > 0$ and denote $8m\underline{D}/M^2 = \chi$;
- (2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik} u_i(t_k, x)$, $0 \leq \theta_{ik} \leq 2$;
- (3) there exist constants $\gamma > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$, where $\lambda = \max_{i=1,\dots,n} (-\chi - 2a_i + \sum_{j=1}^n (\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2)) + (n/\varepsilon_1) \max_{i=1,\dots,n} (l_i^2)$ and $\rho = (n/\varepsilon_2) \max_{i=1,\dots,n} (l_i^2)$;

then, the equilibrium point $u = 0$ of problem (5)–(8) is globally exponentially stable with convergence rate $-(\lambda + h\rho e^{\gamma\tau})/2$.

Remark 10. According to Theorem 5, we know that there must exist constant $\gamma > 0$ satisfying condition (3) of Theorem 9 if there are constants $\varepsilon_1, \varepsilon_2 > 0$ such that $\lambda < -h\rho$.

Theorem 11. *Assume that one has the following:*

- (1) let $\underline{D} = \min\{D_{is} : i = 1, \dots, n; s = 1, \dots, m\} > 0$ and denote $8m\underline{D}/M^2 = \chi$;
- (2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik} u_i(t_k, x)$, $1 - \sqrt{1 + \alpha} \leq \theta_{ik} \leq 1 + \sqrt{1 + \alpha}$, $\alpha \geq 0$;
- (3) $\inf_{k=1,2,\dots} (t_k - t_{k-1}) \geq \mu$;
- (4) there exist constants $\gamma > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} + \ln(1 + \alpha)/\mu < 0$, where $\lambda = \max_{i=1,\dots,n} (-\chi - 2a_i + \sum_{j=1}^n (\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2)) + (n/\varepsilon_1) \max_{i=1,\dots,n} (l_i^2)$ and $\rho = (n/\varepsilon_2) \max_{i=1,\dots,n} (l_i^2)$;

then, the equilibrium point $u = 0$ of problem (5)–(8) is globally exponentially stable with convergence rate $-(1/2)(\lambda + h\rho e^{\gamma\tau} + (\ln(1 + \alpha)/\mu))$.

Further, on the condition that $|P_{ik}(u_i(t_k, x))| \leq \theta_{ik} |u_i(t_k, x)|$, where $\theta_{ik}^2 \leq (\alpha - 1)/2$ and $\alpha \geq 1$, we obtain, for $t = t_k$ ($k = 1, 2, \dots$),

$$\begin{aligned}
& u_i^2(t_k + 0, x) - u_i^2(t_k, x) \\
& = (P_{ik}(u_i(t_k, x)) + u_i(t_k, x))^2 - u_i^2(t_k, x) \\
& \leq 2(u_i(t_k, x))^2 + 2(P_{ik}(u_i(t_k, x)))^2 - u_i^2(t_k, x) \quad (58) \\
& \leq (2 + 2\theta_{ik}^2)(u_i(t_k, x))^2 - u_i^2(t_k, x) \\
& \leq \alpha u_i^2(t_k, x).
\end{aligned}$$

Identical with the proof of Theorem 8, we reach the following.

Theorem 12. *Assume that one has the following:*

- (1) let $\underline{D} = \min\{D_{is} : i = 1, \dots, n; s = 1, \dots, m\} > 0$ and denote $8m\underline{D}/M^2 = \chi$;
- (2) $|P_{ik}(u_i(t_k, x))| \leq \theta_{ik} |u_i(t_k, x)|$, where $\theta_{ik}^2 \leq (\alpha - 1)/2$ and $\alpha \geq 1$;

- (3) $\inf_{k=1,2,\dots}(t_k - t_{k-1}) \geq \mu$;
 (4) there exist constants $\gamma > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $\gamma + \lambda + hpe^{\gamma\tau} > 0$ and $\lambda + hpe^{\gamma\tau} + (\ln(1 + \alpha)/\mu) < 0$, where $\lambda = \max_{i=1,\dots,n}(-\chi - 2a_i + \sum_{j=1}^n(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2)) + (n/\varepsilon_1)\max_{i=1,\dots,n}(l_i^2)$ and $\rho = (n/\varepsilon_2)\max_{i=1,\dots,n}(l_i^2)$;

then, the equilibrium point $u = 0$ of problem (5)–(8) is globally exponentially stable with convergence rate $-(1/2)(\lambda + hpe^{\gamma\tau} + (\ln(1 + \alpha)/\mu))$.

Remark 13. Different from Theorems 5–11, the impulsive part in Theorem 12 could be nonlinear and this will be of more applicability. Actually, Theorems 5–11 can be regarded as the special cases of Theorem 12.

4. Examples

Example 14. Consider system (5)–(8) equipped with $P_{ik}(u_i(t_k, x)) = 1.343u_i(t_k, x)$. Let $n = 2, m = 2, \Omega = [0, 1.5] \times [0, 2]$, $\tau_j(t) = (3/4)\arctan(t)$, $a_1 = a_2 = 6.5$, $(D_{is})_{2 \times 2} = \begin{pmatrix} 1.2 & 2.3 \\ 2.2 & 1.5 \end{pmatrix}$, $(b_{ij})_{2 \times 2} = \begin{pmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{pmatrix}$, $(c_{ij})_{2 \times 2} = \begin{pmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{pmatrix}$, and $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$.

For $M = 2$ and $\underline{D} = 1.2$, we compute $\chi = 4.8$. This, together with $l_i = \sqrt{2}/2$, yields

$$\rho = n \max_{i=1,\dots,n} (l_i^2) = 1, \quad (59)$$

$$\lambda = \max_{i=1,\dots,n} \left(-\chi - 2a_i + \sum_{j=1}^n (b_{ij}^2 + c_{ij}^2) \right) + \rho = -6.461. \quad (60)$$

Let $h = 4$. Since $\lambda = -6.461 < -4 = -h\rho$, we conclude from Theorem 5 that the equilibrium point $u = 0$ of this system is globally exponentially stable.

Example 15. Consider system (5)–(8) equipped with $P_{ik}(u_i(t_k, x)) = \arctan(0.5u_i(t_k, x))$. Let $n = 2, m = 2, \tau_j(t) = (1/\pi)\arctan(t)$, $\Omega = [0, 1.5] \times [0, 2]$, $a_i = 6.5$, $(D_{is})_{2 \times 2} = \begin{pmatrix} 1.2 & 2.3 \\ 2.2 & 3.5 \end{pmatrix}$, $(b_{ij})_{2 \times 2} = \begin{pmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{pmatrix}$, $(c_{ij})_{2 \times 2} = \begin{pmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{pmatrix}$, $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$, and $t_k = t_{k-1} + 2k$.

For $M = 2$ and $\underline{D} = 1.2$, we compute $\chi = 4.8$. This, together with $l_i = \sqrt{2}/2$ and $\varepsilon_1 = \varepsilon_2 = 1$, yields

$$\begin{aligned} \rho &= \frac{n}{\varepsilon_2} \max_{i=1,\dots,n} (l_i^2) = 1, \\ \lambda &= \max_{i=1,\dots,n} \left(-\chi - 2a_i + \sum_{j=1}^n (\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2) \right) \\ &\quad + \frac{n}{\varepsilon_1} \max_{i=1,\dots,n} (l_i^2) = -6.461. \end{aligned} \quad (61)$$

Letting $\tau = 0.5, h = 4, \mu = 2$, and $\alpha = 1.5$, we can find $\gamma = 0.78$ satisfying

$$\begin{aligned} \gamma + \lambda + hpe^{\gamma\tau} &= 0.2269 > 0, \\ \lambda + hpe^{\gamma\tau} + \frac{\ln(1 + \alpha)}{\mu} &= -0.0949 < 0. \end{aligned} \quad (62)$$

It is then concluded from Theorem 12 that this system is globally exponentially stable.

Acknowledgment

The work is supported by National Natural Science Foundation of China under Grant nos. 60904028, 71271118, and 71171116.

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