Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 126457, 5 pages
http://dx.doi.org/10.1155/2013/126457

Research Article

Hermite-Hadamard-Type Inequalities for r-Preinvex Functions

Wasim Ul-Haq and Javed Iqbal

Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan

Correspondence should be addressed to Wasim Ul-Haq; wasim474@hotmail.com

Received 7 May 2013; Revised 24 August 2013; Accepted 24 August 2013

Academic Editor: Zhongxiao Jia

Copyright © 2013 W. Ul-Haq and J. Iqbal. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We aim to find Hermite-Hadamard inequality for r-preinvex functions. Also, it is investigated for the product of an r-preinvex function and s-preinvex function.

1. Introduction

Several interesting generalizations and extensions of classical convexity have been studied and investigated in recent years. Hanson [1] introduced the invex functions as a generalization of convex functions. Later, subsequent works inspired from Hanson's result have greatly found the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. The basic properties and role of preinvex functions in optimization, equilibrium problems and variational inequalities were studied by Noor [2, 3] and Weir and Mond [4]. The well known Hermite-Hadamard inequality has been extensively investigated for the convex functions and their variant forms. Noor [5] and Pachpatte [6] studied these inequalities and related types for preinvex and log-preinvex functions.

The Hermite-Hadamard inequality was investigated in [7] to an r-convex positive function which is defined on an interval $[a,b]$. Certain refinements of the Hadamard inequality for r-convex functions were studied in [8, 9]. Recently, Zabandan et al. [10] extended and refined the work of [8]. Bessenyei [11] studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In this paper, we introduce the r-preinvex functions and establish Hermite-Hadamard inequality for such functions by using the method of [10].

2. Preliminaries

Let $f : K \to \mathbb{R}$ and $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}$ be continuous functions, where $K \subset \mathbb{R}^n$ is a nonempty closed set. We use the notations, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, for inner product and norm, respectively. We require the following well known concepts and results which are essential in our investigations.

Definition 1 (see [2, 4]). Let $u \in K$. Then, the set K is said to be invex at $u \in K$ with respect to $\eta(\cdot,\cdot)$ if

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K, \ t \in [0,1]. \tag{1}$$

K is said to be invex set with respect to $\eta(\cdot,\cdot)$ if it is invex at every $u \in K$. The invex set K is also called a η connected set.

Geometrically, Definition 1 says that there is a path starting from the point u which is contained in K. The point v should not be one of the end points of the path in general, see [12]. This observation plays a key role in our study. If we require that v should be an end point of the path for every pair of points $u, v \in K$, then $\eta(v,u) = v - u$, and consequently, invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not necessarily true; see [4, 13] and the references therein. For the sake of simplicity, we assume that $K = [a, a+\eta(b,a)]$, unless otherwise specified.

Definition 2 (see [4]). The function f on the invex set K is said to be preinvex with respect to η if

$$\begin{aligned} f(u + t\eta(v,u)) &\le (1-t) f(u) + t f(v), \\ &\forall u, v \in K, \ t \in [0,1]. \end{aligned} \tag{2}$$

Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u) =$

$-|u|$ is not a convex function, but it is a preinvex function with respect to η, where

$$\eta(v,u)=\begin{cases} v-u, & \text{if } v\le 0,\ u\le 0 \text{ or } v\ge 0,\ u\ge 0,\\ u-v, & \text{otherwise.}\end{cases} \quad (3)$$

The concepts of the invex and preinvex functions have played very important roles in the development of generalized convex programming, see [14–17]. For more characterizations and applications of invex and preinvex functions, we refer to [3, 15, 18–25].

Antczak [26, 27] introduced and studied the concept of r-invex and r-preinvex functions. Here, we define the following.

Definition 3. A positive function f on the invex set K is said to be r-preinvex with respect to η if, for each $u, v \in K$, $t \in [0,1]$:

$$f(u+t\eta(v,u))\le\begin{cases}\left((1-t)f^r(u)+tf^r(v)\right)^{1/r}, & r\ne 0\\ (f(u))^{1-t}(f(v))^t, & r=0.\end{cases} \quad (4)$$

Note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are classical preinvex functions. It should be noted that if f is r-preinvex function, then f^r is preinvex function ($r>0$).

The well known Hermite-Hadamard inequality for a convex function defined on the interval $[a,b]$ is given by

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}; \quad (5)$$

see [6, 28, 29].

Hermite-Hadamard inequalities for log-convex functions were proved by Dragomir and Mond [30]. Pachpatte [6, 17] also gave some other refinements of these inequalities related to differentiable log-convex functions. Noor [5] proved the following Hermite-Hadamard inequalities for the preinvex and log-preinvex functions, respectively.

(i) Let $f: K=[a,a+\eta(b,a)]\to(0,\infty)$ be a preinvex function on the interval of real numbers K^0 (interior of K) and $a,b\in K^0$ with $a<a+\eta(b,a)$. Then, the following inequalities hold:

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\le\frac{1}{\eta(b,a)}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \quad (6)$$

(ii) Let f be a log-preinvex function defined on the interval $[a,a+\eta(b,a)]$. Then,

$$\frac{1}{\eta(b,a)}\int_a^b f(x)\,dx\le\frac{f(a)-f(b)}{\log f(a)-\log f(b)} = L(f(a),f(b)), \quad (7)$$

where $L(p,q)(p\ne q)$ is the logarithmic mean of positive real numbers p,q.

3. Main Results

Theorem 4. *Let $f: K=[a,a+\eta(b,a)]\to(0,\infty)$ be an r-preinvex function on the interval of real numbers K^0 (interior of K) and $a,b\in K^0$ with $a<a+\eta(b,a)$. Then, the following inequalities hold:*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\le\left[\frac{f^r(a)+f^r(b)}{2}\right]^{1/r}, \quad r\ge 1. \quad (8)$$

Proof. From Jensen's inequality, we obtain

$$\left(\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right)^r\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f^r(x)\,dx. \quad (9)$$

Since f^r is preinvex, then, using Hermite-Hadamard inequality for preinvex functions (see [5]), we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f^r(x)\,dx\le\frac{f^r(a)+f^r(b)}{2}. \quad (10)$$

Hence,

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\le\left[\frac{f^r(a)+f^r(b)}{2}\right]^{1/r}. \quad (11)$$

This completes the proof. □

Corollary 5. *Let $f: K=[a,a+\eta(b,a)]\to(0,\infty)$ be a 1-preinvex function on the interval of real numbers K^0 (interior of K) and $a,b\in K^0$ with $a<a+\eta(b,a)$. Then,*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\le\frac{f(a)+f(b)}{2}. \quad (12)$$

Theorem 6. *Let $f: K=[a,a+\eta(b,a)]\to(0,\infty)$ be an r-preinvex function (with $r\ge 0$) on the interval of real numbers K^0 (interior of K) and $a,b\in K^0$ with $a<a+\eta(b,a)$. Then, the following inequalities hold:*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le\begin{cases}\dfrac{r}{r+1}\left[\dfrac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}\right], & r\ne 0\\ \dfrac{f(a)-f(b)}{\log f(a)-\log f(b)}, & r=0.\end{cases} \quad (13)$$

Proof. For $r=0$, Noor [5] proved this result. We proceed for the case $r>0$. Since f is r-preinvex function, for all $t\in[0,1]$, we have

$$f(a+t\eta(b,a))\le\left((1-t)f^r(a)+tf^r(b)\right)^{1/r}. \quad (14)$$

Therefore,

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx = \int_0^1 f(a+t\eta(b,a))\,dt \le \int_0^1 \left(f^r(a)+t\left(f^r(b)-f^r(a)\right)\right)^{1/r}dt. \quad (15)$$

Putting $u = f^r(a) + t(f^r(b) - f^r(a))$, we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{1}{f^r(b)-f^r(a)}\int_{f^r(a)}^{f^r(b)} (u)^{1/r}dt = \frac{r}{r+1}\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}, \quad (16)$$

which completes the proof. □

Note that for $r = 1$, in Theorem 6, we have the same inequality again as in Corollary 5.

Theorem 7. *Let $f : K = [a, a+\eta(b,a)] \to (0,\infty)$ be an r-preinvex function with $0 \le r \le s$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a+\eta(b,a)$. Then, f is s-preinvex function.*

Proof. To prove this, we need the following inequality:

$$u^{1-t}v^t \le \left((1-t)u^r + tv^r\right)^{1/r} \le \left((1-t)u^s + tv^s\right)^{1/s}, \quad (17)$$

(see [10]) where $0 \le t \le 1$, $0 \le r \le s$. Since f is r-preinvex, by inequality (17), for all $u, v \in K^0$, $t \in [0,1]$, we obtain

$$f(u+t\eta(v,u)) \le \begin{cases} \left((1-t)f^r(u)+tf^r(v)\right)^{1/r} \le \left((1-t)f^s(u)+tf^s(v)\right)^{1/s}, & 0<r\le s \\ (f(u))^{1-t}(f(v))^t \le \left((1-t)f^s(u)+tf^s(v)\right)^{1/s}, & 0=r\le s. \end{cases} \quad (18)$$

Hence, f is s-preinvex function. □

As a special case of Theorem 7, we deduce the following result.

Corollary 8. *Let $f : K = [a, a+\eta(b,a)] \to (0,\infty)$ be an r-preinvex function with $0 \le r \le 1$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a+\eta(b,a)$. Then, f is preinvex function.*

Theorem 9. *Let $f : K = [a, a+\eta(b,a)] \to (0,\infty)$ be an r-preinvex function with $0 \le r \le s$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a+\eta(b,a)$. Then, the following inequalities hold:*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le \begin{cases} \dfrac{r}{r+1}\left[\dfrac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}\right] \le \dfrac{s}{s+1}\left[\dfrac{f^{s+1}(a)-f^{s+1}(b)}{f^s(a)-f^s(b)}\right], & 0<r\le s \\ \dfrac{f(a)-f(b)}{\log f(a)-\log f(b)} \le \dfrac{s}{s+1}\left[\dfrac{f^{s+1}(a)-f^{s+1}(b)}{f^s(a)-f^s(b)}\right], & 0=r\le s. \end{cases} \quad (19)$$

Proof. The left side of these inequalities is clear from Theorem 7. For the right hand side, we have from inequality (17)

$$\left((1-t)f^r(a)+tf^r(b)\right)^{1/r} \le \left((1-t)f^s(a)+tf^s(b)\right)^{1/s}. \quad (20)$$

Integrating both sides with respect to t, we obtain

$$\frac{r}{r+1}\frac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)} \le \frac{s}{s+1}\left[\frac{f^{s+1}(a)-f^{s+1}(b)}{f^s(a)-f^s(b)}\right]. \quad (21)$$

Again, using the other part of (17) and integrating, one can easily deduce the second part of the required inequalities. □

Corollary 10. *Let $f : K = [a, a+\eta(b,a)] \to (0,\infty)$ be an r-preinvex function with $0 \le r \le 1$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a+\eta(b,a)$. Then, the following inequalities hold:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le [f(a)-f(b)]\ln\frac{f(a)}{f(b)} \le \frac{r}{r+1}\left[\frac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}\right] \le \frac{f(a)+f(b)}{2}. \quad (22)$$

Theorem 11. *Let $f, g : K = [a, a+\eta(b,a)] \to (0,\infty)$ be r-preinvex and s-preinvex functions, respectively, with $r, s > 0$*

on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned}
&\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) g(x)\, dx \\
&\quad \leq \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\
&\quad + \frac{1}{2} \frac{s}{s+2} \left[\frac{g^{s+2}(a) - g^{s+2}(b)}{g^s(a) - g^s(b)} \right] \\
&\quad (f(a) \neq f(b),\ g(a) \neq g(b)).
\end{aligned} \tag{23}$$

Proof. Since f is r-preinvex and g is s-preinvex, for all $t \in [0, 1]$, we have

$$\begin{aligned}
&f(a + t\eta(b,a)) \leq ((1-t) f^r(a) + t f^r(b))^{1/r}, \\
&g(a + t\eta(b,a)) \leq ((1-t) g^s(a) + t g^s(b))^{1/s}.
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) g(x)\, dx \\
&\quad = \int_0^1 f(a + t\eta(b,a)) g(a + t\eta(b,a))\, dt \\
&\quad \leq \int_0^1 [(1-t) f^r(a) + t f^r(b)]^{1/r} \\
&\qquad \times [(1-t) g^s(a) + t g^s(b)]^{1/s} dt \\
&\quad = \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\
&\qquad \times [g^s(a) + t(g^s(b) - g^s(a))]^{1/s} dt.
\end{aligned} \tag{25}$$

Now, applying Cauchy's inequality, we obtain

$$\begin{aligned}
&\int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\
&\quad \times [g^s(a) + t(g^s(b) - g^s(a))]^{1/s} dt \\
&\quad \leq \frac{1}{2} \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{2/r} dt \\
&\quad + \frac{1}{2} \int_0^1 [g^s(a) + t(g^s(b) - g^s(a))]^{2/s} dt \\
&\quad = \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\
&\quad + \frac{1}{2} \frac{s}{s+2} \left[\frac{g^{s+2}(a) - g^{s+2}(b)}{g^s(a) - g^s(b)} \right],
\end{aligned} \tag{26}$$

which leads us to the required result. □

Note. By putting $f = g$, $r = s = 2$, in Theorem 11, we have the following inequality:

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f^2(x)\, dx \leq \frac{f^2(a) + f^2(b)}{2}. \tag{27}$$

Theorem 11, for $r = s = 0$, was proved by Noor [31].

Theorem 12. *Let $f, g : K = [a, a + \eta(b, a)] \to (0, \infty)$ be r-preinvex ($r > 0$) and 0-preinvex functions, respectively, on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) g(x)\, dx \\
&\quad \leq \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\
&\quad + \frac{1}{4} \left[\frac{g^2(a) - g^2(b)}{\log g(a) - \log g(b)} \right] \\
&\quad (f(a) \neq f(b)).
\end{aligned} \tag{28}$$

Proof. Since f is r-preinvex and g is 0-preinvex, for all $t \in [0, 1]$, we have

$$\begin{aligned}
&f(a + t\eta(b,a)) \leq ((1-t) f^r(a) + t f^r(b))^{1/r}, \\
&g(a + t\eta(b,a)) \leq (g(a))^{1-t} (g(b))^t.
\end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) g(x)\, dx \\
&\quad = \int_0^1 f(a + t\eta(b,a)) g(a + t\eta(b,a))\, dt \\
&\quad \leq \int_0^1 [(1-t) f^r(a) + t f^r(b)]^{1/r} \\
&\qquad \times \left[(g(a))^{1-t} (g(b))^t\right] dt \\
&\quad = \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\
&\qquad \times \left[(g(a))^{1-t} (g(b))^t\right] dt.
\end{aligned} \tag{30}$$

Now, applying Cauchy's inequality, we obtain

$$\begin{aligned}
&\int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \left[(g(a))^{1-t} (g(b))^t\right] dt \\
&\quad \leq \frac{1}{2} \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{2/r} dt \\
&\quad + \frac{1}{2} \int_0^1 \left[(g(a))^{2-2t} (g(b))^{2t}\right] dt \\
&\quad = \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] + \frac{1}{4} \left[\frac{g^2(a) - g^2(b)}{\log(g(a)/g(b))} \right],
\end{aligned} \tag{31}$$

which is the required result. □

Acknowledgment

The authors would like to thank referees of this paper for their insightful comments which greatly improved the entire presentation of the paper. They would also like to acknowledge Prof. Dr. Ehsan Ali, VC AWKUM, for the financial support on the publication of this article.

References

[1] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[2] M. A. Noor, "Variational-like inequalities," *Optimization*, vol. 30, no. 4, pp. 323–330, 1994.

[3] M. A. Noor, "Invex equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 463–475, 2005.

[4] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.

[5] M. A. Noor, "Hermite-Hadamard integral inequalities for log-preinvex functions," *Journal of Mathematical Analysis and Approximation Theory*, vol. 2, no. 2, pp. 126–131, 2007.

[6] B. G. Pachpatte, *Mathematical Inequalities*, vol. 67 of *North-Holland Mathematical Library*, Elsevier, Amsterdam, The Netherlands, 2005.

[7] C. E. M. Pearce, J. Pečarić, and V. Šimić, "Stolarsky means and Hadamard's inequality," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 99–109, 1998.

[8] N. P. N. Ngoc, N. V. Vinh, and P. T. T. Hien, "Integral inequalities of Hadamard type for r-convex functions," *International Mathematical Forum. Journal for Theory and Applications*, vol. 4, no. 33-36, pp. 1723–1728, 2009.

[9] G.-S. Yang and D.-Y. Hwang, "Refinements of Hadamard's inequality for r-convex functions," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 10, pp. 1571–1579, 2001.

[10] G. Zabandan, A. Bodaghi, and A. Kılıçman, "The Hermite-Hadamard inequality for r-convex functions," *Journal of Inequalities and Applications*, vol. 2012, article 215, 2012.

[11] M. Bessenyei, "Hermite-Hadamard-type inequalities for generalized 3-convex functions," *Publicationes Mathematicae Debrecen*, vol. 65, no. 1-2, pp. 223–232, 2004.

[12] T. Antczak, "Mean value in invexity analysis," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 60, no. 8, pp. 1473–1484, 2005.

[13] X. M. Yang, X. Q. Yang, and K. L. Teo, "Generalized invexity and generalized invariant monotonicity," *Journal of Optimization Theory and Applications*, vol. 117, no. 3, pp. 607–625, 2003.

[14] S. R. Mohan and S. K. Neogy, "On invex sets and preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 3, pp. 901–908, 1995.

[15] M. A. Noor and K. I. Noor, "Some characterizations of strongly preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 697–706, 2006.

[16] M. A. Noor and K. I. Noor, "Hemiequilibrium-like problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 64, no. 12, pp. 2631–2642, 2006.

[17] B. G. Pachpatte, "A note on integral inequalities involving two log-convex functions," *Mathematical Inequalities & Applications*, vol. 7, no. 4, pp. 511–515, 2004.

[18] J. Chudziak and J. Tabor, "Characterization of a condition related to a class of preinvex functions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 16, pp. 5572–5577, 2011.

[19] C.-P. Liu, "Some characterizations and applications on strongly α-preinvex and strongly α-invex functions," *Journal of Industrial and Management Optimization*, vol. 4, no. 4, pp. 727–738, 2008.

[20] M. Soleimani-Damaneh, "Nonsmooth optimization using Mordukhovich's subdifferential," *SIAM Journal on Control and Optimization*, vol. 48, no. 5, pp. 3403–3432, 2010.

[21] M. Soleimani-Damaneh, "Generalized invexity in separable Hilbert spaces," *Topology*, vol. 48, no. 2-4, pp. 66–79, 2009.

[22] M. Soleimani-Damaneh, "The gap function for optimization problems in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 2, pp. 716–723, 2008.

[23] M. Soleimani-Damaneh and M. E. Sarabi, "Sufficient conditions for nonsmooth r-invexity," *Numerical Functional Analysis and Optimization*, vol. 29, no. 5-6, pp. 674–686, 2008.

[24] M. Soleimani-Damaneh, "Optimality for nonsmooth fractional multiple objective programming," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 10, pp. 2873–2878, 2008.

[25] J. Yang and X. Yang, "Two new characterizations of preinvex functions," *Dynamics of Continuous, Discrete & Impulsive Systems B*, vol. 19, no. 3, pp. 405–410, 2012.

[26] T. Antczak, "r-preinvexity and r-invexity in mathematical programming," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 551–566, 2005.

[27] T. Antczak, "A new method of solving nonlinear mathematical programming problems involving r-invex functions," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 1, pp. 313–323, 2005.

[28] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Type Inequalities*, RGMIA Monograph, Victoria University, Melbourne, Australia, 2000.

[29] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1992.

[30] S. S. Dragomir and B. Mond, "Integral inequalities of Hadamard type for log-convex functions," *Demonstratio Mathematica*, vol. 31, no. 2, pp. 355–364, 1998.

[31] M. A. Noor, "On Hadamard integral inequalities involving two log-preinvex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 3, article 75, 2007.