Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 971560, 7 pages
doi:10.1155/2012/971560

Research Article

f-Orthomorphisms and f-Linear Operators on the Order Dual of an f-Algebra

Ying Feng, Jin Xi Chen, and Zi Li Chen

Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, China

Correspondence should be addressed to Jin Xi Chen, jinxichen@home.swjtu.edu.cn

Received 1 August 2012; Revised 24 September 2012; Accepted 27 September 2012

Academic Editor: Alberto Fiorenza

We consider the f-orthomorphisms and f-linear operators on the order dual of an f-algebra. In particular, when the f-algebra has the factorization property (not necessarily unital), we prove that the orthomorphisms, f-orthomorphisms, and f-linear operators on the order dual are precisely the same class of operators.

1. Introduction

Let A be an f-algebra with ${}^{\circ}(A^{\sim}) = \{0\}$. Recall that we can define a multiplication on $(A^{\sim})_n^{\sim}$, the order continuous part of the order bidual of A, with respect to which $(A^{\sim})_n^{\sim}$ can also be made an f-algebra. This is done in three steps:

(1) $A \times A^{\sim} \to A^{\sim}$

$(a, f) \mapsto f \cdot a : (f \cdot a)(b) = f(ab)$ for $b \in A$,

(2) $(A^{\sim})_n^{\sim} \times A^{\sim} \to A^{\sim}$

$(F, f) \mapsto F \cdot f : (F \cdot f)(a) = F(f \cdot a)$ for $a \in A$,

(3) $(A^{\sim})_n^{\sim} \times (A^{\sim})_n^{\sim} \to (A^{\sim})_n^{\sim}$

$(F, G) \mapsto F \cdot G : (F \cdot G)(f) = F(G \cdot f)$ for $f \in A^{\sim}$.

With the so-called Arens multiplication defined in step (3), $(A^{\sim})_n^{\sim}$ is an Archimedean (and hence commutative) f-algebra. Moreover, if A has a multiplicative unit, then $(A^{\sim})_n^{\sim} = (A^{\sim})^{\sim}$, the whole order bidual of A. The mapping $V : (A^{\sim})_n^{\sim} \to \mathrm{Orth}(A^{\sim})$ defined by $V(F) = V_F$ for all $F \in (A^{\sim})_n^{\sim}$, where $V_F(f) = F \cdot f$ for every $f \in A^{\sim}$, is an algebra and Riesz isomorphism. See [1, 2] for details.

Let A be an f-algebra. A Riesz space L with ${}^{\circ}(L^{\sim}) = \{0\}$ is said to be an (left) f-module over A (cf. [2, 3]) if L is a left module over A and satisfies the following two conditions:

(i) for each $a \in A^{+}$ and $x \in L^{+}$, we have $ax \in L^{+}$,

(ii) if $x \perp y$, then for each $a \in A$, we have $a \cdot x \perp y$.

When A is an f-algebra with unit e, saying L is a unital f-module over A implies that the left multiplication satisfies $e \cdot x = x$ for all $x \in L$. From Corollary 2.3 in [2], we know that if L is an f-module over A, then $L^{\sim}$ is an f-module over A (and $(A^{\sim})^{\sim}_{n}$). The f-module L over A with unit e is said to be *topologically full* with respect to A if for two arbitrary vectors x, y satisfying $0 \le y \le x$ in L, there exists a net $0 \le a_{\alpha} \le e$ in A such that $a_{\alpha} \cdot x \to y$ in $\sigma(L, L^{\sim})$. If L is topologically full with respect to A, then $L^{\sim}$ is topologically full with respect to $(A^{\sim})^{\sim}_{n}$ [2, Proposition 3.12].

Let A be a unital f-algebra, and, L, M be f-modules over A. $T \in L_b(L, M)$ is called an f-linear operator if $T(a \cdot x) = a \cdot Tx$ for each $a \in A$ and $x \in L$. The collection of all f-linear operators will be denoted by $L_b(L, M; A)$. For each $x \in L$ and $f \in L^{\sim}$, we can define $\psi_{x,f} \in A^{\sim}$ by $\psi_{x,f}(a) = f(a \cdot x)$ for all $a \in A$. Let $S(x) := \{\psi_{x,f} : f \in L^{\sim}\}$. Then $S(x)$ is an order ideal in $A^{\sim}$ [2]. $T \in L_b(L, M)$ is said to be an f-orthomorphism if $S(Tx) \subseteq S(x)$ for each $x \in L$. The collection of all f-orthomorphisms will be denoted by $\mathrm{Orth}(L, M; A)$. Turan [2] showed that $\mathrm{Orth}(L, M; A) = L_b(L, M; A)$ whenever M is topologically full with respect to A.

Clearly, $A^{\sim}$ is an f-module over the f-algebras A and $(A^{\sim})^{\sim}_{n}$, respectively. If A is unital, then A is topologically full with respect to itself ([2, Proposition 2.6]). From the above remarks we know that $A^{\sim}$ is topologically full with respect to $(A^{\sim})^{\sim}_{n}$, and hence, the f-orthomorphisms and f-linear operators are precisely the same class of operators, that is,

$$\mathrm{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})^{\sim}_{n}) = L_b(A^{\sim}, A^{\sim}; (A^{\sim})^{\sim}_{n}). \tag{*}$$

An f-algebra A is said to be *square-root closed* whenever for any $a \in A$ there exists $b \in A$ such that $|a| = b^2$. An immediate example is that a uniformly complete f-algebra with unit element is square-root closed [4]. However, a square-root closed f-algebra is not necessarily unital. For instance, c_0, with the familiar coordinatewise operations and ordering, is a square-root closed f-algebra without unit. We recall that an Archimedean f-algebra A is said to have the *factorization property* if, given $a \in A$, there exist $b, c \in A$ such that $a = bc$. It should be noted that if A is unital or square-root closed, then A has the factorization property.

In this paper, we do not have to assume that the f-algebras are unital. We modify the definition of the f-orthomorphism introduced by Turan [2, Definition 3.7] and consider the f-orthomorphisms and f-linear operators on the order dual of an f-algebra. In particular, when the f-algebra with separating order dual has the factorization property, we prove that the orthomorphisms, f-orthomorphisms, and f-linear operators on the order dual are precisely the same class of operators, that is, the above equality (*) still holds.

Our notions are standard. For the theory of Riesz spaces, positive operators, and f-algebras, we refer the reader to the monographs [5–7].

2. f-Orthomorphisms on the Order Dual

Let A be an f-algebra with separating order dual (and hence A Archimedean!) and $f \in A^{\sim}$. We consider the mapping $T_f : (A^{\sim})^{\sim}_{n} \to A^{\sim}$ defined by $T_f(F) = F \cdot f$ for all $F \in (A^{\sim})^{\sim}_{n}$.

It should be noted that the mapping $V : (A^{\sim})_n^{\sim} \to \mathrm{Orth}(A^{\sim})$ defined by $V(F) = V_F$ for all $F \in (A^{\sim})_n^{\sim}$, where $V_F(f) = F \cdot f$ for every $f \in A^{\sim}$, is an algebra and Riesz isomorphism (cf. [2, Proposition 2.2]).

Theorem 2.1. *For $0 \le f \in A^{\sim}$, T_f is an interval preserving lattice homomorphism.*

Proof. Clearly, T_f is linear and positive. Since the mapping V is a lattice homomorphism and $V_F, V_G \in \mathrm{Orth}(A^{\sim})$ for $F, G \in (A^{\sim})_n^{\sim}$, we have

$$\begin{aligned} T_f(F \vee G) = (F \vee G) \cdot f &= V_{F\vee G}(f) \\ &= (V(F \vee G))(f) \\ &= (V(F) \vee V(G))(f) \\ &= \big(V(F)(f)\big) \vee \big(V(G)(f)\big) \\ &= F \cdot f \vee G \cdot f \\ &= T_f(F) \vee T_f(G). \end{aligned} \tag{2.1}$$

Hence, T_f is a lattice homomorphism.

Next, we show that T_f is an interval preserving operator. We identify x with its canonical image x'' in $(A^{\sim})_n^{\sim}$ and denote the restriction of T_f to A by $T_f|_A$. Then

$$T_F|_A(x) = T_f(x'') = x'' \cdot f = f \cdot x. \tag{2.2}$$

Thus, for each $F \in (A^{\sim})_n^{\sim}$ and $x \in A$, we see that

$$\Big((T_f|_A)'(F)\Big)(x) = F\big((T_f|_A)(x)\big) = F(f \cdot x) = (F \cdot f)(x) = \big(T_f(F)\big)(x), \tag{2.3}$$

which implies that $(T_f|_A)'$ is the same as T_f on $(A^{\sim})_n^{\sim}$. Since $(T_f|_A)'$ is interval preserving (cf. [5, Theorem 7.8]), T_f is likewise an interval preserving operator. □

Corollary 2.2. *For $f \in A^{\sim}, F \in (A^{\sim})_n^{\sim}$, one has $|F \cdot f| = |F| \cdot |f|$. Furthermore, if $f \perp g$ in $A^{\sim}$, $F \cdot f \perp G \cdot g$ holds for any $F, G \in (A^{\sim})_n^{\sim}$.*

Proof. Since V_F is an orthomorphism on $A^{\sim}$, we have $V_F(f^+) \perp V_F(f^-)$ for each $f \in A^{\sim}$, that is, $F \cdot (f^+) \perp F \cdot (f^-)$. From Theorem 2.1, we know that

$$\begin{aligned} \left|F \cdot f\right| &= \left|F \cdot f^+\right| + \left|F \cdot f^-\right| \\ &= \left|T_{f^+}(F)\right| + \left|T_{f^-}(F)\right| \\ &= T_{f^+}(|F|) + T_{f^-}(|F|) \\ &= |F| \cdot f^+ + |F| \cdot f^- = |F| \cdot \left|f\right|. \end{aligned} \tag{2.4}$$

Let $f \perp g$ in $A^\sim$. Then we have

$$\begin{aligned} \left|F \cdot f\right| \wedge \left|G \cdot g\right| &= |F| \cdot \left|f\right| \wedge |G| \cdot \left|g\right| \\ &\leq \left((|F| + |G|) \cdot \left|f\right|\right) \wedge \left((|F| + |G|) \cdot \left|g\right|\right) = 0, \end{aligned} \tag{2.5}$$

which implies that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^\sim)_n^\sim$. □

Following the above discussion, we now consider $R(f) = \{F \cdot f : F \in (A^\sim)_n^\sim\}$, the image of $(A^\sim)_n^\sim$ under T_f.

Corollary 2.3. *If A is an f-algebra and $f \in (A^\sim)$, then $R(f) = R(|f|)$, and $R(f)$ is an order ideal in $A^\sim$.*

Proof. First, since $T_{|f|}$ is an interval preserving lattice homomorphism, we can easily see that $R(|f|)$ is an order ideal in $A^\sim$. By Corollary 2.2 we conclude that $R(f) \subseteq R(|f|)$.

Now, to complete the proof we only need to prove that $R(|f|) \subseteq R(f)$. To this end, let $P_1 : A^\sim \to B_{f^+}$, $P_2 : A^\sim \to B_{f^-}$ be band projections, where B_{f^+} and B_{f^-} are the bands generated by f^+ and f^- in $A^\sim$, respectively. If $\pi = P_1 - P_2$, we have

$$\pi \in \operatorname{Orth}(A^\sim), \qquad \pi(f) = \left|f\right|, \qquad \pi(\left|f\right|) = f. \tag{2.6}$$

In addition, $\pi(f) \cdot a = \pi(f \cdot a)$ for all $a \in A$ (cf. Theorem 3.1). Since π is an orthomorphism on $A^\sim$ and hence order continuous (cf. [5, Theorem 8.10]), we have $\pi'((A^\sim)_n^\sim) \subseteq (A^\sim)_n^\sim$. For all $a \in A$ and all $F \in (A^\sim)_n^\sim$, from

$$\begin{aligned} \left(F \cdot \left|f\right|\right)(a) &= \left(F \cdot \pi(f)\right)(a) \\ &= F\left(\pi(f) \cdot a\right) \\ &= F\left(\pi(f \cdot a)\right) \\ &= \left(\pi'(F) \cdot f\right)(a), \end{aligned} \tag{2.7}$$

it follows that $F \cdot |f| = \pi'(F) \cdot f$ for all $F \in (A^\sim)_n^\sim$, which implies that $R(|f|) \subseteq R(f)$, as desired. □

Next, we give a necessary and sufficient condition for $R(f) \perp R(g)$ when A has the factorization property. First, we need the following lemma.

Lemma 2.4. *Let A be an f-algebra with the factorization property, and $f \in A^\sim$. If $f \cdot x = 0$ for each $x \in A$, then $f = 0$.*

Proof. Since A has the factorization property, for each $a \in A^+$, there exist $x, y \in A$ such that $a = xy$. Hence, from

$$f(a) = f(xy) = (f \cdot x)(y) = 0, \tag{2.8}$$

it follows easily that $f = 0$ holds. □

Theorem 2.5. *Let A be an f-algebra with the factorization property. If $f, g \in A^\sim$, then $f \perp g$ if and only if $R(f) \perp R(g)$.*

Proof. If $f \perp g$ in $A^\sim$, then it follows from Corollary 2.2 that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^\sim)^\sim_n$. This implies that $R(f) \perp R(g)$.

Conversely, if $R(f)$ and $R(g)$ are disjoint, then for each $F \in ((A^\sim)^\sim_n)^+$ we have

$$\begin{aligned} F \cdot (|f| \wedge |g|) &= V_F(|f| \wedge |g|) \\ &= V_F(|f|) \wedge V_F(|g|) \\ &= F \cdot |f| \wedge F \cdot |g| \\ &= |F \cdot f| \wedge |F \cdot g| = 0. \end{aligned} \tag{2.9}$$

In particular, for any $x \in A$, its canonical image $x'' \in (A^\sim)^\sim_n$ also satisfies $x'' \cdot (|f| \wedge |g|) = (|f| \wedge |g|) \cdot x = 0$. By the preceding lemma, we have $|f| \wedge |g| = 0$, that is, $f \perp g$, as desired. □

Now, we give the definition of the so-called f-orthomorphism.

Definition 2.6. Let A be an f-algebra and $T \in L_b(A^\sim)$. T is called an f-orthomorphism on $A^\sim$ if $R(Tf) \subseteq R(f)$ for each $f \in A^\sim$. The collection of all f-orthomorphisms on $A^\sim$ will be denoted by $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$.

The next result deals with the relationship between the f-orthomorphisms and the orthomorphisms on the order dual of an f-algebra with the factorization property. Note that $\mathrm{Orth}(A^\sim)$ is a band in $L_b(A^\sim)$.

Theorem 2.7. *Let A be an f-algebra. Then $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$ is a linear subspace of $L_b(A^\sim)$ and $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$.*

If A, in addition, has the factorization property, then $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n) = \mathrm{Orth}(A^\sim)$.

Proof. First, we can easily see that $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$ is a linear subspace of $L_b(A^\sim)$. To prove $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$, let $\pi \in \mathrm{Orth}(A^\sim)$. We claim that $F \cdot \pi(f) = \pi'(F) \cdot f$ for all $F \in (A^\sim)^\sim_n$ and all $f \in A^\sim$. To this end, let $F \in (A^\sim)^\sim_n$, $f \in A^\sim$, and $x \in A$ be arbitrary. Since $(A^\sim)^\sim_n$ is a commutative f-algebra, by Theorem 3.1, we have

$$\begin{aligned} (\pi'(F) \cdot f)(x) = \pi'(F)(f \cdot x) = F(\pi(f \cdot x)) &= F(\pi(x'' \cdot f)) \\ &= F(x'' \cdot (\pi(f))) \\ &= (F \cdot x'')(\pi(f)) \\ &= (x'' \cdot F)(\pi(f)) \\ &= x''(F \cdot \pi(f)) = (F \cdot \pi(f))(x). \end{aligned} \tag{2.10}$$

Thus, $F \cdot \pi(f) = \pi'(F) \cdot f$. This implies that $R(\pi(f)) \subseteq R(f)$ for each $f \in A^\sim$, that is, $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$.

If A has the factorization property, we prove that $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n) \subseteq \mathrm{Orth}(A^\sim)$ holds. To this end, take $T \in \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$ and $f, g \in A^\sim$ satisfying $f \perp g$ in $A^\sim$. Then, it follows from Theorem 2.5 that $R(f) \perp R(g)$. Since $T \in \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)^\sim_n)$, we have

$R(T(f)) \subset R(f)$. Therefore, $R(T(f)) \perp R(g)$, which implies that $T(f) \perp g$, and hence T is an orthomorphism on $A^\sim$, as desired. □

3. f-Linear Operators on the Order Dual

Let A be an f-algebra with separating order dual and $T \in L_b(A^\sim)$. Recall that T is called to be f-linear with respect to $(A^\sim)_n^\sim$ if $T(G \cdot f) = G \cdot T(f)$ for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. The set of all f-linear operators on $A^\sim$ will be denoted by $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. It follows from [3, Lemma 4.4] that $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a band in $L_b(A^\sim)$.

Theorem 3.1. *Let A be an f-algebra with separating order dual. Then $Orth(A^\sim) \subseteq L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$.*

Proof. Clearly $\mathrm{Orth}(A^\sim)$ is commutative since $\mathrm{Orth}(A^\sim)$ is an Archimedean f-algebra. To complete the proof, let $\pi \in \mathrm{Orth}(A^\sim)$. We have

$$\pi(G \cdot f) = \pi(V_G(f)) = V_G(\pi(f)) = G \cdot (\pi(f)), \tag{3.1}$$

for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. Hence, $\pi \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. □

The following result deals with the order adjoint of an f-linear operator on the order dual of an f-algebra. It should be noted that the order adjoint of an order-bounded operator is order continuous (cf. [5, Theorem 5.8]).

Lemma 3.2. *Let $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. Then the order adjoint T' of T satisfies $T'(F) \cdot f = F \cdot T(f)$ for all $F \in (A^\sim)_n^\sim$ and $f \in A^\sim$. In particular, $G \cdot T'(F) = T'(G \cdot F)$ for all $F, G \in (A^\sim)_n^\sim$.*

Proof. Since $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$, and $(A^\sim)_n^\sim$ is a commutative f-algebra, we have

$$\begin{aligned}(T'(F) \cdot f)(x) = T'(F)(f \cdot x) &= F(T(f \cdot x)) \\ &= F(T(x'' \cdot f)) \\ &= F(x'' \cdot (T(f))) \\ &= (F \cdot x'')(T(f)) \\ &= (x'' \cdot F)(T(f)) \\ &= x''(F \cdot T(F)) = (F \cdot T(f))(x),\end{aligned} \tag{3.2}$$

for all $F \in (A^\sim)_n^\sim$, $f \in A^\sim$, and $x \in A$, which implies that $T'(F) \cdot f = F \cdot T(f)$.

Let $F,\ G \in (A^\sim)_n^\sim$ be given. Then for $f \in A^\sim$, from

$$\begin{aligned}(G \cdot T'(F))(f) = G(T'(F) \cdot f) &= G(F \cdot T(f)) \\ &= (G \cdot F)(T(f)) \\ &= (T'(G \cdot F))(f),\end{aligned} \tag{3.3}$$

it follows that $G \cdot T'(F) = T'(G \cdot F)$. This completes the proof. □

Theorem 3.3.

$$L_b(A^\sim, A^\sim; (A^\sim)_n^\sim) \subseteq Orth(A^\sim, A^\sim; (A^\sim)_n^\sim). \tag{3.4}$$

Proof. For $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$, we know that $|T|$ is also f-linear with respect to $(A^\sim)_n^\sim$. Assume that $0 \le G \in (A^\sim)_n^\sim$ and $f \in A^\sim$. So by Lemma 3.2, we have

$$0 \le G \cdot (|T(f)|) \le G \cdot (|T||f|) = (|T|'(G)) \cdot |f| = T_{|f|}(|T|'(G)). \tag{3.5}$$

Since $T_{|f|}$ is interval preserving, there exists $F \in (A^\sim)_n^\sim$ such that $0 \le F \le |T|'(G)$ and $G \cdot (|T(f)|) = F \cdot |f|$. It is now immediate that $R(|T(f)|) \subseteq R(|f|)$, and hence, $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim) \subseteq \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$, as desired. □

Combining Theorems 3.1, 3.3, and 2.7, we have the following result.

Theorem 3.4. *If A is an f-algebra with separating order dual, then*

$$Orth(A^\sim) \subseteq L_b(A^\sim, A^\sim; (A^\sim)_n^\sim) \subseteq Orth(A^\sim, A^\sim; (A^\sim)_n^\sim). \tag{3.6}$$

In particular, if, in addition, A has the factorization property, then

$$Orth(A^\sim) = L_b(A^\sim, A^\sim; (A^\sim)_n^\sim) = Orth(A^\sim, A^\sim; (A^\sim)_n^\sim). \tag{3.7}$$

Funding

The authors were supported in part by the Fundamental Research funds for the Central Universities (SWJTU11CX154, SWJTU12ZT13).

Acknowledgment

The authors would like to thank the reviewer for his/her kind comments and valuable suggestions which have improved this paper. Corollary 2.3 in its present formulation and the proof are essentially due to the reviewer. In particular, he/she suggested that the authors should consider their questions under the condition of "A has the factorization property," which is weaker than the hypothesis "A is square-root closed" used originally in this paper.

References

[1] C. B. Huijsmans and B. de Pagter, "The order bidual of lattice ordered algebras," *Journal of Functional Analysis*, vol. 59, no. 1, pp. 41–64, 1984.
[2] B. Turan, "On f-linearity and f-orthomorphisms," *Positivity*, vol. 4, no. 3, pp. 293–301, 2000.
[3] W. A. J. Luxemburg and B. de Pagter, "Maharam extensions of positive operators and f-modules," *Positivity*, vol. 6, no. 2, pp. 147–190, 2002.
[4] C. B. Huijsmans and B. de Pagter, "Ideal theory in f-algebras," *Transactions of the American Mathematical Society*, vol. 269, no. 1, pp. 225–245, 1982.
[5] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, vol. 119 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1985.
[6] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer, Berlin, Germany, 1991.
[7] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, The Netherlands, 1983.