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Research Article

Necessary and Sufficient Condition for the Existence of Solutions to a Discrete Second-Order Boundary Value Problem

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This paper is concerned with the existence of solutions for the discrete second-order boundary value problem $\Delta^2 u(t-1) + \lambda_1 u(t) + g(\Delta u(t)) = f(t)$, $t \in \{1, 2, \ldots, T\}$, $u(0) = u(T+1) = 0$, where $T > 1$ is an integer, $f : \{1, \ldots, T\} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and λ_1 is the first eigenvalue of the eigenvalue problem $\Delta^2 u(t-1) + \lambda u(t) = 0$, $t \in \mathbb{T}$, $u(0) = u(T+1) = 0$.

1. Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ and $p : [0, \pi] \to \mathbb{R}$ be continuous. The nonlinear two-point boundary value problem of ordinary differential equation

$$\begin{gathered} u''(t) + u(t) + g(u') = p(t), \quad t \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

is very important in applications. Let us mention the problems arising in viscosity, nonlinear oscillations, electric circuits, and so forth. The term $g(u')$ may be regarded as a nonlinear damping term in resonance problems and its appears, for example, in Rayleigh's equation (which is closely connected with a theory of oscillation of violin string), in oscillations of a simple pendulum under the action of viscous damping, in dry (Coulomb) friction (which occurs when the surfaces of two solids are contact and relative motion without lubrication), and in some cases of van der Pol oscillator, see [1–4] and the references therein.

Since the pioneer work of Landesman and Lazer [5], the problems of the type

$$u''(t) + u(t) + g(u) = f(t), \quad t \in (0, \pi),$$
$$u(0) = u(\pi) = 0, \tag{1.2}$$

(where g is independent of u') have been extensively studied in the past forty years, see Iannacci and Nkashama [6] and the references therein.

It has been remarked (see [7, 8]) that conditions of the Landesmen-Lazer type are not appropriated to yield the existence of solutions to (1.1). Thus, it is usually much more difficult to deal with (1.1) than to deal with (1.2), see Kannan et al. [7], Cañada and Drábek [8], Habets and Sanchez [9], Drábek et al. [10], and Del Toro and Roca [11].

In [8], Cañada and Drábek used the well-known Lyapunov-Schmidt method and the Schauder fixed point theorem to find a necessary and sufficient condition for the existence of solutions of (1.1). To wit, they proved

Theorem A (See [8, Theorem 3.1]). *Let $p : [0, \pi] \to \mathbb{R}$ be continuous and let*

$$p(t) = s\sqrt{\frac{2}{\pi}}\sin t + \tilde{p}(t), \quad s \in \mathbb{R}, \qquad \int_0^{\pi} \tilde{p}(t)\sin t\, dt = 0. \tag{1.3}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and bounded with $g(-\infty) = g(+\infty)$ and $g(\xi) < g(+\infty)$ for $\xi \in \mathbb{R}$, where

$$g(-\infty) := \lim_{s \to -\infty} g(s), \qquad g(+\infty) := \lim_{s \to +\infty} g(s). \tag{1.4}$$

Then for any $\tilde{p} \in C[0, \pi]$ with $\int_0^{\pi} \tilde{p}(t)\sin t\, dt = 0$, there exists a real number $g_{\tilde{p}} < 2\sqrt{2/\pi}\, g(+\infty)$ such that (1.1) has at least one solution $u \in C^2[0, \pi]$ if and only if

$$s \in \left[g_{\tilde{p}},\ 2\sqrt{\frac{2}{\pi}}g(+\infty)\right). \tag{1.5}$$

It is the purpose of this paper to establish the similar results for the discrete analogue of (1.1) of the form

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g(\Delta u(t)) = f(t), \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0, \tag{1.6}$$

where $T > 1$ is an integer, $\mathbb{T} := \{1, \ldots, T\}$, $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, $f : \mathbb{T} \to \mathbb{R}$, λ_1 is the first eigenvalue of the linear eigenvalue problem

$$\Delta^2 u(t-1) + \lambda u(t) = 0, \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0. \tag{1.7}$$

Finally, it is worth remarking that the existence of solutions for nonlinear problem

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g(u(t)) = f(t), \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \tag{1.8}$$

which is a discrete analogue of (1.2), has been studied by Rodriguez [12] and Ma [13]. For other recent results on the existence of solutions of discrete problems, see [14–21] and the reference therein.

The rest of this paper is arranged as follows. In Section 2, we give some preliminaries and develop the methods of lower and upper solutions for the more generalized problems, that is, the case of the nonlinearity $g = g(t, u, \Delta u)$; in Section 3, we state our main result and provide the proof.

2. Preliminaries

Recall that $\mathbb{T} = \{1, 2, \ldots, T\}$. Let $\widehat{\mathbb{T}} = \{0, 1, \ldots, T+1\}$. Let $X := \{u \mid u : \widehat{\mathbb{T}} \to \mathbb{R}\}$, $Y := \{u \mid u : \mathbb{T} \to \mathbb{R}\}$ be equipped with the norm

$$\|u\|_X = \max_{k \in \widehat{\mathbb{T}}} |u(k)|, \qquad \|u\|_Y = \max_{k \in \mathbb{T}} |u(k)|, \tag{2.1}$$

respectively. It is easy to see that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

Assume that $g_0 : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, bounded by a constant $M > 0$:

$$\left| g_0(t, \eta, \xi) \right| \leq M \tag{2.2}$$

for $t \in \mathbb{T}$ and $(\eta, \xi) \in \mathbb{R}^2$. Consider the following problem:

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g_0(t, u(t), \Delta u(t)) = f(t), \quad t \in \mathbb{T}, \tag{2.3}$$

$$u(0) = u(T+1) = 0. \tag{2.4}$$

Definition 2.1. If $x \in X$ satisfies

$$\Delta^2 x(t-1) + \lambda_1 x(t) \geq f(t) - g_0(t, x(t), \Delta x(t)), \quad t \in \mathbb{T}, \\ x(0) \leq 0, \quad x(T+1) \leq 0, \tag{2.5}$$

then one says $x(t)$ is a lower solution of (2.3), (2.4). If $y \in X$ satisfies

$$\Delta^2 y(t-1) + \lambda_1 y(t) \leq f(t) - g_0(t, y(t), \Delta y(t)), \quad t \in \mathbb{T}, \\ y(0) \geq 0, \quad y(T+1) \geq 0, \tag{2.6}$$

then one says $y(t)$ is an upper solution of (2.3), (2.4).

Theorem 2.2. *Suppose that $x(t)$, $y(t)$ are the lower and upper solutions of* (2.3), (2.4), *respectively, and $x(t) \le y(t)$, $t \in \mathbb{T}$. Then BVP* (2.3) *and* (2.4) *have at least one solution $u(t)$ satisfies*

$$x(t) \le u(t) \le y(t). \tag{2.7}$$

Proof. Define the function $p : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ by

$$p(t,u(t)) = \begin{cases} x(t), & u(t) < x(t), \\ u(t), & x(t) \le u(t) \le y(t), \\ y(t), & u(t) > y(t). \end{cases} \tag{2.8}$$

Set $f^*(t,u,v) = f(t) - g_0(t,u,v) - \lambda_1 u$. Consider the auxiliary problems:

$$\begin{gathered} \Delta^2 u(t-1) = f^*\big(t, p(t,u), \Delta p(t,u)\big), \qquad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0. \end{gathered} \tag{2.9}$$

From (2.8) and the boundness of g_0, we know $f^*(t,p(t,u),\Delta p(t,u))$ is bounded. So, by the Schauder fixed point theorem, (2.9) has a solution $u \in X$.

Now, we only prove $u(t) \le y(t)$, the other case $u(t) \ge x(t)$ is similar.

Set $z(t) = u(t) - y(t)$. Suppose that $z(t) > 0$, for $t \in \{t_0+1, t_0+2, \dots, t_0+p\}$, and $z(t_0) \le 0$, $z(t_0+p+1) \le 0$, where $t_0 \in \{0,1,\dots,T\}$, $p \in \{1,2,\dots,T\}$.

On the other hand, by the definition of upper solution, for $t \in \{t_0+1, t_0+2, \dots, t_0+p\}$,

$$\begin{aligned} \Delta^2 y(t-1) &\le f^*\big(t, y(t), \Delta y(t)\big) \\ &= f(t) - \lambda_1 y(t) - g_0\big(t, y(t), \Delta y(t)\big) \\ &= f(t) - \lambda_1 p(t,u) - g_0\big(t, p(t,u(t)), \Delta p(t,u(t))\big) \\ &= \Delta^2 u(t-1). \end{aligned} \tag{2.10}$$

Then

$$\begin{gathered} \Delta^2 z(t-1) \ge 0, \quad t \in \big\{t_0+1, t_0+2, \dots, t_0+p\big\}, \\ z(t_0) \le 0, \quad z\big(t_0+p+1\big) \le 0. \end{gathered} \tag{2.11}$$

Now, by the convexity of z on $\{t_0+1, t_0+2, \dots, t_0+p\}$, we get $z(t) \le 0$, $t \in \{t_0+1, t_0+2, \dots, t_0+p\}$, that is, $u(t) \le y(t), t \in \{t_0+1, t_0+2, \dots, t_0+p\}$. This contradicts $u(t) > y(t)$, $t \in \{t_0+1, t_0+2, \dots, t_0+p\}$. Thus, $u(t) \le y(t)$, $t \in \widehat{\mathbb{T}}$. □

Lemma 2.3. *See* $\sum_{t=1}^{T} \sin^2(\pi t/(T+1)) = (T+1)/2$.

Proof. Let $\omega = \cos(2\pi/(T+1)) + i\sin(2\pi/(T+1))$. Then $\omega^{T+1} = 1$ and $(1-\omega)(1+\omega+\omega^2+\cdots+\omega^T) = 0$. Since $1-\omega \ne 0$, we have $\sum_{t=1}^{T}\cos(2\pi t/(T+1)) = -1$. This together with the fact that $\sum_{t=1}^{T}\sin^2(\pi t/(T+1)) = \sum_{t=1}^{T}(1-\cos(2\pi t/(T+1)))/2$ implies the assertion holds. □

Now, let $\varphi_1(t) := \sqrt{2/(T+1)}\sin(\pi t/(T+1))$, $t \in \mathbb{T}$, denote the positive eigenfunction corresponding to the first eigenvalue $\lambda_1 = 4\sin^2(\pi/2(T+1))$ of (1.7). Then by Lemma 2.3, $\sum_{t=1}^{T}\varphi_1^2(t) = 1$.

Since $\varphi_1(t)$ is located on $\sqrt{2/(T+1)}\sin t,\ t \in [0,\pi]$, by the direct computation, we can obtain the following result.

Lemma 2.4. *If T is an odd number, then*

$$\Delta\varphi_1(t) > 0, \quad \text{for } t \in \left\{0, \ \dots \ \left[\frac{T}{2}\right]\right\}, \qquad \Delta\varphi_1(t) < 0, \quad \text{for } t \in \left\{\left[\frac{T}{2}\right] + 1, \ \dots, \ T\right\}, \tag{2.12}$$

if T is an even number, then $\Delta\varphi_1(T/2) = 0$,

$$\Delta\varphi_1(t) > 0, \quad \text{for } t \in \left\{0, \ \dots \ \frac{T}{2} - 1\right\}, \qquad \Delta\varphi_1(t) < 0, \quad \text{for } t \in \left\{\frac{T}{2} + 1, \ \dots, \ T\right\}. \tag{2.13}$$

Define the operator $L : D(L) \subset X \to Y$ by

$$Lu(t) = \Delta^2 u(t-1) + \lambda_1 u(t), \tag{2.14}$$

where $D(L) = \{u \in X \mid u(0) = u(T+1) = 0\}$.

Define $N : X \to Y$ by

$$(Nu)(t) = f(t) - g_0(t, u(t), \Delta u(t)). \tag{2.15}$$

Then (2.3), (2.4) is equivalent to the operator equation $Lu = Nu$.

In Theorem 2.2, we established the methods of lower and upper solutions under well order. Now, we can also develop the methods of lower and upper solutions for (2.3), (2.4) when $x(t) \le y(t)$ is not necessary, its proofs are based on the following lemma, that is, the connectivity properties of the solution sets of parameterized families of compact vector fields, they are a direct consequence of Mawhin [22, Lemma 2.3].

Lemma 2.5 (see [22, Lemma 2.3]). *Let E be a Banach space and $C \subset E$ a nonempty, bounded, closed convex subset. Suppose that $T : [a,b] \times C \to C$ is completely continuous. Then the set*

$$S = \{(\lambda, x) \mid T(\lambda, x) = x, \quad \lambda \in [a,b]\} \tag{2.16}$$

contains to be a closed connected subset Σ which connects $\{a\} \times C$ to $\{b\} \times C$.

Theorem 2.6. *Assume that $x(t)$, $y(t)$ are the lower solution and the upper solution of* (2.3), (2.4), *respectively. Then* (2.3) *and* (2.4) *have at least one solution.*

Proof. Define the projections $P : X \to X$, $Q : Y \to \mathbb{R}$ by

$$(Pu)(t) = \left[\sum_{t=1}^{T} u(t)\varphi_1(t)\right]\varphi_1(t), \qquad (Qy)(t) = \sum_{t=1}^{T} y(t)\varphi_1(t). \tag{2.17}$$

Then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, and $X = (\operatorname{Ker} P \oplus \operatorname{Ker} L), Y = (\operatorname{Im} L \oplus \operatorname{Im} Q)$. Now, the operator equation $Lu = Nu$ is equivalent to the alternative system

$$\begin{gathered} u - Pu = K(I - Q)Nu, \\ QNu = 0, \end{gathered} \tag{2.18}$$

where K is the inverse of mapping $L : (D(L) \cap \operatorname{Ker} P) \to \operatorname{Im} L$.

Writing $u \in D(L)$ in the form $u(t) = c\psi_1(t) + w(t)$, $c \in \mathbb{R}$, $\sum_{t=1}^{T} \psi_1(t)w(t) = 0$, (2.3) and (2.4) are equivalent to the system

$$w = K(I - Q)N\big(c\psi_1(\cdot) + w\big), \tag{2.19}$$

$$QN\big(c\psi_1(\cdot) + w\big) = 0. \tag{2.20}$$

Since X is finite dimensional, it is easy to see that $K(I-Q)N$ is completely continuous, by the Schauder fixed point theorem and the fact $N(c\psi_1(t) + w(t))$ is bounded, we get that for any fixed $c \in \mathbb{R}$, $W(c) := \{w \in X \cap \operatorname{Ker} P \mid (c, w) \text{ satisfies } (2.19)\} \neq \emptyset$ and $W(c)$ is bounded. Then there exist positive constants $\alpha > \tau$ such that $-\tau\psi_1 \leq w \leq \tau\psi_1$ for all $w \in W(c)$, $(\alpha-\tau)\psi_1 \geq x(t)$ and $-(\alpha - \tau)\psi_1 \leq y(t)$. Let

$$\gamma(c, w) = \sum_{t=1}^{T} N\big(c\psi_1(t) + w(t)\big)\psi_1(t) \tag{2.21}$$

for all $(c, w) \in \mathbb{R} \times W(c)$. Observe that Lemma 2.5 is applicable. Hence there exists a connected subset of $\{(c, w) \in R \times (X \cap \operatorname{Ker} P) \mid (c, w) \text{ satisfies } (2.19)\}$, $\Sigma_\alpha(\psi_1)$, which connected $\{-\alpha\} \times W(-\alpha)$ and $\{\alpha\} \times W(\alpha)$. Since $\gamma : \Sigma_\alpha(\psi_1) \to \mathbb{R}$ is continuous, $I := \gamma(\Sigma_\alpha(\psi_1))$ is an interval. If $0 \in I$, then (2.3) and (2.4) have a solution. If $I \subset (0, \infty)$, then every $c\psi_1 + w$ with $(c, w) \in \Sigma_\alpha(\psi_1)$ is an upper solution. Indeed, it is obvious that

$$\begin{gathered} L\big(c\psi_1(t) + w(t)\big) - N\big(c\psi_1(t) + w(t)\big) = -\gamma(c, w) \leq 0, \qquad t \in \mathbb{T}, \\ (c\psi_1 + w)(0) = (c\psi_1 + w)(T + 1) = 0. \end{gathered} \tag{2.22}$$

By construction, $\alpha\psi_1 + w$ with $(\alpha, w) \in \Sigma_\alpha(\psi_1)$ satisfies $x(t) \leq \alpha\psi_1 + w$. Hence, from Theorem 2.2, (2.3) and (2.4) have a solution. A similar argument applies if $I \subset (-\infty, 0)$. □

Theorem 2.7. *Suppose that f satisfies*

$$f(t) = s\psi_1(t) + \tilde{f}(t), \qquad s \in \mathbb{R}, \tag{2.23}$$

where $\tilde{f}$ satisfies

$$\sum_{t=1}^{T} \tilde{f}(t)\psi_1(t) = 0. \tag{2.24}$$

Then, there exists a nonempty, connected, and bounded set $J_{\tilde{f}} \subset \mathbb{R}$ *such that* (2.3) *and* (2.4) *have at least one solution* $u \in X$ *if and only if* $s \in J_{\tilde{f}}$.

Proof. As the proof of Theorem 2.6, (2.3) and (2.4) are equivalent to the system (2.19), (2.20). Since N is bounded, applying the Schauder fixed point theorem we obtain that for any fixed $c \in \mathbb{R}$, there exits at least one $w_c \in X$ such that (2.19) holds.

Now, (2.20) becomes

$$\sum_{t=1}^{T} g_0\big(t, c\psi_1(t) + w_c(t), c\Delta\psi_1(t) + \Delta w_c(t)\big)\psi_1(t) = s. \tag{2.25}$$

Hence, for a given $\tilde{f}$, $\sum_{t=1}^{T} \tilde{f}(t)\psi_1(t) = 0$, (2.3), (2.4) with $f(t) = c\psi_1(t) + \tilde{f}(t)$ has at least one solution if and only if s belongs to the range of the (multivalued, in general) function $\Gamma_{\tilde{f}} : \mathbb{R} \to \Gamma_{\tilde{f}}(\mathbb{R})$,

$$\Gamma_{\tilde{f}}(c) = \sum_{t=1}^{T} g_0\big(t, c\psi_1(t) + w_c(t), c\Delta\psi_1(t) + \Delta w_c(t)\big)\psi_1(t), \tag{2.26}$$

where $w_c \in \{w \in D(L) : w \text{ is a solution of (2.19) for fixed } c\}$. But $J_{\tilde{f}} \equiv \Gamma_{\tilde{f}}(\mathbb{R})$ is a connected set. In fact, let s_1 and s_2 belong to $J_{\tilde{f}}$ and $s_1 \le s_2$. Then (2.3), (2.4) with $f_1 = s_1\psi_1 + \tilde{f}$ and $f_2 = s_2\psi_1 + \tilde{f}$ has solutions u_1 and u_2, respectively. If we consider (2.3), (2.4) with $f = s\psi_1 + \tilde{f}$, where $s \in [s_1, s_2]$, then u_1 is an upper solution and u_2 is a lower solution to this problem. By Theorem 2.6, there exists at least one solution, that is, s belongs to $J_{\tilde{f}}$. Moreover, since g is bounded, the range of $\Gamma_{\tilde{f}}$ is bounded. □

3. Main Results

In this section, we deal with (1.6). First, let us make the following assumptions:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function and satisfies $g(+\infty) = g(-\infty)$ and $g(\xi) < g(+\infty)$ for any $\xi \in \mathbb{R}$,

(H2) $f : \mathbb{T} \to \mathbb{R}$ satisfies

$$f(t) = s\psi_1(t) + \tilde{f}(t), \quad s \in \mathbb{R}, \qquad \sum_{t=1}^{T} \tilde{f}(t)\psi_1(t) = 0. \tag{3.1}$$

Theorem 3.1. *Suppose that* (H1), (H2) *hold. Then there exists a real number* $g_{\tilde{f}}$, $g_{\tilde{f}} < \sqrt{2/(T+1)}g(+\infty)\sum_{t=1}^{T}\sin(\pi t/(T+1))$, *such that* (1.6) *has at least one solution* $u \in X$ *if and only if*

$$s \in \left[g_{\tilde{f}}, \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}\right). \tag{3.2}$$

Proof. Note that $\psi_1(t) = \sqrt{2/(T+1)}\sin(\pi t/(T+1))$. Due to the consideration in the the proof of Theorem 2.7. It is sufficient to show that for a given $\tilde{f}$ with $\sum_{t=1}^{T}\tilde{f}(t)\psi_1(t) = 0$, we have

$$\Gamma_{\tilde{f}}(\mathbb{R}) = \left[g_{\tilde{f}}, \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}\right). \tag{3.3}$$

The (possibly multivalued) function $\Gamma_{\tilde{f}}$ has the following form:

$$\Gamma_{\tilde{f}}(c) = \sqrt{\frac{2}{T+1}}\sum_{t=1}^{T}g\left(c\Delta\psi_1(t) + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}, \tag{3.4}$$

where $c \in \mathbb{R}$ and w_c verify (2.19). From the boundedness of g and (2.19), there exists a constant $D > 0$ (independent of c) such that $\|w_c\|_X \le D$ for any $c \in \mathbb{R}$, furthermore,

$$\max_{t\in\{0,1,\dots,T\}}|\Delta w_c(t)| \le 2D. \tag{3.5}$$

Now, we divide the proof into two cases.

Case 1. T is an odd number. By Lemma 2.4, we obtain that

$$\begin{aligned}
&\sum_{t=1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad= \sum_{t=1}^{[T/2]}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+ \sum_{t=[T/2]+1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad\longrightarrow \sum_{t=1}^{[T/2]}g(\pm\infty)\sin\frac{\pi t}{T+1} + \sum_{s=[T/2]+1}^{T}g(\mp\infty)\sin\frac{\pi t}{T+1},
\end{aligned} \tag{3.6}$$

as $c \longrightarrow \pm\infty$. Due to $g(+\infty) = g(-\infty)$, we get

$$\Gamma_{\tilde{f}}(c) \longrightarrow \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}. \tag{3.7}$$

The assumption $g(\xi) < g(+\infty)$, $\xi \in \mathbb{R}$, and (3.5) yields

$$\Gamma_{\tilde{f}}(c) < \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1} \tag{3.8}$$

for any $c \in \mathbb{R}$.

Case 2. T is an even number. By Lemma 2.4, we know that

$$\Delta\varphi_1\left(\frac{T}{2}\right) = \sqrt{\frac{2}{T+1}}\Delta\sin\frac{\pi T}{2(T+1)} = 2\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2T/2+1)}{2(T+1)} = 0. \tag{3.9}$$

Hence,

$$\begin{aligned}
&\sum_{t=1}^{T} g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad= \sum_{t=1}^{(T/2)-1} g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+ \sum_{t=(T/2)+1}^{T} g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+ g\left(\Delta w_c\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}.
\end{aligned} \tag{3.10}$$

By (3.5) and the assumption $g(\xi) < g(+\infty)$, $\xi \in \mathbb{R}$, we know that for any $c \in \mathbb{R}$, $g(\Delta w_c(T/2)) < g(+\infty)$. Thus, for any $c \in \mathbb{R}$,

$$\Gamma_{\tilde{f}}(c) < \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}. \tag{3.11}$$

It is sufficient to prove that this infimum is achieved. Let us denote

$$g_{\tilde{f}} = \inf_{c\in\mathbb{R}}\Gamma_{\tilde{f}}(c). \tag{3.12}$$

Suppose that $\{s_n\} \subset \Gamma_{\tilde{f}}(\mathbb{R})$ satisfies $s_n \to g_{\tilde{f}}$ and $\{c_n\}$ is the corresponding minimizing sequence, that is, $u_n(t) = c_n\sqrt{2/(T+1)}\sin(\pi t/(T+1)) + w_{c_n}(t)$, are the solution of (1.1), with the right-hand sides $f_n(t) = s_n\sqrt{2/(T+1)}\sin(\pi t/(T+1)) + \tilde{f}(t)$.

We claim that $\{c_n\}$ is bounded. In fact, if $c_n \to \infty$ as $n \to \infty$, then we can get two contradictions in the following two cases.

Case 1. If T is an odd number, then by (2.20),

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} = s_n, \tag{3.13}$$

letting $n \to \infty$ in (3.13), we get

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T} g(+\infty)\sin\frac{\pi t}{T+1} = g_{\tilde{f}}. \tag{3.14}$$

From (H1), we arrive for any $\xi \in \mathbb{R}$, $g(\xi) < g(+\infty)$, which together with (3.14) implies that

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T} g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1} < g_{\tilde{f}}. \tag{3.15}$$

This contradicts (3.12).

Case 2. If T is an even number, then by (2.20) and $\Delta\varphi_1(T/2) = 0$, we get

$$\begin{aligned} s_n = {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}} g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}. \end{aligned} \tag{3.16}$$

This implies that

$$\begin{aligned} g_{\tilde{f}} \geq {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g(+\infty)\sin\frac{\pi t}{T+1} + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g(+\infty)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\inf_{w_{c_n}\in X} g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}. \end{aligned} \tag{3.17}$$

On the other hand, by (3.12) and (H1), we get that for any fixed $n \in \mathbb{N}$,

$$\begin{aligned} g_{\tilde{f}} \leq {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\inf_{w_{c_n}\in X} g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)} \end{aligned}$$

$$< \sqrt{\frac{2}{T+1}} \sum_{t=1}^{(T/2)-1} g(+\infty) \sin \frac{\pi t}{T+1} + \sqrt{\frac{2}{T+1}} \sum_{t=(T/2)+1}^{T} g(+\infty) \sin \frac{\pi t}{T+1}$$
$$+ \sqrt{\frac{2}{T+1}} \inf_{w_{c_n} \in X} g\left(\Delta w_{c_n}\left(\frac{T}{2} \right) \right) \sin \frac{\pi T}{2(T+1)}. \tag{3.18}$$

Now, we obtain a contradiction. Thus, c_n is bounded.

Since X is finite dimensional and w_{c_n} is bounded, we obtain that $c_n \to c$, $w_{c_n} \to w_c$ (at least for a subsequence), and $u(t) = c\sqrt{2/(T+1)}\sin(\pi t/(T+1)) + w_c(t)$ is a solution of (1.1) with $f(t) = g_{\tilde{f}}\sqrt{2/(T+1)}\sin(\pi t/(T+1)) + \tilde{f}(t)$. Hence, the infimum is achieved in c. □

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