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Research Article

Iterative Algorithms Approach to Variational Inequalities and Fixed Point Problems

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We consider a general variational inequality and fixed point problem, which is to find a point x^* with the property that (GVF): $x^* \in \mathrm{GVI}(C, A)$ and $g(x^*) \in \mathrm{Fix}(S)$ where $\mathrm{GVI}(C, A)$ is the solution set of some variational inequality $\mathrm{Fix}(S)$ is the fixed points set of nonexpansive mapping S, and g is a nonlinear operator. Assume the solution set Ω of (GVF) is nonempty. For solving (GVF), we suggest the following method $g(x_{n+1}) = \beta g(x_n) + (1-\beta) SP_C[\alpha_n F(x_n) + (1-\alpha_n)(g(x_n) - \lambda A x_n)]$, $n \geq 0$. It is shown that the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which is the unique solution of the variational inequality $\langle F(x^*) - g(x^*), g(x) - g(x^*) \rangle \leq 0$, for all $x \in \Omega$.

1. Introduction

Let $A : C \to H$ and $g : C \to C$ be two nonlinear mappings. We concern the following generalized variational inequality of finding $u \in C$, $g(u) \in C$ such that

$$\langle g(v) - g(u), Au \rangle \geq 0, \quad \forall g(v) \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $\mathrm{GVI}(C, A, g)$. It has been shown that a large class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering, and applied sciences can be studied in the unified and general framework of the general variational inequalities (1.1), see [1–16] and the references therein. Noor [17] has introduced a new type of variational inequality involving two nonlinear operators, which is called the general variational inequality. It is worth mentioning that this general variational inequality is remarkably different from the so-called general variational inequality which was introduced by Noor [18] in 1988. Noor [17] proved that the general variational inequalities are equivalent to nonlinear projection equations and the Wiener-Hopf equations by using the projection technique. Using this equivalent formulation, Noor [17] suggested and analyzed some iterative algorithms for solving the special general variational inequalities and further proved that these algorithms have strong convergence.

For $g = I$, where I is the identity operator, problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [19] in 1964. This field has been extensively studied due to a wide range of applications in industry, finance, economics, social, pure and applied sciences. For related works, please see [20–35]. Our main purposes in the present paper is devoted to study this topic.

Motivated and inspired by the works in this field, in this paper, we consider a general variational inequality and fixed point problem, which is to find a point x^* with the property that

$$x^* \in \mathrm{GVI}(C, A), \quad g(x^*) \in \mathrm{Fix}(S), \tag{GVF}$$

where $\mathrm{Fix}(S)$ is the fixed points set of nonexpansive mapping S. Assume the solution set Ω of (GVF) is nonempty. For solving (GVF), we suggest the following method

$$g(x_{n+1}) = \beta g(x_n) + (1-\beta) S P_C \left[\alpha_n F(x_n) + (1-\alpha_n)\left(g(x_n) - \lambda A x_n\right)\right], \quad n \geq 0. \tag{1.3}$$

It is shown that the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which is the unique solution of the following variational inequality

$$\left\langle F(x^*) - g(x^*), g(x) - g(x^*) \right\rangle \leq 0, \quad \forall x \in \Omega. \tag{1.4}$$

Our results contain some interesting results as special cases.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{2.1}$$

for all $x, y \in C$. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S. A mapping $F : C \to H$ is said to be L-Lipschitz continuous, if there exists a constant $L > 0$ such that $\|F(x) - F(y)\| \leq L\|x - y\|$

for all $x, y \in C$. A mapping $A : C \to H$ is said to be α-inverse strongly g-monotone if and only if

$$\langle Ax - Ay, g(x) - g(y) \rangle \geq \alpha \| Ax - Ay \|^2, \tag{2.2}$$

for some $\alpha > 0$ and for all $x, y \in C$. A mapping $g : C \to C$ is said to be strongly monotone if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \gamma \| x - y \|^2, \tag{2.3}$$

for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multivalued mapping B is said to be a monotone operator on H if and only if

$$\langle x - y, u - v \rangle \geq 0, \tag{2.4}$$

for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if and only if its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\| u - u_0 \| = \inf\{ \| u - x \| : x \in C \}. \tag{2.5}$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\| P_C x - P_C y \| \leq \| x - y \|$ for all $x, y \in H$;

(ii) $\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2$ for every x, $y \in H$;

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$, $y \in C$.

It is easy to see that the following is true:

$$u \in \mathrm{GVI}(C, A, g) \Longleftrightarrow g(u) = P_C(g(u) - \lambda A(u)), \quad \forall \lambda > 0. \tag{2.6}$$

We use the following notation:

(i) $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;

(ii) $x_n \to x$ stands for the strong convergence of (x_n) to x.

We need the following lemmas for the next section.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \to C$ be a nonlinear mapping and let the mapping $A : C \to H$ be α-inverse strongly g-monotone. Then, for any $\lambda > 0$, one has*

$$\| P_C[g(x) - \lambda Ax] - P_C[g(y) - \lambda Ay] \|^2 \leq \| g(x) - g(y) \|^2 + \lambda(\lambda - 2\alpha) \| Ax - Ay \|^2, \quad x, y \in C. \tag{2.7}$$

Proof. Consider the following:

$$\begin{aligned}
&\left\|P_C[g(x)-\lambda Ax]-P_C[g(y)-\lambda Ay]\right\|^2 \\
&\quad \le \left\|g(x)-g(y)-\lambda(Ax-Ay)\right\|^2 \\
&\quad = \left\|g(x)-g(y)\right\|^2-2\lambda\langle Ax-Ay,g(x)-g(y)\rangle+\lambda^2\left\|Ax-Ay\right\|^2 \\
&\quad \le \left\|g(x)-g(y)\right\|^2-2\lambda\alpha\left\|Ax-Ay\right\|^2+\lambda^2\left\|Ax-Ay\right\|^2 \\
&\quad \le \left\|g(x)-g(y)\right\|^2+\lambda(\lambda-2\alpha)\left\|Ax-Ay\right\|^2.
\end{aligned} \tag{2.8}$$

If $\lambda \in [0,2\alpha]$, we have

$$\left\|P_C[g(x)-\lambda Ax]-P_C[g(y)-\lambda Ay]\right\| \le \left\|g(x)-g(y)-\lambda(Ax-Ay)\right\| \le \left\|g(x)-g(y)\right\|. \tag{2.9}$$

□

Lemma 2.2 (see [36]). *Let C be a closed convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Then $\text{Fix}(S)$ is a closed convex subset of C and the mapping $I-S$ is demiclosed at 0, that is, whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I-S)x_n \to 0$, then $(I-S)x = 0$.*

Lemma 2.3 (see [37]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.4 (see [38]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n\gamma_n, \tag{2.10}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

In this section, we will prove our main results.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \to H$ be an L-Lipschitz continuous mapping, $g : C \to C$ be a weakly continuous and γ-strongly monotone mapping such that $R(g) = C$. Let $A : C \to H$ be an α-inverse strongly g-monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Suppose that $\Omega \neq \emptyset$. Let $\beta \in (0,1)$ and $\gamma \in (L,2\alpha)$. For given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$g(x_{n+1}) = \beta g(x_n) + (1-\beta)SP_C\left[\alpha_n F(x_n) + (1-\alpha_n)\left(g(x_n) - \lambda Ax_n\right)\right], \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ *satisfies* (C1): $\lim_{n\to\infty}\alpha_n = 0$ *and* (C2): $\sum_n \alpha_n = \infty$. *Then the sequence* $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* \in \Omega$ *which is the unique solution of the following variational inequality:*

$$\langle F(x^*) - g(x^*), g(x) - g(x^*) \rangle \leq 0, \quad \forall x \in \Omega. \tag{3.2}$$

Proof. First, we show the solution set of variational inequality (3.2) is singleton. Assume $\tilde{x} \in \Omega$ also solves (3.2). Then, we have

$$\langle F(x^*) - g(x^*), g(\tilde{x}) - g(x^*) \rangle \leq 0, \qquad \langle F(\tilde{x}) - g(\tilde{x}), g(x^*) - g(\tilde{x}) \rangle \leq 0. \tag{3.3}$$

It follows that

$$\begin{aligned} &\langle F(\tilde{x}) - g(\tilde{x}) - F(x^*) + g(x^*), g(x^*) - g(\tilde{x}) \rangle \leq 0 \\ &\quad \Longrightarrow \|g(x^*) - g(\tilde{x})\|^2 \leq \langle F(x^*) - F(\tilde{x}), g(x^*) - g(\tilde{x}) \rangle \\ &\quad \Longrightarrow \|g(x^*) - g(\tilde{x})\|^2 \leq \langle F(x^*) - F(\tilde{x}), g(x^*) - g(\tilde{x}) \rangle \leq \|F(x^*) - F(\tilde{x})\| \|g(x^*) - g(\tilde{x})\| \\ &\quad \Longrightarrow \|g(x^*) - g(\tilde{x})\| \leq \|F(x^*) - F(\tilde{x})\|. \end{aligned} \tag{3.4}$$

Since g is γ-strongly monotone, we have

$$\gamma \|x - y\|^2 \leq \langle g(x) - g(y), x - y \rangle \leq \|g(x) - g(y)\| \|x - y\|, \quad \forall x, y \in C. \tag{3.5}$$

Hence,

$$\gamma \|x - y\| \leq \|g(x) - g(y)\|, \quad \forall x, y \in C. \tag{3.6}$$

In particular, $\gamma\|x^* - \tilde{x}\| \leq \|g(x^*) - g(\tilde{x})\|$. By (3.4), we deduce

$$\gamma\|x^* - \tilde{x}\| \leq \|g(x^*) - g(\tilde{x})\| \leq \|F(x^*) - F(\tilde{x})\| \leq L\|x^* - \tilde{x}\|, \tag{3.7}$$

which implies that $\tilde{x} = x^*$ because of $L < \gamma$ by the assumption. Therefore, the solution of variational inequality (3.2) is unique.

Pick up any $u \in \Omega$. It is obvious that $u \in \mathrm{GVI}(C, A, g)$ and $g(u) \in \mathrm{Fix}(S)$. Set $u_n = P_C[\alpha_n F(x_n) + (1 - \alpha_n)(g(x_n) - \lambda A x_n)]$, $n \geq 0$. From (2.6), we know $g(u) = P_C[g(u) - \mu A u]$ for any $\mu > 0$. Hence, we have

$$g(u) = P_C[g(u) - (1 - \alpha_n)\lambda A u] = P_C[\alpha_n g(u) + (1 - \alpha_n)(g(u) - \lambda A u)], \quad \forall n \geq 0. \tag{3.8}$$

From (3.6), (3.8), and Lemma 2.1, we get

$$
\begin{aligned}
\|u_n - g(u)\| &= \big\|P_C\left[\alpha_n F(x_n) + (1-\alpha_n)\big(g(x_n) - \lambda A x_n\big)\right] \\
&\qquad - P_C\left[\alpha_n g(u) + (1-\alpha_n)\big(g(u) - \lambda A u\big)\right]\big\| \\
&\le \alpha_n \big\|F(x_n) - g(u)\big\| + (1-\alpha_n)\big\|\big(g(x_n) - \lambda A x_n\big) - \big(g(u) - \lambda A u\big)\big\| \\
&\le \alpha_n \|F(x_n) - F(u)\| + \alpha_n \big\|F(u) - g(u)\big\| + (1-\alpha_n)\big\|g(x_n) - g(u)\big\| \\
&\le \alpha_n L \|x_n - u\| + \alpha_n \big\|F(u) - g(u)\big\| + (1-\alpha_n)\big\|g(x_n) - g(u)\big\| \\
&\le \frac{\alpha_n L}{\gamma}\big\|g(x_n) - g(u)\big\| + \alpha_n \big\|F(u) - g(u)\big\| + (1-\alpha_n)\big\|g(x_n) - g(u)\big\| \\
&= \left[1 - \left(1 - \frac{L}{\gamma}\right)\alpha_n\right]\big\|g(x_n) - g(u)\big\| + \alpha_n \big\|F(u) - g(u)\big\|.
\end{aligned}
\tag{3.9}
$$

It follows from (3.1) that

$$
\begin{aligned}
\big\|g(x_{n+1}) - g(u)\big\| &\le \beta\big\|g(x_n) - g(u)\big\| + (1-\beta)\big\|Su_n - Sg(u)\big\| \\
&\le \beta\big\|g(x_n) - g(u)\big\| + (1-\beta)\big\|u_n - g(u)\big\| \\
&\le \beta\big\|g(x_n) - g(u)\big\| + (1-\beta)\left[1 - \left(1 - \frac{L}{\gamma}\right)\alpha_n\right]\big\|g(x_n) - g(u)\big\| \\
&\quad + (1-\beta)\alpha_n\big\|F(u) - g(u)\big\| \\
&= \left[1 - \left(1 - \frac{L}{\gamma}\right)(1-\beta)\alpha_n\right]\big\|g(x_n) - g(u)\big\| \\
&\quad + \left(1 - \frac{L}{\gamma}\right)(1-\beta)\alpha_n \frac{\big\|F(u) - g(u)\big\|}{1 - L/\gamma}.
\end{aligned}
\tag{3.10}
$$

This indicates by induction that

$$
\big\|g(x_{n+1}) - g(u)\big\| \le \max\left\{\big\|g(x_n) - g(u)\big\|, \frac{\big\|F(u) - g(u)\big\|}{1 - L/\gamma}\right\}.
\tag{3.11}
$$

Hence, $\{g(x_n)\}$ is bounded. By (3.6), we have $\|x_n - u\| \le (1/\gamma)\|g(x_n) - g(u)\|$. This implies that $\{x_n\}$ is bounded. Consequently, $\{F(x_n)\}$, $\{Ax_n\}$, $\{u_n\}$, and $\{Su_n\}$ are all bounded.

Note that we can rewrite (3.1) as $g(x_{n+1}) = \beta g(x_n) + (1-\beta)Su_n$ for all n. Next, we will use Lemma 2.3 to prove that $\|x_{n+1} - x_n\| \to 0$. In fact, we firstly have

$$
\begin{aligned}
\|Su_n - Su_{n-1}\| &= \big\|SP_C\left[\alpha_n F(x_n) + (1-\alpha_n)\big(g(x_n) - \lambda A x_n\big)\right] \\
&\qquad - SP_C\left[\alpha_{n-1} F(x_{n-1}) + (1-\alpha_{n-1})\big(g(x_{n-1}) - \lambda A x_{n-1}\big)\right]\big\| \\
&\le \big\|\left[\alpha_n F(x_n) + (1-\alpha_n)\big(g(x_n) - \lambda A x_n\big)\right] \\
&\qquad - \left[\alpha_{n-1} F(x_{n-1}) + (1-\alpha_{n-1})\big(g(x_{n-1}) - \lambda A x_{n-1}\big)\right]\big\| \\
&\le \alpha_n \|F(x_n) - F(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|F(x_{n-1})\|
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\alpha_n)\left\|g(x_n)-\lambda A x_n-\left(g(x_{n-1})-\lambda A x_{n-1}\right)\right\| \\
&+ |\alpha_n-\alpha_{n-1}|\left\|g(x_{n-1})-\lambda A x_{n-1}\right\| \\
&\le \alpha_n L\|x_n-x_{n-1}\| + (1-\alpha_n)\left\|g(x_n)-g(x_{n-1})\right\| \\
&+ |\alpha_n-\alpha_{n-1}|\left(\|F(x_{n-1})\|+\left\|g(x_{n-1})-\lambda A x_{n-1}\right\|\right) \\
&\le \alpha_n\left(\frac{L}{\gamma}\right)\left\|g(x_n)-g(x_{n-1})\right\| + (1-\alpha_n)\left\|g(x_n)-g(x_{n-1})\right\| \\
&+ |\alpha_n-\alpha_{n-1}|\left(\|F(x_{n-1})\|+\left\|g(x_{n-1})-\lambda A x_{n-1}\right\|\right) \\
&= \left[1-\left(1-\frac{L}{\gamma}\right)\alpha_n\right]\left\|g(x_n)-g(x_{n-1})\right\| \\
&+ |\alpha_n-\alpha_{n-1}|\left(\|F(x_{n-1})\|+\left\|g(x_{n-1})-\lambda A x_{n-1}\right\|\right).
\end{aligned}
\tag{3.12}
$$

It follows that

$$
\|Su_n - Su_{n-1}\| - \left\|g(x_n)-g(x_{n-1})\right\| \le |\alpha_n-\alpha_{n-1}|\left(\|F(x_{n-1})\|+\left\|g(x_{n-1})-\lambda A x_{n-1}\right\|\right). \tag{3.13}
$$

Since $\alpha_n \to 0$ and the sequences $\{F(x_n)\}, \{g(x_n)\}$, and $\{Ax_n\}$ are bounded, we have

$$
\limsup_{n\to\infty}\left(\|Su_n - Su_{n-1}\| - \left\|g(x_n)-g(x_{n-1})\right\|\right) \le 0. \tag{3.14}
$$

By Lemma 2.3, we obtain

$$
\lim_{n\to\infty}\left\|Su_n - g(x_n)\right\| = 0. \tag{3.15}
$$

Hence,

$$
\lim_{n\to\infty}\left\|g(x_{n+1})-g(x_n)\right\| = \lim_{n\to\infty}(1-\beta)\left\|Su_n-g(x_n)\right\| = 0. \tag{3.16}
$$

This together with (3.6) imply that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.17}
$$

By the convexity of the norm and (3.9), we have

$$
\begin{aligned}
\left\|g(x_{n+1})-g(u)\right\|^2 &= \left\|\beta(g(x_n)-g(u)) + (1-\beta)(Su_n - Sg(u))\right\|^2 \\
&\le \beta\left\|g(x_n)-g(u)\right\|^2 + (1-\beta)\left\|Su_n - Sg(u)\right\|^2 \\
&\le \beta\left\|g(x_n)-g(u)\right\|^2 + (1-\beta)\left\|u_n - g(u)\right\|^2
\end{aligned}
$$

$$\begin{aligned}
&\le \beta\|g(x_n)-g(u)\|^2 \\
&\quad + (1-\beta)\left[\alpha_n\|F(x_n)-g(u)\| \right.\\
&\qquad\qquad \left. +(1-\alpha_n)\|(g(x_n)-\lambda Ax_n)-(g(u)-\lambda Au)\|\right]^2 \\
&\le \beta\|g(x_n)-g(u)\|^2 \\
&\quad + (1-\beta)\Big[\alpha_n\|F(x_n)-g(u)\|^2 \\
&\qquad\qquad +(1-\alpha_n)\|(g(x_n)-\lambda Ax_n)-(g(u)-\lambda Au)\|^2\Big].
\end{aligned} \tag{3.18}$$

From Lemma 2.1, we derive

$$\|(g(x_n)-\lambda Ax_n)-(g(u)-\lambda Au)\|^2 \le \|g(x_n)-g(u)\|^2 + \lambda(\lambda-2\alpha)\|Ax_n-Au\|^2. \tag{3.19}$$

Thus,

$$\begin{aligned}
\|g(x_{n+1})-g(u)\|^2 &\le \beta\|g(x_n)-g(u)\|^2 \\
&\quad + (1-\beta)\Big[\alpha_n\|F(x_n)-g(u)\|^2 \\
&\qquad\qquad +(1-\alpha_n)\Big(\|g(x_n)-g(u)\|^2 + \lambda(\lambda-2\alpha)\|Ax_n-Au\|^2\Big)\Big] \\
&= (1-\beta)\alpha_n\|F(x_n)-g(u)\|^2 + [1-(1-\beta)\alpha_n]\|g(x_n)-g(u)\|^2 \\
&\quad + (1-\beta)(1-\alpha_n)\lambda(\lambda-2\alpha)\|Ax_n-Au\|^2.
\end{aligned} \tag{3.20}$$

So,

$$\begin{aligned}
&(1-\beta)(1-\alpha_n)\lambda(2\alpha-\lambda)\|Ax_n-Au\|^2 \\
&\quad \le (1-\beta)\alpha_n\|F(x_n)-g(u)\|^2 + \|g(x_n)-g(u)\|^2 - \|g(x_{n+1})-g(u)\|^2 \\
&\quad \le (1-\beta)\alpha_n\|F(x_n)-g(u)\|^2 \\
&\qquad + (\|g(x_n)-g(u)\| + \|g(x_{n+1})-g(u)\|)\|g(x_{n+1})-g(x_n)\|.
\end{aligned} \tag{3.21}$$

Since $\alpha_n \to 0$, $\|g(x_{n+1})-g(x_n)\| \to 0$ and $\liminf_{n\to\infty}(1-\beta)(1-\alpha_n)\lambda(2\alpha-\lambda) > 0$, we obtain

$$\lim_{n\to\infty}\|Ax_n-Au\| = 0. \tag{3.22}$$

Set $y_n = g(x_n) - \lambda A x_n - (g(u) - \lambda A u)$ for all n. By using the property of projection, we get

$$\begin{aligned}
\|u_n - g(u)\|^2 &= \| P_C[\alpha_n F(x_n) + (1-\alpha_n)(g(x_n) - \lambda A x_n)] \\
&\quad - P_C[\alpha_n g(u) + (1-\alpha_n)(g(u) - \lambda A u)] \|^2 \\
&\le \langle \alpha_n (F(x_n) - g(u)) + (1-\alpha_n) y_n, u_n - g(u) \rangle \\
&= \frac{1}{2}\Big\{ \|\alpha_n (F(x_n) - g(u)) + (1-\alpha_n) y_n\|^2 + \|u_n - g(u)\|^2 \\
&\quad - \|\alpha_n (F(x_n) - g(u)) + (1-\alpha_n) y_n - u_n + g(u)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \alpha_n \|F(x_n) - g(u)\|^2 + (1-\alpha_n)\|g(x_n) - g(u)\|^2 + \|u_n - g(u)\|^2 \\
&\quad - \|\alpha_n (F(x_n) - g(u) - y_n) + g(x_n) - u_n - \lambda (A x_n - A u)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \alpha_n \|F(x_n) - g(u)\|^2 + (1-\alpha_n)\|g(x_n) - g(u)\|^2 + \|u_n - g(u)\|^2 \\
&\quad - \|g(x_n) - u_n\|^2 - \lambda^2 \|A x_n - A u\| - \alpha_n^2 \|F(x_n) - g(u) - y_n\|^2 \\
&\quad + 2\lambda\alpha_n \langle A x_n - A u, F(x_n) - g(u) - y_n \rangle + 2\lambda \langle g(x_n) - u_n, A x_n - A u \rangle \\
&\quad - 2\alpha_n \langle g(x_n) - u_n, F(x_n) - g(u) - y_n \rangle \Big\}.
\end{aligned} \tag{3.23}$$

It follows that

$$\begin{aligned}
\|u_n - g(u)\|^2 &\le \alpha_n \|F(x_n) - g(u)\|^2 + (1-\alpha_n)\|g(x_n) - g(u)\|^2 - \|g(x_n) - u_n\|^2 \\
&\quad + 2\lambda\alpha_n \|A x_n - A u\| \|F(x_n) - g(u) - y_n\| + 2\lambda \|g(x_n) - u_n\| \|A x_n - A u\| \\
&\quad + 2\alpha_n \|g(x_n) - u_n\| \|F(x_n) - g(u) - y_n\|.
\end{aligned} \tag{3.24}$$

From (3.18) and (3.24), we have

$$\begin{aligned}
\|g(x_{n+1}) - g(u)\|^2 &\le \beta \|g(x_n) - g(u)\|^2 + (1-\beta)\|u_n - g(u)\|^2 \\
&\le \beta \|g(x_n) - g(u)\|^2 + (1-\beta)\alpha_n \|F(x_n) - g(u)\|^2 \\
&\quad + (1-\alpha_n)(1-\beta)\|g(x_n) - g(u)\|^2 - (1-\beta)\|g(x_n) - u_n\|^2 \\
&\quad + 2\lambda(1-\beta)\alpha_n \|A x_n - A u\| \|F(x_n) - g(u) - y_n\| \\
&\quad + 2\lambda(1-\beta)\|g(x_n) - u_n\| \|A x_n - A u\|
\end{aligned}$$

$$
\begin{aligned}
&+2(1-\beta)\alpha_n\|g(x_n)-u_n\|\|F(x_n)-g(u)-y_n\| \\
&\le \|g(x_n)-g(u)\|^2+\alpha_n\|F(x_n)-g(u)\|^2-(1-\beta)\|g(x_n)-u_n\|^2 \\
&\quad +2\lambda\alpha_n\|Ax_n-Au\|\|F(x_n)-g(u)-y_n\|+2\lambda\|g(x_n)-u_n\|\|Ax_n-Au\| \\
&\quad +2\alpha_n\|g(x_n)-u_n\|\|F(x_n)-g(u)-y_n\|.
\end{aligned}
\tag{3.25}
$$

Then, we obtain

$$
\begin{aligned}
(1-\beta)\|g(x_n)-u_n\|^2 \le{}& (\|g(x_n)-g(u)\|+\|g(x_{n+1})-g(u)\|)\|g(x_{n+1})-g(x_n)\| \\
&+\alpha_n\|F(x_n)-g(u)\|^2+2\lambda\alpha_n\|Ax_n-Au\|\|F(x_n)-g(u)-y_n\| \\
&+2\lambda\|g(x_n)-u_n\|\|Ax_n-Au\|+2\alpha_n\|g(x_n)-u_n\|\|F(x_n)-g(u)-y_n\|.
\end{aligned}
\tag{3.26}
$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|g(x_{n+1})-g(x_n)\|=0$ and $\lim_{n\to\infty}\|Ax_n-Au\|=0$, we have

$$
\lim_{n\to\infty}\|g(x_n)\ -u_n\|=0. \tag{3.27}
$$

Next, we prove $\limsup_{n\to\infty}\langle F(x^*)-g(x^*),u_n-g(x^*)\rangle\le 0$ where x^* is the unique solution of (3.2). We take a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle F(x^*)-g(x^*),u_n-g(x^*)\rangle &= \lim_{i\to\infty}\langle F(x^*)-g(x^*),u_{n_i}-g(x^*)\rangle \\
&= \lim_{i\to\infty}\langle F(x^*)-g(x^*),g(x_{n_i})-g(x^*)\rangle.
\end{aligned}
\tag{3.28}
$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z\in C$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup z$. This implies that $g(x_{n_i})\rightharpoonup g(z)$ due to the weak continuity of g. Now, we show $z\in\Omega$. First, we note that from (3.15) and (3.27) that $\|g(x_n)-Sg(x_n)\|\to 0$. Hence, $\lim_{i\to\infty}\|g(x_{n_i})-Sg(x_{n_i})\|=0$. By the demiclosedness principle of the nonexpansive mapping (see Lemma 2.2), we deduce $g(z)\in\mathrm{Fix}(S)$. Next, we only need to prove $z\in\mathrm{GVI}(C,A,g)$. Set

$$
Tv=\begin{cases} Av+N_C(v), & v\in C,\\ \emptyset, & v\notin C.\end{cases} \tag{3.29}
$$

By [39], we know that T is maximal g-monotone. Let $(v,w)\in G(T)$. Since $w-Av\in N_C(v)$ and $x_n\in C$, we have

$$
\langle g(v)-g(x_n),w-Av\rangle\ge 0. \tag{3.30}
$$

From $u_n = P_C[\alpha_n F(x_n) + (1-\alpha_n)(g(x_n) - \lambda A x_n)]$, we get

$$\left\langle g(v) - u_n, u_n - \left[\alpha_n F(x_n) + (1-\alpha_n)\left(g(x_n) - \lambda A x_n\right)\right]\right\rangle \geq 0. \tag{3.31}$$

It follows that

$$\left\langle g(v) - u_n, \frac{u_n - g(x_n)}{\lambda} + A x_n - \frac{\alpha_n}{\lambda}\left(F(x_n) - g(x_n) + \lambda A x_n\right)\right\rangle \geq 0. \tag{3.32}$$

Then,

$$\begin{aligned}
\langle g(v) - g(x_{n_i}), w\rangle &\geq \langle g(v) - g(x_{n_i}), Av\rangle \\
&\geq \langle g(v) - g(x_{n_i}), Av\rangle - \left\langle g(v) - u_{n_i}, \frac{u_{n_i} - g(x_{n_i})}{\lambda}\right\rangle \\
&\quad - \langle g(v) - u_{n_i}, A x_{n_i}\rangle + \frac{\alpha_{n_i}}{\lambda}\langle g(v) - u_{n_i}, F(x_{n_i}) - g(x_{n_i}) + \lambda A x_{n_i}\rangle \\
&= \langle g(v) - g(x_{n_i}), Av - A x_{n_i}\rangle + \langle g(v) - g(x_{n_i}), -A x_{n_i}\rangle \\
&\quad - \left\langle g(v) - u_{n_i}, \frac{u_{n_i} - g(x_{n_i})}{\lambda}\right\rangle - \langle g(v) - u_{n_i}, A x_{n_i}\rangle \\
&\quad + \frac{\alpha_{n_i}}{\lambda}\langle g(v) - u_{n_i}, F(x_{n_i}) - g(x_{n_i}) + \lambda A x_{n_i}\rangle \\
&\geq -\left\langle g(v) - u_{n_i}, \frac{u_{n_i} - g(x_{n_i})}{\lambda}\right\rangle - \langle g(x_{n_i}) - u_{n_i}, A x_{n_i}\rangle \\
&\quad + \frac{\alpha_{n_i}}{\lambda}\langle g(v) - u_{n_i}, F(x_{n_i}) - g(x_{n_i}) + \lambda A x_{n_i}\rangle.
\end{aligned} \tag{3.33}$$

Since $\|g(x_{n_i}) - u_{n_i}\| \to 0$ and $g(x_{n_i}) \rightharpoonup g(z)$, we deduce that $\langle g(v) - g(z), w\rangle \geq 0$ by taking $i \to \infty$ in (3.33). Thus, $z \in T^{-1}0$ by the maximal g-monotonicity of T. Hence, $z \in \mathrm{GVI}(C, A, g)$. Therefore, $z \in \Omega$. From (3.28), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle &= \lim_{i\to\infty}\langle F(x^*) - g(x^*), g(x_{n_i}) - g(x^*)\rangle \\
&= \langle F(x^*) - g(x^*), g(z) - g(x^*)\rangle \leq 0.
\end{aligned} \tag{3.34}$$

We take $u = x^*$ in (3.23) to get

$$\begin{aligned}
\|u_n - g(x^*)\|^2 &\le \alpha_n\langle F(x_n) - g(x^*), u_n - g(x^*)\rangle \\
&\quad + (1-\alpha_n)\langle g(x_n) - \lambda A x_n - (g(x^*) - \lambda A x^*), u_n - g(x^*)\rangle \\
&\le \alpha_n\langle F(x_n) - F(x^*), u_n - g(x^*)\rangle + \alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&\quad + (1-\alpha_n)\|g(x_n) - \lambda A x_n - (g(x^*) - \lambda A x^*)\|\|u_n - g(x^*)\| \\
&\le \alpha_n L\|x_n - x^*\|\|u_n - g(x^*)\| + \alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&\quad + (1-\alpha_n)\|g(x_n) - g(x^*)\|\|u_n - g(x^*)\| \\
&\le \alpha_n\left(\frac{L}{\gamma}\right)\|g(x_n) - g(x^*)\|\|u_n - g(x^*)\| + \alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&\quad + (1-\alpha_n)\|g(x_n) - g(x^*)\|\|u_n - g(x^*)\| \\
&= \left[1 - \left(1 - \frac{L}{\gamma}\right)\alpha_n\right]\|g(x_n) - g(x^*)\|\|u_n - g(x^*)\| \\
&\quad + \alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&= \frac{1 - (1 - L/\gamma)\alpha_n}{2}\|g(x_n) - g(x^*)\|^2 + \frac{1}{2}\|u_n - g(x^*)\|^2 \\
&\quad + \alpha_n\langle F(x^*) - g(x^*), u_n - g(u)\rangle.
\end{aligned} \tag{3.35}$$

It follows that

$$\|u_n - g(x^*)\|^2 \le \left[1 - \left(1 - \frac{L}{\gamma}\right)\alpha_n\right]\|g(x_n) - g(x^*)\|^2 + 2\alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle. \tag{3.36}$$

Therefore,

$$\begin{aligned}
\|g(x_{n+1}) - g(x^*)\|^2 &\le \beta\|g(x_n) - g(x^*)\|^2 + (1-\beta)\|u_n - g(x^*)\|^2 \\
&\le \beta\|g(x_n) - g(x^*)\|^2 + (1-\beta)\left[1 - \left(1 - \frac{L}{\gamma}\right)\alpha_n\right]\|g(x_n) - g(x^*)\|^2 \\
&\quad + 2(1-\beta)\alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&= \left[1 - \left(1 - \frac{L}{\gamma}\right)(1-\beta)\alpha_n\right]\|g(x_n) - g(x^*)\|^2 \\
&\quad + 2(1-\beta)\alpha_n\langle F(x^*) - g(x^*), u_n - g(x^*)\rangle \\
&= \left[1 - \left(1 - \frac{L}{\gamma}\right)(1-\beta)\alpha_n\right]\|g(x_n) - g(x^*)\|^2
\end{aligned}$$

$$+\left(1-\frac{L}{\gamma}\right)(1-\beta)\alpha_n\left(\frac{2}{1-L/\gamma}\langle F(x^*)-g(x^*),u_n-g(x^*)\rangle\right)$$
$$=(1-\gamma_n)\left\|g(x_n)-g(x^*)\right\|^2+\delta_n\gamma_n, \tag{3.37}$$

where $\gamma_n=(1-L/\gamma)(1-\beta)\alpha_n$ and $\delta_n=(2/(1-L/\gamma))\langle F(x^*)-g(x^*),u_n-g(x^*)\rangle$. From condition $(C2)$, we have $\sum_n\gamma_n=\infty$. By (3.34), we have $\limsup_{n\to\infty}\delta_n\le 0$. We can therefore apply Lemma 2.4 to conclude that $g(x_n)\to g(x^*)$ and $x_n\to x^*$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F: C\to H$ be an L-contraction. Let $A: C\to H$ be an α-inverse strongly monotone mapping and let $S: C\to C$ be a nonexpansive mapping. Suppose that $\Omega\neq\emptyset$. Let $\beta\in(0,1)$ and $\gamma\in(L,2\alpha)$. For given $x_0\in C$, let $\{x_n\}\subset C$ be a sequence generated by*

$$x_{n+1}=\beta x_n+(1-\beta)SP_C[\alpha_nF(x_n)+(1-\alpha_n)(x_n-\lambda Ax_n)],\quad n\ge 0, \tag{3.38}$$

where $\{\alpha_n\}\subset(0,1)$ satisfies $(C1)$: $\lim_{n\to\infty}\alpha_n=0$ and $(C2)$: $\sum_n\alpha_n=\infty$. Then the sequence $\{x_n\}$ generated by (3.38) converges strongly to $x^*\in\Omega$ which is the unique solution of the following variational inequality:

$$\langle F(x^*)-x^*,x-x^*\rangle\le 0,\quad \forall x\in\Omega. \tag{3.39}$$

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