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Research Article

Fine Spectra of Symmetric Toeplitz Operators

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The fine spectra of 2-banded and 3-banded infinite Toeplitz matrices were examined by several authors. The fine spectra of n-banded triangular Toeplitz matrices and tridiagonal symmetric matrices were computed in the following papers: Altun, "On the fine spectra of triangular toeplitz operators" (2011) and Altun, "Fine spectra of tridiagonal symmetric matrices" (2011). Here, we generalize those results to the $(2n+1)$-banded symmetric Toeplitz matrix operators for arbitrary positive integer n.

1. Introduction and Preliminaries

The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum, and the residual spectrum. Some other parts also arise by examining the surjectivity of the operator and continuity of the inverse operator. Such subparts of the spectrum are called the fine spectra of the operator.

The spectra and fine spectra of linear operators defined by some particular limitation matrices over some sequence spaces were studied by several authors. We introduce the knowledge in the existing literature concerning the spectrum and the fine spectrum. Wenger [1] examined the fine spectrum of the integer power of the Cesàro operator over c, and Rhoades [2] generalized this result to the weighted mean methods. Reade [3] worked on the spectrum of the Cesàro operator over the sequence space c_0. Gonzáles [4] studied the fine spectrum of the Cesàro operator over the sequence space ℓ_p. Okutoyi [5] computed the spectrum of the Cesàro operator over the sequence space bv. Recently, Rhoades and Yildirim [6] examined the fine spectrum of factorable matrices over c_0 and c. Akhmedov and Başar [7, 8] have determined the fine spectrum of the Cesàro operator over the sequence spaces c_0, ℓ_∞ and ℓ_p. Altun and Karakaya [9] computed the fine spectra of Lacunary matrices over c_0 and c. Furkan et al. [10] determined the fine spectrum of $B(r,s,t)$ over the sequence spaces

c_0 and c, where $B(r, s, t)$ is a lower triangular triple-band matrix. Later, Altun [11] computed the fine spectra of triangular Toeplitz matrices over c_0 and c.

The fine spectrum of the difference operator Δ over c_0 and c was studied by Altay and Başar [12]. Recently, the fine spectra of Δ over ℓ_p and bv_p are studied by Akhmedov and Başar [13, 14], where bv_p is the space of p-bounded variation sequences, introduced by Başar and Altay [15] with $1 \leq p < \infty$. The fine spectrum with respect to the Goldberg's classification of the operator $B(r, s, t)$ over ℓ_p and bv_p with $1 < p < \infty$ has recently been studied by Furkan et al. [16]. Quite recently, Akhmedov and El-Shabrawy [17] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over c. In 2010, Srivastava and Kumar [18] have determined the spectra and the fine spectra of the generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$ and they have just generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ (see [19]).

In this work, our purpose is to determine the spectra of the operator, for which the corresponding matrix is a $(2n + 1)$-banded symmetric Toeplitz matrix, over the sequence spaces c_0, c, ℓ_1 and ℓ_∞. We will also give the fine spectra results for the spaces c_0 and c.

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T, that is,

$$\mathcal{R}(T) = \{y \in Y : y = Tx; x \in X\}. \tag{1.1}$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathfrak{D}(T) \to X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. With T, we associate the operator

$$T_\lambda = T - \lambda I, \tag{1.2}$$

where λ is a complex number and I is the identity operator on $\mathfrak{D}(T)$. If T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is,

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{1.3}$$

and call it the *resolvent operator* of T_λ. If $\lambda = 0$, we will simply write T^{-1}. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we will be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We will also ask for what λ's the domain of T_λ^{-1} is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [20, pages 370-371]).

Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathfrak{D}(T) \to X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* $\rho(T)$ of T is the set of all regular values λ of T. Its complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows: the *point spectrum* $\sigma_p(T)$ is the set such that T_λ^{-1} does not exist. A $\lambda \in \sigma_p(T)$ is called an *eigenvalue* of T. The *continuous spectrum* $\sigma_c(T)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2). The *residual spectrum* $\sigma_r(T)$ is the set such that T_λ^{-1} exists but does not satisfy (R3).

From Goldberg [21], if $T \in B(X)$, X a Banach space, then there are three possibilities for $\mathcal{R}(T)$, the range of T:

(I) $\mathcal{R}(T) = X$,

(II) $\overline{\mathcal{R}(T)} = X$, but $\mathcal{R}(T) \neq X$,

(III) $\overline{\mathcal{R}(T)} \neq X$,

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists, but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled as I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, and III_3. If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in \mathrm{II}_1$, then λ is in the resolvent set $\rho(T, X)$ of T, the set of all regular values of T on X. The other classification gives rise to the fine spectrum of T. For example, we will write $\lambda \in \mathrm{III}_1\sigma(T, X)$ if T satisfies III and 1.

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. We will write ℓ_∞, c and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. By ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$. Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in γ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.4}$$

By $(\mu : \gamma)$, we denote the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right side of (1.4) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$.

Let an $(n+1)$-tuple $t = (t_0, t_1, \ldots, t_n) \in \mathbb{C}^{n+1}$ be given. A symmetric infinite Toeplitz matrix is a $(2n+1)$-band matrix of the form

$$S = S(t) = \begin{bmatrix} t_0 & t_1 & \cdots & \cdots & t_n & 0 & 0 & \cdots \\ t_1 & t_0 & t_1 & \cdots & \cdots & t_n & 0 & \cdots \\ \vdots & t_1 & t_0 & t_1 & \cdots & \cdots & t_n & \cdots \\ \vdots & \vdots & t_1 & t_0 & t_1 & \cdots & \cdots & \cdots \\ t_n & \vdots & \vdots & t_1 & t_0 & t_1 & \cdots & \cdots \\ 0 & t_n & \vdots & \vdots & t_1 & t_0 & t_1 & \cdots \\ 0 & 0 & t_n & \vdots & \vdots & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.5}$$

The spectral results are clear when S is a multiple of the identity matrix, so for the sequel we will have $n \geq 1$ and $t_n \neq 0$.

Let R be the right shift operator:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.6}$$

and L be the left shift operator:

$$L = R^t = R^{-1}. \tag{1.7}$$

Let $F(z) = t_n[z^n + z^{-n}] + t_{n-1}[z^{n-1} + z^{-(n-1)}] + \cdots + t_1[z + z^{-1}] + t_0 = P(z)/z^n$, where P is the palindromic polynomial $P(z) = t_n z^{2n} + t_{n-1} z^{2n-1} + \cdots + t_0 z^n + t_1 z^{n-1} + t_2 z^{n-2} + \cdots + t_n$. Then, we can see that $S = F(R)$ and we will call F and P as the function and polynomial associated to the operator S, respectively. We also have

$$S = L^n P(R). \tag{1.8}$$

The roots of $P(z)$ are nonzero and symmetric, that is, if α is a root, α^{-1} is also a root. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1}$ be the roots of $P(z)$ such that $|\alpha_k| \leq 1$ for $k = 1, 2, \ldots, n$. Then

$$S = t_n L^n (R - \alpha_1 I)(R - \alpha_2 I) \cdots (R - \alpha_n I)\left(R - \alpha_1^{-1} I\right)\left(R - \alpha_2^{-1} I\right) \cdots \left(R - \alpha_n^{-1} I\right). \tag{1.9}$$

Now, by induction, we can see that

$$L^n (R - \alpha_1 I)(R - \alpha_2 I) \cdots (R - \alpha_n I) = (I - \alpha_1 L)(I - \alpha_2 L) \cdots (I - \alpha_n L). \tag{1.10}$$

Let D be the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ and ∂D be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. We have the following two lemmas as a consequence of the corresponding results in [22] and [23], respectively.

Lemma 1.1. *$(I - \alpha L) \in (c_0, c_0)$ is onto if and only if α is not on the unit circle.*

Lemma 1.2. *$(R - \alpha I) \in (c_0, c_0)$ is onto if and only if α is outside the unit disc.*

Theorem 1.3. *$S \in (c_0, c_0)$ is onto if and only if P has no root on the unit circle.*

Proof. Suppose P has a root on the unit circle. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1}$ be the roots of $P(z)$ such that $|\alpha_k| \leq 1$ for $k = 1, 2, \ldots, n$. We have

$$S = t_n(I - \alpha_1 L)(I - \alpha_2 L) \cdots (I - \alpha_n L)\left(R - \alpha_1^{-1} I\right)\left(R - \alpha_2^{-1} I\right) \cdots \left(R - \alpha_n^{-1} I\right). \tag{1.11}$$

Since the matrix operators $(I-\alpha_1 L), (I-\alpha_2 L), \ldots, (I-\alpha_n L)$ commute with each other, without loss of generality, we can suppose α_1 is a root on the unit circle. Clearly, all the operators $(I - \alpha_1 L), (I - \alpha_2 L), \ldots, (I - \alpha_n L), (R - \alpha_1^{-1} I), (R - \alpha_2^{-1} I), \ldots, (R - \alpha_n^{-1} I)$ are in (c_0, c_0). But, by Lemma 1.1 the operator $(I - \alpha_1 L)$ is not onto. So, S cannot be onto.

Suppose, now, P has no root on the unit circle. That means $|\alpha_k| < 1$ for $k = 1, 2, \ldots, n$. Then all the operators $(I-\alpha_1 L), (I-\alpha_2 L), \ldots, (I-\alpha_n L), (R-\alpha_1^{-1} I), (R-\alpha_2^{-1} I), \ldots, (R-\alpha_n^{-1} I)$ are onto by Lemma 1.1 and Lemma 1.2. Hence, $S = t_n(I-\alpha_1 L)(I-\alpha_2 L) \cdots (I-\alpha_n L)(R-\alpha_1^{-1} I)(R-\alpha_2^{-1} I) \cdots (R-\alpha_n^{-1} I)$ is onto. $\square$

Theorem 1.4 (cf. [24]). *Let T be an operator with the associated matrix $A = (a_{nk})$.*

(i) *$T \in B(c)$ if and only if*

$$\|A\| := \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty, \tag{1.12}$$

$$a_k := \lim_{n \to \infty} a_{nk} \quad \text{exists for each } k, \tag{1.13}$$

$$a := \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} \quad \text{exists}. \tag{1.14}$$

(ii) *$T \in B(c_0)$ if and only if* (1.12) *and* (1.13) *with $a_k = 0$ for each k.*

(iii) *$T \in B(\ell_\infty)$ if and only if* (1.12). *In these cases, the operator norm of T is*

$$\|T\|_{(\ell_\infty:\ell_\infty)} = \|T\|_{(c:c)} = \|T\|_{(c_0:c_0)} = \|A\|. \tag{1.15}$$

(iv) *$T \in B(\ell_1)$ if and only if*

$$\left\|A^t\right\| = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty. \tag{1.16}$$

In this case, the operator norm of T is $\|T\|_{(\ell_1:\ell_1)} = \|A^t\|$.

Corollary 1.5. *$S(t) \in B(\mu)$ for $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$ and*

$$\|S(t)\|_{(\mu,\mu)} = |t_0| + 2(|t_1| + |t_2| + \cdots + |t_n|). \tag{1.17}$$

Theorem 1.6. *Let X be a Banach space and $T \in B(X)$. Then $\lambda \in \mathbb{C}$ is in the spectrum $\sigma(T, X)$ if and only if $T - \lambda I$ is not bijective.*

Proof. Suppose $T - \lambda I$ is not bijective. Then $T - \lambda I$ is not 1-1 or not onto. If it is not 1-1, then $\lambda \in \sigma_p(T, X) \subset \sigma(T, X)$. Suppose now $T - \lambda I$ is 1-1. Then it is not onto and by Lemma 7.2-3 of [20], λ cannot be in $\rho(T, X)$. Hence, $\lambda \in \sigma(T, X)$.

Now, suppose $T - \lambda I$ is bijective. Then by the open mapping theorem $(T - \lambda I)^{-1}$ is continuous. Hence, λ is not in the spectrum $\sigma(T, X)$. □

Corollary 1.7. *Let X be a Banach space and $T \in B(X)$. Then $\lambda \in \rho(T, X)$ if and only if T_λ is bijective.*

2. The Spectra and Fine Spectra

Lemma 2.1 (Lemma 3.4 of [11]). *Let $z_1, z_2, \ldots, z_r$ be distinct complex numbers with $|z_i| = 1$ for $1 \le i \le r$. Let $0 \ne x = (x_k)$ be a sequence satisfying*

$$\begin{aligned} x_k = {} & \left(\alpha_{1,0} + \alpha_{1,1}k + \cdots + \alpha_{1,m_1-1}k^{m_1-1}\right) z_1^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right) z_2^k \\ & + \cdots + \left(\alpha_{r,0} + \alpha_{r,1}k + \cdots + \alpha_{r,m_r-1}k^{m_r-1}\right) z_r^k, \end{aligned} \tag{2.1}$$

for $k = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants forming the polynomials $P_i(k) = \alpha_{i,0} + \alpha_{i,1}k + \cdots + \alpha_{i,m_i-1}k^{m_i-1} \ne 0$ for $1 \le i \le r$ and $0 \le j \le m_i - 1$. Then $x \notin c_0$.

Lemma 2.2. *Let $z_1, z_2, \ldots, z_r$ be distinct complex numbers. Let $0 \ne x = (x_k) \in c$ be a sequence satisfying*

$$\begin{aligned} x_k = {} & \left(\alpha_{1,0} + \alpha_{1,1}k + \cdots + \alpha_{1,m_1-1}k^{m_1-1}\right) z_1^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right) z_2^k \\ & + \cdots + \left(\alpha_{r,0} + \alpha_{r,1}k + \cdots + \alpha_{r,m_r-1}k^{m_r-1}\right) z_r^k, \end{aligned} \tag{2.2}$$

for $k = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants forming the polynomials $P_i(k) = \alpha_{i,0} + \alpha_{i,1}k + \cdots + \alpha_{i,m_i-1}k^{m_i-1} \ne 0$ for $1 \le i \le r$ and $0 \le j \le m_i - 1$. Then $|z_i| \le 1$ for $1 \le i \le r$, and the existence of a $t \le r$ with $|z_t| = 1$ implies $z_t = 1$ and P_t is a constant.

Proof. Let $|z_1| \ge |z_2| \ge \cdots \ge |z_r|$. To prove $|z_i| \le 1$ for all $i \le r$, suppose it is not true. Then let $a := |z_1| > 1$. Let $s \le r$ be the largest positive integer with $|z_1| = |z_2| = \cdots = |z_s|$. Then $(x_k / a^k) \in c_0$. Let

$$\begin{aligned} u_k = {} & \left(\alpha_{1,0} + \alpha_{1,1}k + \cdots + \alpha_{1,m_1-1}k^{m_1-1}\right)\left(\frac{z_1}{a}\right)^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right)\left(\frac{z_2}{a}\right)^k \\ & + \cdots + \left(\alpha_{s,0} + \alpha_{s,1}k + \cdots + \alpha_{s,m_s-1}k^{m_s-1}\right)\left(\frac{z_s}{a}\right)^k. \end{aligned} \tag{2.3}$$

We have $0 = (x_k/a^k - u_k)$ for $s = r$. If $s < r$, we have

$$\begin{aligned}\frac{x_k}{a^k} - u_k = {} & \left(\alpha_{s+1,0} + \alpha_{s+1,1}k + \cdots + \alpha_{s+1,m_{s+1}-1}k^{m_{s+1}-1}\right)\left(\frac{z_{s+1}}{a}\right)^k \\ & + \left(\alpha_{s+2,0} + \alpha_{s+2,1}k + \cdots + \alpha_{s+2,m_{s+2}-1}k^{m_{s+2}-1}\right)\left(\frac{z_{s+2}}{a}\right)^k \\ & + \cdots + \left(\alpha_{r,0} + \alpha_{r,1}k + \cdots + \alpha_{r,m_r-1}k^{m_r-1}\right)\left(\frac{z_r}{a}\right)^k.\end{aligned} \tag{2.4}$$

Since $|z_j/a| < 1$ for $s+1 \le j \le r$, we have $(x_k/a^k - u_k) \in c_0$. Then $(u_k) \in c_0$ but this contradicts with Lemma 2.1. Hence, we have $|z_i| \le 1$ for $1 \le i \le r$.

Now, let us prove the second part. Suppose, there exist a positive integer $q \le r$ such that $|z_i| = 1$ for all $i \le q$. For any i, P_i is constant means $m_i = 1$. Suppose $m = \max\{m_1, \ldots, m_q\} > 1$. Without loss of generality let $m_1 \ge m_2 \ge \cdots \ge m_q$. Let $q_0 \le q$ be the largest integer satisfying $m_1 = m_2 = \cdots = m_{q_0} = m$. Then $(x_k/k^{m-1}) \in c_0$. This means, since $|z_i|$ have modulus less than 1, $(v_k) \in c_0$, where

$$v_k = \alpha_{1,m-1}z_1^k + \alpha_{2,m-1}z_2^k + \cdots + \alpha_{q_0,m-1}z_{q_0}^k. \tag{2.5}$$

But, this again contradicts with Lemma 2.1. Hence, we have $m_1 = m_2 = \cdots = m_q = 1$. Now, we have $(w_k) \in c$, where

$$w_k = \alpha_{1,0}z_1^k + \alpha_{2,0}z_2^k + \cdots + \alpha_{q,0}z_q^k. \tag{2.6}$$

Suppose, one of the elements in $\{z_1, z_2, \ldots, z_q\}$ is equal to 1, say $z_1 = 1$. Then, $(w_{k+1} - w_k) \in c_0$, where

$$w_{k+1} - w_k = \alpha_{2,0}(z_2 - 1)z_2^k + \cdots + \alpha_{q,0}(z_q - 1)z_q^k, \tag{2.7}$$

and this again contradicts with Lemma 2.1. Hence, $q \le 1$ and 1 is the unique candidate for z_t with modulus 1. □

Theorem 2.3. *$\sigma_p(S, \mu) = \emptyset$ for $\mu \in \{\ell_1, c_0, c\}$.*

Proof. Since $\ell_1 \subset c_0 \subset c$, it is enough to show that $\sigma_p(S, c) = \emptyset$. Let λ be an eigenvalue of the operator S. An eigenvector $x = (x_0, x_1, \ldots) \in c$ corresponding to this eigenvalue satisfies the linear system of equations:

$$\begin{gathered} t_0 x_0 + t_1 x_1 + t_2 x_2 + \cdots + t_n x_n = \lambda x_0 \\ t_1 x_0 + t_0 x_1 + t_1 x_2 + \cdots + t_n x_{n+1} = \lambda x_1 \\ t_2 x_0 + t_1 x_1 + t_0 x_2 + \cdots + t_n x_{n+2} = \lambda x_2 \\ \vdots \\ t_n x_0 + t_{n-1} x_1 + t_{n-2} x_2 + \cdots + t_n x_{2n} = \lambda x_n \\ t_n x_1 + t_{n-1} x_2 + t_{n-2} x_3 + \cdots + t_n x_{2n+1} = \lambda x_{n+1} \\ t_n x_2 + t_{n-1} x_3 + t_{n-2} x_4 + \cdots + t_n x_{2n+2} = \lambda x_{n+2} \\ \vdots \end{gathered} \tag{2.8}$$

Since $t_n \neq 0$ we can write this system of equations in the form:

$$\begin{gathered} d_0 x_0 + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n = 0 \\ d_1 x_0 + d_0 x_1 + d_1 x_2 + \cdots + d_n x_{n+1} = 0 \\ d_2 x_0 + d_1 x_1 + d_0 x_2 + \cdots + d_n x_{n+2} = 0 \\ \vdots \\ d_n x_0 + d_{n-1} x_1 + d_{n-2} x_2 + \cdots + d_n x_{2n} = 0 \\ d_n x_1 + d_{n-1} x_2 + d_{n-2} x_3 + \cdots + d_n x_{2n+1} = 0 \\ d_n x_2 + d_{n-1} x_3 + d_{n-2} x_4 + \cdots + d_n x_{2n+2} = 0 \\ \vdots \end{gathered} \tag{2.9}$$

where $d_0 = (t_0 - \lambda)/t_n$, $d_n = 1$ and $d_j = t_j/t_n$ for $1 \leq j \leq n-1$. This system of equations, by change of variables $u_{n+k} = x_k$ for $k = 0, 1, 2, \ldots$, is equivalent to system of equations

$$d_n u_k + d_{n-1} u_{k+1} + \cdots + d_0 u_{n+k} + d_1 u_{n+k+1} + \cdots + d_n u_{2n+k} = 0, \quad k = 0, 1, 2, \ldots, \tag{2.10}$$

with the initial conditions $u_0 = u_1 = \cdots = u_{n-1} = 0$.

We see that this is a $2n$-th order linear homogenous difference equation with the corresponding characteristic polynomial

$$P(z) = z^{2n} + d_{n-1} z^{2n-1} + \cdots + d_1 z^{n+1} + d_0 z^n + d_1 z^{n-1} + \cdots + d_{n-1} z + 1. \tag{2.11}$$

Suppose $P(z)$ has r distinct roots $z_1, z_2, \ldots, z_r$ with multiplicities $m_1, m_2, \ldots, m_r$. Then, any solution (u_k) of the system of equations satisfies

$$\begin{aligned} u_k = {} & \left(\alpha_{1,0} + \alpha_{1,1}k + \cdots + \alpha_{1,m_1-1}k^{m_1-1}\right)z_1^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right)z_2^k \\ & + \cdots + \left(\alpha_{r,0} + \alpha_{r,1}k + \cdots + \alpha_{r,m_r-1}k^{m_r-1}\right)z_r^k. \end{aligned} \tag{2.12}$$

Observe that if ζ is a root of $P(z)$, then $1/\zeta$ is also a root. There are two cases.

Case 1 (1 is not a root of $P(z)$). Since $(u_k) \in c$, by Lemma 2.2 we can write

$$\begin{aligned} u_k = {} & \left(\alpha_{1,0} + \alpha_{1,1}k + \cdots + \alpha_{1,m_1-1}k^{m_1-1}\right)z_1^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right)z_2^k \\ & + \cdots + \left(\alpha_{q,0} + \alpha_{q,1}k + \cdots + \alpha_{q,m_q-1}k^{m_q-1}\right)z_q^k, \end{aligned} \tag{2.13}$$

where $|z_i| < 1$ for $1 \le i \le q$. By the symmetry of the roots we have $m_1 + m_2 + m_3 + \cdots + m_q = n$. Now, using the initial conditions $u_0 = u_1 = \cdots = u_{n-1} = 0$ we have

$$V\alpha = 0, \tag{2.14}$$

where $\alpha = [\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,m_1-1}, \alpha_{2,0}, \alpha_{2,1}, \ldots, \alpha_{2,m_2-1}, \ldots, \alpha_{q,0}, \alpha_{q,1}, \ldots, \alpha_{q,m_q-1}]^t$ and V is the generalized Vandermonde matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ z_1 & z_1 & \cdots & z_1 & z_2 & z_2 & \cdots & z_2 & \cdots & \cdots & z_q & z_q & \cdots & z_q \\ z_1^2 & 2z_1^2 & \cdots & 2^{m_1-1}z_1^2 & z_2^2 & 2z_2^2 & \cdots & 2^{m_2-1}z_2^2 & \cdots & \cdots & z_q^2 & 2z_q^2 & \cdots & 2^{m_q-1}z_q^2 \\ z_1^3 & 3z_1^3 & \cdots & 3^{m_1-1}z_1^3 & z_2^3 & 3z_2^3 & \cdots & 3^{m_2-1}z_2^3 & \cdots & \cdots & z_q^3 & 3z_q^3 & \cdots & 3^{m_q-1}z_q^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{n-1} & \mathfrak{A} & \cdots & \mathfrak{B} & z_2^{n-1} & \mathfrak{C} & \cdots & \mathfrak{D} & \cdots & \cdots & z_q^{n-1} & \mathfrak{E} & \cdots & \mathfrak{F} \end{bmatrix}, \tag{2.15}$$

where $\mathfrak{A}$ denotes $(n-1)z_1^{n-1}$, $\mathfrak{B}$ denotes $(n-1)^{m_1-1}z_1^{n-1}$, $\mathfrak{C}$ denotes $(n-1)z_2^{n-1}$, $\mathfrak{D}$ denotes $(n-1)^{m_2-1}z_2^{n-1}$, $\mathfrak{E}$ denotes $(n-1)z_q^{n-1}$, and $\mathfrak{F}$ denotes $(n-1)^{m_q-1}z_q^{n-1}$.

The determinant of the matrix V was explicitly given in [25, 26]:

$$\det V = \left[\prod_{i=1}^{q}\left(\prod_{j=0}^{m_i-1} j!\right) z_i^{\binom{m_i}{2}}\right]\left[\prod_{1\le i<j\le q}\left(z_j - z_i\right)^{m_i m_j}\right]. \tag{2.16}$$

An inductive proof of this formula is given by Chen and Li [27]. Since zero is not a root of our polynomial P, we have $\det V \neq 0$; hence, we conclude $\alpha = 0$, which means the sequences $(u_k) = 0$ and $(x_k) = 0$. Hence, there is no eigenvalue in this case.

Case 2 (1 is a root of $P(z)$). Since $(u_k) \in c$, by Lemma 2.2 we can write

$$u_k = \alpha_{1,0}1^k + \left(\alpha_{2,0} + \alpha_{2,1}k + \cdots + \alpha_{2,m_2-1}k^{m_2-1}\right)z_2^k + \cdots + \left(\alpha_{q,0} + \alpha_{q,1}k + \cdots + \alpha_{q,m_q-1}k^{m_q-1}\right)z_q^k, \tag{2.17}$$

where $|z_i| < 1$ for $2 \le i \le q$. By the symmetry of the roots we have $p := 1+m_2+m_3+\cdots+m_k \le n$. Now, using the initial conditions $u_0 = u_1 = \cdots = u_{n-1} = 0$ we have

$$W\alpha = 0, \tag{2.18}$$

where $\alpha = [\alpha_{1,0}, \alpha_{2,0}, \alpha_{2,1}, \ldots, \alpha_{2,m_2-1}, \ldots, \alpha_{q,0}, \alpha_{q,1}, \ldots, \alpha_{q,m_q-1}]^t$ and W is an $n \times p$ submatrix of a generalized $n \times n$ Vandermonde matrix. Since the determinant of generalized Vandermonde matrix with nonzero roots is not zero, we have that the columns of W are linearly independent. So again we can conclude that $\alpha = 0$, which again will mean that there is no eigenvalue. □

If $T : \mu \to \mu$ (μ is ℓ_1 or c_0) is a bounded linear operator represented by the matrix A, then it is known that the adjoint operator $T^* : \mu^* \to \mu^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 and the dual space ℓ_1^* of ℓ_1 is isometrically isomorphic to the Banach space ℓ_∞.

Lemma 2.4 (see [21, page 59]). *T has a dense range if and only if T^* is one to one.*

Corollary 2.5. *If $T \in (\mu : \mu)$ then $\sigma_r(T, \mu) = \sigma_p(T^*, \mu^*) \setminus \sigma_p(T, \mu)$.*

Theorem 2.6. *$\sigma_r(S, c_0) = \emptyset$.*

Proof. $\sigma_p(S, \ell_1) = \emptyset$ by Theorem 2.3. Now using Corollary 2.5 we have $\sigma_r(S, c_0) = \sigma_p(S^*, c_0^*) \setminus \sigma_p(S, c_0) = \sigma_p(S, \ell_1) \setminus \sigma_p(S, c_0) = \emptyset$. □

If $T : c \to c$ is a bounded matrix operator represented by the matrix A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}, \tag{2.19}$$

where χ is the limit of the sequence of row sums of A minus the sum of the limits of the columns of A, and b is the column vector whose kth entry is the limit of the kth column of A for each $k \in \mathbb{N}$. For $S : c \to c$, the matrix S^* is of the form

$$\begin{bmatrix} 2(t_1 + t_2 + \cdots + t_n) + t_0 & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} F(1) & 0 \\ 0 & S \end{bmatrix}. \tag{2.20}$$

Theorem 2.7. *$\sigma_r(S, c) = \{t_0 + 2(t_1 + t_2 + \cdots + t_n)\} = \{F(1)\}$.*

Proof. Let $x = (x_0, x_1, \ldots) \in \mathbb{C} \oplus \ell_1$ be an eigenvector of S^* corresponding to the eigenvalue λ. Then we have $(2[t_1 + t_2 + \cdots + t_n] + t_0)x_0 = \lambda x_0$ and $Sx' = \lambda x'$ where $x' = (x_1, x_2, \ldots)$. By Theorem 2.3 $x' = (0, 0, \ldots)$. Then $x_0 \neq 0$. So $\lambda = 2[t_1 + t_2 + \cdots + t_n] + t_0$ is the only value that satisfies $(2[t_1 + t_2 + \cdots + t_n] + t_0)x_0 = \lambda x_0$. Hence, $\sigma_p(S^*, c^*) = \{2[t_1 + t_2 + \cdots + t_n] + t_0\}$. Then $\sigma_r(S, c) = \sigma_p(S^*, c^*) \setminus \sigma_p(S, c) = \{2[t_1 + t_2 + \cdots + t_n] + t_0\}$. □

We will write $F(1)$ instead of $\{F(1)\}$ for the sequel.

Lemma 2.8. $\sigma(S,\ell_1) = \sigma(S,c_0) = \sigma(S,c) = \sigma(S,\ell_\infty)$.

Proof. We will use the fact that the spectrum of a bounded operator over a Banach space is equal to the spectrum of the adjoint operator. The adjoint operator is the transpose of the matrix for c_0 and ℓ_1. So $\sigma(S,c_0) = \sigma(S^*,c_0^*) = \sigma(S,\ell_1) = \sigma(S^*,\ell_1^*) = \sigma(S,\ell_\infty)$. We know by Cartlidge [28] that if a matrix operator A is bounded on c, then $\sigma(A,c) = \sigma(A,\ell_\infty)$. Hence, we have $\sigma(S,c_0) = \sigma(S,\ell_1) = \sigma(S,\ell_\infty) = \sigma(S,c)$. □

Theorem 2.9. *$\sigma(S,\mu) = F(\partial D)$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$.*

Proof. Let us first consider S as an operator on c_0. By Theorems 1.6 and 2.3 $\lambda \in \sigma(S,c_0)$ if and only if $S - \lambda I$ is not onto over c_0. By Theorem 1.3 $S - \lambda I$ is not onto over c_0 if and only if the polynomial $P(z) - \lambda z^n$ has a root on the unit circle. $P(z) - \lambda z^n$ has a root on the unit circle if and only if $\lambda = P(z)/z^n = F(z)$ for some $z \in \partial D$. We have $\lambda = F(z)$ for some $z \in \partial D$ if and only if $\lambda \in F(\partial D)$. Hence, $\sigma(S,c_0) = F(\partial D)$. Finally, we apply Lemma 2.8. □

Corollary 2.10. *$S \in (c,c)$ is onto if and only if P has no root on the unit circle.*

The spectrum σ is the disjoint union of σ_p, σ_r and σ_c, so we have the following theorem as a consequence of Theorems 2.3, 2.6, 2.7, and 2.9.

Theorem 2.11. *$\sigma_c(S,c_0) = F(\partial D)$ and $\sigma_c(S,c) = F(\partial D) \setminus F(1)$.*

As a result of Theorems 1.3, 2.3, 2.6, 2.7, and 2.9 and Corollary 2.10, we have the following.

Theorem 2.12. *$F(\partial D) = II_2\sigma(S,c_0)$, $F(\partial D) \setminus F(1) = II_2\sigma(S,c)$ and $F(1) = III_2\sigma(S,c)$.*

3. Some Applications

Now, let us give an application of Theorem 2.9. Consider the system of equations

$$y_k = t_0 x_k + \sum_{j=1}^{n} t_j \left(x_{k+j} + x_{k-j}\right) \quad k = 0,1,2,\ldots, \tag{3.1}$$

where $x_k = 0$ for negative k.

Theorem 3.1. *Let $P(z) = t_n z^{2n} + t_{n-1} z^{2n-1} + \cdots + t_0 z^n + t_1 z^{n-1} + t_2 z^{n-2} + \cdots + t_n$, where $t_0, t_1, \ldots, t_n$ are complex numbers such that the complex sequences $x = (x_n)$ and $y = (y_n)$ are solutions of system* (3.1)*. Then the following are equivalent:*

(i) *boundedness of (y_n) always implies a unique bounded solution (x_n),*

(ii) *convergence of (y_n) always implies a unique convergent solution (x_n),*

(iii) *$y_n \to 0$ always implies a unique solution (x_n) with $x_n \to 0$,*

(iv) *$\sum |y_n| < \infty$ always implies a unique solution (x_n) with $\sum |x_n| < \infty$,*

(v) *P has no root on the unit circle ∂D.*

Proof. The system of equations (3.1) holds, so we have $Sx = y$. Then P is the polynomial associated to S. Let F be the function associated to S. Let us prove only (i)⇔(v) and omit the proofs of (ii)⇔(v), (iii)⇔(v), (iv)⇔(v) since they are similarly proved. Suppose boundedness of (y_n) always implies a unique bounded solution (x_n). Then the operator $S - 0I = S \in (\ell_\infty, \ell_\infty)$ is bijective. So, $\lambda = 0$ is not in the spectrum $\sigma(S, \ell_\infty)$ by Theorem 1.6, which means $0 \notin F(\partial D)$ and $0 \notin P(\partial D)$.

For the reverse implication, suppose $P(z)$ has no root on the unit circle ∂D. Then $F(z)$ has no zero on the unit circle. So, $\lambda = 0$ is in the resolvent set $\rho(S, \ell_\infty)$. Now, by Theorem 1.6, $S = S - 0I$ is bijective on ℓ_∞, which means that the boundedness of (y_n) implies a bounded unique solution (x_n). □

Example 3.2. We can see that

$$F(\partial D) = \{t_0 + 2t_1 \cos\theta + 2t_2 \cos 2\theta + \cdots + 2t_n \cos n\theta : \theta \in [0, \pi]\}. \tag{3.2}$$

When $n = 1$, S is a tridiagonal matrix, that is,

$$S = \begin{bmatrix} q & r & 0 & 0 & 0 & 0 & \cdots \\ r & q & r & 0 & 0 & 0 & \cdots \\ 0 & r & q & r & 0 & 0 & \cdots \\ 0 & 0 & r & q & r & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.3}$$

then $\sigma(S, \mu) = F(\partial D)$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$, where $F = q + r(z + z^{-1})$. Therefore,

$$\sigma(S, \mu) = \{q + 2r\cos\theta : \theta \in [0, \pi]\} = [q - 2r, q + 2r], \tag{3.4}$$

which is one of the main results of [29]. $[q-2r, q+2r]$ is the closed line segment in the complex plane with endpoints $q - 2r$ and $q + 2r$.

Example 3.3. When $n = 2$, S is a pentadiagonal matrix, that is,

$$S = \begin{bmatrix} q & r & s & 0 & 0 & 0 & 0 & \cdots \\ r & q & r & s & 0 & 0 & 0 & \cdots \\ s & r & q & r & s & 0 & 0 & \cdots \\ 0 & s & r & q & r & s & 0 & \cdots \\ 0 & 0 & s & r & q & r & s & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.5}$$

then $F = q + r(z + z^{-1}) + s(z^2 + z^{-2})$ and

$$\sigma(S, \mu) = \{q + 2r\cos\theta + 2s\cos 2\theta : \theta \in [0, \pi]\}. \tag{3.6}$$

So the spectrum is a line segment if r is a real multiple of s. It can be proved that, the spectrum is a closed connected part of a parabola if r is not a real multiple of s. For example, if $q = r = 1$ and $s = i$ (the complex number i) we have

$$\sigma(S, \mu) = \{1 + 2\cos\theta + 2i\cos 2\theta : \theta \in [0, \pi]\} = \left\{(x, y) \in \mathbb{R}^2 : y = x^2 - 2x - 1, x \in [-1, 3]\right\}. \tag{3.7}$$

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