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Research Article

On the Parametric Stokes Phenomenon for Solutions of Singularly Perturbed Linear Partial Differential Equations

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We study a family of singularly perturbed linear partial differential equations with irregular type $\epsilon t^2\partial_t\partial_z^S X_i(t,z,\epsilon) + (\epsilon t+1)\partial_z^S X_i(t,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)t^s\partial_t^{k_0}\partial_z^{k_1}X_i(t,z,\epsilon)$ in the complex domain. In a previous work, Malek (2012), we have given sufficient conditions under which the Borel transform of a formal solution to the above mentioned equation with respect to the perturbation parameter ϵ converges near the origin in $\mathbb{C}$ and can be extended on a finite number of unbounded sectors with small opening and bisecting directions, say $\kappa_i \in [0,2\pi)$, $0 \le i \le \nu - 1$ for some integer $\nu \ge 2$. The proof rests on the construction of neighboring sectorial holomorphic solutions to the first mentioned equation whose differences have exponentially small bounds in the perturbation parameter (Stokes phenomenon) for which the classical Ramis-Sibuya theorem can be applied. In this paper, we introduce new conditions for the Borel transform to be analytically continued in the larger sectors $\{\epsilon \in \mathbb{C}^*/\arg(\epsilon) \in (\kappa_i,\kappa_{i+1})\}$, where it develops isolated singularities of logarithmic type lying on some half lattice. In the proof, we use a criterion of analytic continuation of the Borel transform described by Fruchard and Schäfke (2011) and is based on a more accurate description of the Stokes phenomenon for the sectorial solutions mentioned above.

1. Introduction

We consider a family of singularly perturbed linear partial differential equations of the form

$$\epsilon t^2\partial_t\partial_z^S X_i(t,z,\epsilon) + (\epsilon t+1)\partial_z^S X_i(t,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)t^s\partial_t^{k_0}\partial_z^{k_1}X_i(t,z,\epsilon) \tag{1.1}$$

for given initial conditions

$$\left(\partial_z^j X_i\right)(t,0,\epsilon) = \varphi_{i,j}(t,\epsilon), \quad 0 \le i \le \nu-1,\ 0 \le j \le S-1, \tag{1.2}$$

where ϵ is a complex perturbation parameter, S is some positive integer, ν is some positive integer larger than 2, and $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$ with the property that there exists an integer $b > 1$ with

$$S \geq b(s - k_0 + 2) + k_1, \quad s \geq 2k_0 \tag{1.3}$$

for all $(s, k_0, k_1) \in \mathcal{S}$, and the coefficients $b_{s,k_0,k_1}(z,\epsilon)$ belong to $\mathcal{O}\{z,\epsilon\}$ where $\mathcal{O}\{z,\epsilon\}$ denotes the space of holomorphic functions in (z,ϵ) near the origin in $\mathbb{C}^2$. In this work, we make the assumption that the coefficients of (1.1) factorize in the form $b_{s,k_0,k_1}(z,\epsilon) = \epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon)$ where $\widetilde{b}_{s,k_0,k_1}(z,\epsilon)$ belong to $\mathcal{O}\{z,\epsilon\}$. The initial data $\varphi_{i,j}(t,\epsilon)$ are assumed to be holomorphic functions on a product of two sectors $\mathcal{T} \times \mathcal{E}_i$, where $\mathcal{T}$ is a fixed bounded sector centered at 0 and $\mathcal{E}_i$, $0 \leq i \leq \nu - 1$, are sectors with opening larger than π centered at the origin whose union form a covering of $\mathcal{V} \setminus \{0\}$, where $\mathcal{V}$ is some neighborhood of 0. For all $\epsilon \neq 0$, this family belongs to a class of partial differential equations which have a so-called irregular singularity at $t = 0$ (in the sense of [1]).

In the previous work [2], we have given sufficient conditions on the initial data $\varphi_{i,j}(t,\epsilon)$, for the existence of a formal series

$$\widehat{X}(t,z,\epsilon) = \sum_{k\geq 0} \frac{H_k(t,z)\epsilon^k}{k!} \in \mathcal{O}(\mathcal{T})\{z\}[[\epsilon]] \tag{1.4}$$

solution of (1.1), with holomorphic coefficients $H_k(t,z)$ on $\mathcal{T} \times D(0,\delta)$ for some disc $D(0,\delta)$, with $\delta > 0$, such that, for all $0 \leq i \leq \nu - 1$, the solution $X_i(t,z,\epsilon)$ of the problem (1.1), (1.2) defines a holomorphic function on $\mathcal{T} \times D(0,\delta) \times \mathcal{E}_i$ which is the 1-sum of $\widehat{X}$ on $\mathcal{E}_i$. In other words, for all fixed $(t,z) \in \mathcal{T} \times D(0,\delta)$, the Borel transform of $\widehat{X}$ with respect to ϵ defined as $\mathcal{B}(\widehat{X})(s) = \sum_{k\geq 0} H_k(t,z)s^k/(k!^2)$ is holomorphic on some disc $D(0,s_0)$ and can be analytically continued (with exponential growth) to sectors $\mathcal{G}_{\kappa_i}$, centered at 0, with infinite radius and with the bisecting direction $\kappa_i \in [0,2\pi)$ of the sector $\mathcal{E}_i$. But in general, due to the fact that the functions X_i do not coincide on the intersections $\mathcal{E}_i \cap \mathcal{E}_{i+1}$ (known as the Stokes phenomenon), the Borel transform cannot be analytically extended to the whole sectors $S_{\kappa_i,\kappa_{i+1}} = \{s \in \mathbb{C}^*/\arg(s) \in (\kappa_i, \kappa_{i+1})\}$ for all $0 \leq i \leq \nu - 1$, where by convention $\kappa_\nu = \kappa_0$, $\mathcal{E}_\nu = \mathcal{E}_0$, and $X_\nu = X_0$.

In this work, we address the question of the possibility of analytic continuation, location of singularities, and behaviour near these singularities of the Borel transform within the sector $S_{\kappa_i,\kappa_{i+1}}$. More precisely, our goal is to give stronger conditions on the initial data $\varphi_{i,j}(t,\epsilon)$ under which the Borel transform $\mathcal{B}(\widehat{X})(s)$ can be analytically continued to the full-punctured sector $S_{\kappa_i,\kappa_{i+1}}$ except a half lattice of points $\lambda k/t$, $k \in \mathbb{N} \setminus \{0\}$, depending on t and some well-chosen complex number $\lambda \in \mathbb{C}^*$ and moreover develop logarithmic singularities at $\lambda k/t$ (Theorem 5.8).

In a recent paper of Fruchard and Schäfke, see [3], an analogous study has been performed for formal WKB solutions $y(x,\epsilon) = \exp((x^2/2 - x^3/3)/\epsilon)x^{-1/2}(x-1)^{-1/2}\widehat{v}(x,\epsilon)$ to the singularly perturbed Schrödinger equation

$$\epsilon^2 y''(x,\epsilon) = x^2(x-1)^2 y(x,\epsilon), \tag{1.5}$$

where $\widehat{v}(x,\epsilon) = \sum_{n\geq 0} y_n(x)\epsilon^n$ is a formal series with holomorphic coefficients y_n on some domain avoiding 0 and 1. The authors show that the Borel transform of $\widehat{v}$ with respect to ϵ converges near the origin and can be analytically continued along any path avoiding some lattices of points depending on $(x^2/2 - x^3/3)$. We also mention that formal parametric Stokes phenomenon for 1-dimensional stationary linear Schrödinger equation $\epsilon^2 y''(z) = Q(z)y(z)$, where $Q(z)$ is a polynomial, has been investigated by several other authors using WKB analysis, see [4–6]. In a more general framework, analytic continuation properties related with the Stokes phenomenon have been studied by several authors in different contexts. For nonlinear systems of ODEs with irregular singularity at ∞ of the form $y\prime(z) = f(z,y(z))$ and for nonlinear systems of difference equations $y(z+1) = g(z,y(z))$, under nonresonance conditions, we refer to [7, 8]. For linearizations procedures for holomorphic germs of $(\mathbb{C},0)$ in the resonant case, we make mention to [9, 10]. For analytic conjugation of vector fields in $\mathbb{C}^2$ to normal forms, we indicate [11, 12]. For Hamiltonian nonlinear first-order partial differential equations, we notice [13].

In the proof of our main result, we will use a criterion for the analytic continuation of the Borel transform described by Fruchard and Schäfke in [3] (Theorem (FS) in Theorem 5.8). Following this criterion, in order to prove the analytic continuation of the Borel transform $\mathcal{B}(\widehat{X})(s)$, say, on the sector S_{κ_0,κ_1}, for any fixed $(t,z) \in \mathcal{T} \times D(0,\delta)$, we need to have a complete description of the Stokes relation between the solutions X_0 and X_1 of the form

$$X_1(t,z,\epsilon) - X_0(t,z,\epsilon) = \sum_{h=1}^{m} e^{-a_h/\epsilon} X_{h,0}(t,z,\epsilon) + \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon}\right) \tag{1.6}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, for some integer $m \geq 1$, where $\{a_h\}_{1\leq h\leq m}$ is a set of aligned complex numbers such that $\arg(a_h) = \alpha \in (\kappa_0,\kappa_1)$ with $|a_k| < C$ (for some $C > 0$) and $X_{h,0}(t,z,\epsilon)$, $h \geq 1$, are the 1-sums of some formal series $\widehat{G}_h(\epsilon) \in \mathcal{O}(\mathcal{T} \times D(0,\delta))[[\epsilon]]$ on $\mathcal{E}_0$. If the relation (1.6) holds, then $\mathcal{B}(\widehat{X})(s)$ can be analytically continued along any path in the punctured sector $(S_{\kappa_0,\kappa_1} \cap D(0,C)) \setminus \{a_h\}_{1\leq h\leq m}$ and has logarithmic growth as s tends to a_h in a sector. Actually, under suitable conditions on the initial data $\varphi_{i,j}(t,\epsilon)$, we have shown that such a relation holds for $a_k = \lambda k/t$, for some well-chosen $\lambda \in \mathbb{C}^*$ and for all $k \geq 1$, see (5.145) in Theorem 5.8. In order to establish such a Stokes relation (1.6), we proceed in several steps.

In the first step, following the same strategy as in [2], using the linear map $T \mapsto T/\epsilon = t$, we transform the problem (1.1) into an auxiliary regularly perturbed singular linear partial differential equation which has an irregular singularity at $T = 0$ and whose coefficients have poles with respect to ϵ at the origin, see (4.9). Then, for $\lambda \in \mathbb{C}^*$, we construct a formal transseries expansion of the form

$$\widehat{Y}(T,z,\epsilon) = \sum_{h\geq 0} \frac{\exp(-\lambda h/T)}{h!} \widehat{Y}_h(T,z,\epsilon) \tag{1.7}$$

solution of the problem (4.9), (4.10), where each $\widehat{Y}_h(T,z,\epsilon) = \sum_{m\geq 0} Y_{h,m}(z,\epsilon)T^m/m!$ is a formal series in T with coefficients $Y_{h,m}(z,\epsilon)$, which are holomorphic on a punctured polydisc $D(0,\delta) \times (D(0,\epsilon_0) \setminus \{0\})$. We show that the Borel transform of each $\widehat{Y}_h(T,z,\epsilon)$ with respect to T, defined by $V_h(\tau,z,\epsilon) = \sum_{m\geq 0} Y_{h,m}(z,\epsilon)\tau^m/(m!^2)$, satisfies an integrodifferential Cauchy problem with rational coefficients in τ, holomorphic with respect to (τ,z) near the origin and meromorphic in ϵ with a pole at zero, see (4.20), (4.21). For well-chosen λ and suitable initial data, we show that each $V_h(\tau, z, \epsilon)$ defines a holomorphic function near the origin with respect to (τ, z) and on a punctured disc with respect to ϵ and can be analytically continued to functions $V_{h,i}(\tau, z, \epsilon)$ defined on the products $S_i \times D(0,\delta) \times \mathcal{E}_i$, where S_i, $0 \le i \le \nu - 1$ are suitable open sectors with small opening and infinite radius. Moreover, the functions $V_{h,i}(\tau, z, \epsilon)$ have exponential growth rate with respect to (τ, ϵ), namely, there exist $A, B, K > 0$ such that

$$\sup_{z \in D(0,\delta)} |V_{h,i}(\tau, z, \epsilon)| \le A h! B^h e^{K|\tau|/|\epsilon|} \tag{1.8}$$

for all (τ, z, ϵ) in their domain of definition and all $h \ge 0$ (Proposition 4.12). In order to get these estimates, we use the Banach spaces depending on two parameters $\beta \in \mathbb{N}$ and ϵ with norms $\|\cdot\|_{\beta,\epsilon}$ of functions $v(\tau)$ bounded by $\exp(K_\beta|\tau|/|\epsilon|)$ for some bounded sequence K_β already introduced in [2]. If one expands the functions $V_{h,i}(\tau, z, \epsilon) = \sum_{\beta \ge 0} v_{h,i,\beta}(\tau, \epsilon) z^\beta/\beta!$ with respect to z, we show that the generating function $\sum_{h \ge 0, \beta \ge 0} \|v_{h,i,\beta}(\tau, \epsilon)\|_{\beta,\epsilon} u^h x^\beta/(h!\beta!)$ can be majorized by a series $W_i(u, x)$ which satisfies a Cauchy problem of Kowalevski type (4.47), (4.48) and is therefore convergent near the origin in $\mathbb{C}^2$.

We construct a sequence of actual functions $Y_{h,i}(T, z, \epsilon)$, $h \ge 0$, $0 \le i \le \nu - 1$ as Laplace transform of the functions $V_{h,i}(\tau, z, \epsilon)$ with respect to τ along a halfline $L_i = \mathbb{R}_+ e^{\sqrt{-1}\gamma} \subset S_i \cup \{0\}$. We show that the functions $X_{h,i}(t, z, \epsilon) = Y_{h,i}(\epsilon t, z, \epsilon)$ are holomorphic functions on the domain $\mathcal{T} \times D(0,\delta) \times \mathcal{E}_i$ and that the functions $G_{h,i}(\epsilon) := X_{h,i+1}(t, z, \epsilon) - X_{h,i}(t, z, \epsilon)$ are exponentially flat as ϵ tends to 0 on $\mathcal{E}_{i+1} \cap \mathcal{E}_i$ as $\mathcal{O}(\mathcal{T} \times D(0,\delta))$-valued functions. In the proof, we use, as in [2], a deformation of the integration's path in $X_{h,i}$ and the estimates (1.8). Using the Ramis-Sibuya theorem (Theorem (RS) in Proposition 4.15), we deduce that each $X_{h,i}(t, z, \epsilon)$ is the 1-sum of a formal series $\widehat{G}_h(\epsilon) \in \mathcal{O}(\mathcal{T} \times D(0,\delta))[[\epsilon]]$ on $\mathcal{E}_i$, for $0 \le i \le \nu - 1$ (Proposition 4.15). We notice that the functions $X_{0,i}(t, z, \epsilon)$ actually coincide with the functions $X_i(t, z, \epsilon)$ mentioned above solving the problem (1.1), (1.2). We deduce that, for a suitable choice of λ, the function

$$Z_0(t, z, \epsilon) = \sum_{h \ge 0} \frac{\exp(-\lambda h/\epsilon t)}{h!} X_{h,0}(t, z, \epsilon) \tag{1.9}$$

solves (1.1) on the domain $\mathcal{T} \times D(0,\delta) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$.

In the second part of the proof, we establish the connection formula $X_{0,1}(t, z, \epsilon) = Z_0(t, z, \epsilon)$ which is exactly the Stokes relation (1.6) on $\mathcal{T} \times D(0,\delta) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$ (Proposition 5.2). The strategy we follow consists in expressing both functions $X_{0,1}$ and Z_0 as Laplace transforms of objects that are no longer functions in general but distributions supported on $\mathbb{R}_+$ which are called staircase distributions in the terminology of [8]. We stress the fact that such representations of transseries expansions as generalized Laplace transforms were introduced for the first time by Costin in the paper [8]. Notice that similar arguments have been used in the work [14] to study the Stokes phenomenon for sectorial holomorphic solutions to linear integro-differential equations with irregular singularity.

In Lemma 5.5, we show that Z_0 can be written as a generalized Laplace transform in the direction $\arg(\lambda)$ of a staircase distribution $\mathbb{V}(r, z, \epsilon) = \sum_{\beta \ge 0} \mathbb{V}_\beta(r, \epsilon) z^\beta/\beta! \in \mathfrak{D}'(\sigma, \epsilon, \delta)$, which is a convergent series in z on $D(0,\delta)$ with coefficients $\mathbb{V}_\beta(r, \epsilon)$ in some Banach spaces of staircase distributions $\mathfrak{D}'_{\beta,\sigma,\epsilon}$ on $\mathbb{R}_+$ depending on the parameters β and ϵ (see Definition 2.3). We observe that the distribution $\mathbb{V}(r, z, \epsilon)$ solves moreover an integro-differential Cauchy problem with rational coefficients in r, holomorphic with respect to z near the origin and meromorphic with respect to ϵ at zero, see (5.80), (5.81). The idea of proof consists in showing that each function $X_{h,0}(t,z,\epsilon)$ can be expressed as a Laplace transform in a sequence of directions ζ_n tending to $\arg(\lambda)$ of a sequence of staircase distributions $\mathbb{V}_{h,n}(r,z,\epsilon)$ (which are actually convergent series in z with coefficients that are C^∞ functions in r on $\mathbb{R}_+$ with exponential growth). Moreover, each distribution $\mathbb{V}_{h,n}(r,z,\epsilon)$ solves an integro-differential Cauchy problem (5.37), (5.38), whose coefficients tend to the coefficients of an integro-differential equation (5.39), (5.40), as n tends to ∞, having a unique staircase distribution solution $\mathbb{V}_{h,\infty}(r,z,\epsilon)$. Under the hypothesis that the initial data (5.38) converge to (5.40) as $n \to +\infty$, we show that the sequence $\mathbb{V}_{h,n}(r,z,\epsilon)$ converges to $\mathbb{V}_{h,\infty}(r,z,\epsilon)$ in the Banach space $\mathfrak{D}'(\sigma,\epsilon,\delta)$ with precise norm estimates with respect to h and n (Lemma 5.3). In order to show this convergence, we use a majorazing series method together with a version of the classical Cauchy-Kowalevski theorem (Proposition 2.22) in some spaces of analytic functions near the origin in $\mathbb{C}^2$ with dependence on initial conditions and coefficients applied to the auxiliary problem (5.66), (5.68). Using a continuity property of the Laplace transform (3.5), we show that each function $X_{h,0}(t,z,\epsilon)$ can be actually expressed as the Laplace transform of $\mathbb{V}_{h,\infty}(r,z,\epsilon)$ in the direction $\arg(\lambda)$ and finally that Z_0 itself is the Laplace transform of some staircase distribution $\mathbb{V}(r,z,\epsilon)$ solving (5.80), (5.81).

On the other hand, in Lemma 5.7, under suitable conditions on $\varphi_{1,j}(t,\epsilon)$, $0 \leq j \leq S-1$, we can also write $X_{0,1}(t,z,\epsilon)$ as a generalized Laplace transform in the direction $\arg(\lambda)$ of the staircase distribution mentioned above $\mathbb{V}(r,z,\epsilon)$ solving (5.80), (5.81). Therefore, the equality $X_{0,1}(t,z,\epsilon) = Z_0(t,z,\epsilon)$ holds on $\mathcal{T} \times D(0,\delta) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$. The method of proof consists again in showing that $X_{0,1}(t,z,\epsilon)$ can be written as Laplace transform in a sequence of directions ξ_n tending to $\arg(\lambda)$ of a sequence of staircase distributions $\mathbb{V}_n(r,z,\epsilon)$ (which are actually convergent series in z with coefficients that are C^∞ functions in r on $\mathbb{R}_+$ with exponential growth). Moreover, each distribution $\mathbb{V}_n(r,z,\epsilon)$ solves an integro-differential Cauchy problem (5.98), (5.99), whose coefficients tend to the coefficients of the integro-differential equation (5.80). Under the assumption that the initial data (5.99) converge to the initial data (5.81), we show that the sequence $\mathbb{V}_n(r,z,\epsilon)$ converges to the solution of (5.80), (5.81) (i.e., $\mathbb{V}(r,z,\epsilon)$) in the Banach space $\mathfrak{D}'(\sigma,\epsilon,\delta)$, as $n \to +\infty$, see Lemma 5.6. This convergence result is obtained again by using a majorazing series technique which reduces the problem to the study of some linear differential equation (5.106), (5.109), whose coefficients and initial data tend to zero as $n \to +\infty$. Finally, by continuity of the Laplace transform, $X_{0,1}(t,z,\epsilon)$ can be written as the Laplace transform of $\mathbb{V}(r,z,\epsilon)$ in direction $\arg(\lambda)$.

After Theorem 5.8, we give an application to the construction of solutions to some specific singular linear partial differential equations in $\mathbb{C}^3$ having logarithmic singularities at the points $(\lambda k/t, t, z)$, for $k \in \mathbb{N} \setminus \{0\}$. We show that under the hypothesis that the coefficients b_{s,k_0,k_1} are polynomials in ϵ, the Borel transform $\mathcal{B}(\widehat{X})(s)$ turns out to solve the linear partial differential equation (5.149). We would like to mention that there exists a huge literature on the study of complex singularities and analytic continuation of solutions to linear partial differential equations starting from the fundamental contributions of Leray in [15]. Several authors have considered Cauchy problems $a(x,D)u(x) = 0$, where $a(x,D)$ is a differential operator of some order $m \geq 1$, for initial data $\partial_{x_0}^h u_{|x_0=0} = w_h$, $0 \leq h < m$. Under specific hypotheses on the symbol $a(x,\xi)$, precise descriptions of the solutions of these problems are given near the singular locus of the initial data w_h. For meromorphic initial data, we may refer to [16–18] and for more general ramified multivalued initial data, we may cite [19–23].

The layout of this work is as follows.

In Section 2, we introduce Banach spaces of formal series whose coefficients belong to spaces of staircase distributions and we study continuity properties for the actions of multiplication by C^∞ functions and integro-differential operators on these spaces. In this section, we also exhibit a Cauchy Kowalevski theorem for linear partial differential problems in some space of analytic functions near the origin in $\mathbb{C}^2$ with dependence of their solutions on the coefficients and initial data which will be useful to show the connection formula (5.28) stated in Section 5.

In Section 3, we recall the definition of a Laplace transform of a staircase distribution as introduced in [8] and we give useful commutation formulas with respect to multiplication by polynomials, exponential functions, and derivation.

In Section 4, we construct formal and analytic transseries solutions to the singularly perturbed partial differential equation with irregular singularity (1.1).

In Section 5, we establish the crucial connection formula relying on the analytic transseries solution $Z_0(t,z,\epsilon)$ and the solution $X_{0,1}(t,z,\epsilon)$ of (1.1). Finally, we state the main result of the paper which asserts that the Borel transform $\mathcal{B}(\widehat{X})(s)$ in the perturbation parameter ϵ of the formal solution $\widehat{X}(t,z,\epsilon)$ of (1.1) can be analytically continued along any path in the punctured sector $S_{\kappa_0,\kappa_1} \setminus \cup_{h\geq 1}\{\lambda h/t\}$ and has logarithmic growth as s tends to $\lambda h/t$ in a sector for all $h \geq 1$.

2. Banach Spaces of Formal Series with Coefficients in Spaces of Staircase Distributions: A Cauchy Problem in Spaces of Analytic Functions

2.1. Weighted Banach Spaces of Distributions

We define $\mathfrak{D}(\mathbb{R}_+)$ to be the space of complex valued C^∞-functions with compact support in $\mathbb{R}_+$, where $\mathbb{R}_+$ is the set of the positive real numbers $x > 0$. We also denote by $\mathfrak{D}'(\mathbb{R}_+)$ the space of distributions on $\mathbb{R}_+$. For $f \in \mathfrak{D}'(\mathbb{R}_+)$, we write $f^{(k)}$ the k-derivative of f in the sense of distribution, for $k \geq 0$, with the convention $f^{(0)} = f$.

Definition 2.1. A distribution $f \in \mathfrak{D}'(\mathbb{R}_+)$ is called staircase if f can be written in the form

$$f = \sum_{k=0}^{\infty} (\Delta_k(f))^{(k)}, \tag{2.1}$$

for unique integrable functions $\Delta_k(f) \in L^1(\mathbb{R}_+)$ such that the support $\mathrm{supp}(\Delta_k(f))$ of $\Delta_k(f)$ is in $[k, k+1]$ for all $k \geq 0$.

Remark 2.2. Given a compact set $K \in \mathbb{R}_+$, a general distribution $\Lambda \in \mathfrak{D}'(\mathbb{R}_+)$ can always be written as a k-derivative of a continuous function on $\mathbb{R}_+$ restricted to the test functions with support in K, where k depends on K, see [24].

Definition 2.3. Let $\sigma > 0$ be a real number, $b > 1$ an integer and let $r_b(\beta) = \sum_{n=0}^{\beta} 1/(n+1)^b$ for all integers $\beta \geq 0$. Let $\mathcal{E}$ be an open sector centered at 0 and let $\epsilon \in \mathcal{E}$. We denote by $L_{\beta,\sigma,\epsilon}$ the vector space of all locally integrable functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\|f(r)\|_{\beta,\sigma,\epsilon} := \int_0^\infty |f(\tau)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta)\tau\right) d\tau \tag{2.2}$$

is finite. We denote by $\mathfrak{D}'_{\beta,\sigma,\epsilon}$ the vector space of staircase distributions $f = \sum_{k=0}^{\infty}(\Delta_k(f))^{(k)}$ such that

$$\|f\|_{\beta,\sigma,\epsilon,d} = \sum_{k=0}^{+\infty}\left(\frac{\sigma}{|\epsilon|}r_b(\beta)\right)^k \|\Delta_k(f)\|_{\beta,\sigma,\epsilon} \tag{2.3}$$

is finite.

Remark 2.4. Let ϵ, σ, β such that $|\epsilon| < \sigma r_b(\beta)$. If $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$, then $f \in \mathfrak{D}'_{\beta',\sigma,\epsilon}$ for all $\beta' \geq \beta$ and we have that $h \mapsto ||f||_{h,\sigma,\epsilon,d}$ is a decreasing sequence on $[\beta, +\infty)$. Likewise, if $f \in \mathfrak{D}'_{\beta,\tilde{\sigma},\epsilon}$, then $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$ for all $\sigma \geq \tilde{\sigma}$ and we have that $\sigma \mapsto ||f||_{\beta,\sigma,\epsilon,d}$ is a decreasing sequence on $[\tilde{\sigma}, +\infty)$.

Let $\mathscr{H}$ be the Heaviside one-step function defined by $\mathscr{H}(r) = 1$, if $r \geq 0$ and $\mathscr{H}(r) = 0$, if $r < 0$. Let $\not{D}$ the operator defined on distributions $T \in \mathfrak{D}'(\mathbb{R}_+)$ by $\not{D}T = \mathscr{H} * T$. For a subset $A \subset \mathbb{R}$, we denote by 1_A the function which is equal to 1 on A and 0 elsewhere.

The proofs of the following Lemmas 2.5 and 2.6, Propositions 2.7, 2.8, 2.9, and Corollary 2.10 are given in the appendix of [25], see also [8].

Lemma 2.5. *Let $k \geq 0$ and $f = F^{(k)} \in \mathfrak{D}'(\mathbb{R}_+)$, where $F \in L^1(\mathbb{R}_+)$ and* $\operatorname{supp}(F) \subset [k, +\infty)$. *Then f is a staircase distribution and the decomposition of f has the following terms $\Delta_0 = \Delta_1 = \cdots = \Delta_{k-1} = 0$, $\Delta_k = F1_{[k,k+1]}$ and for $n \geq 1$, $\Delta_{k+n} = G_n 1_{[k+n,k+n+1]}$ where $G_n = \not{D}(G_{n-1}1_{[k+n,+\infty)})$ and $G_0 = F$.*

Lemma 2.6. *Let f be as in Lemma 2.5 and ϵ, σ, β such that $|\epsilon| < \sigma r_b(\beta)$. Then, one has*

$$\|\Delta_{k+n}\|_{\beta,\sigma,\epsilon} \leq \left(\frac{\sigma}{|\epsilon|}r_b(\beta)\right)^{-n}\|F\|_{\beta,\sigma,\epsilon}, \tag{2.4}$$

if $n = 0, 1, 2$ and for $n \geq 3$,

$$\|\Delta_{k+n}\|_{\beta,\sigma,\epsilon} \leq e^{(2-n)(\sigma/|\epsilon|)r_b(\beta)}\frac{n^{n-1}}{(n-1)!}\|F\|_{\beta,\sigma,\epsilon}. \tag{2.5}$$

Proposition 2.7. *Let $f \in L_{\beta,\sigma/2,\epsilon}$ and ϵ, σ, β such that $|\epsilon| < \sigma r_b(\beta)/2$. Then f belongs to $\mathfrak{D}'_{\beta,\sigma,\epsilon}$ and the decomposition* (2.1) *of f has the following terms $\Delta_n = G_n 1_{[n,n+1]}$ with $G_n = \not{D}(G_{n-1}1_{[n,+\infty)})$ and $G_0 = f$, for $n \geq 0$. Moreover, there exists a universal constant $C_1 > 0$ such that $||f||_{\beta,\sigma,\epsilon,d} \leq C_1||f||_{\beta,\sigma/2,\epsilon}$.*

Proposition 2.8. *The set $\mathfrak{D}(\mathbb{R}_+)$ of C^∞-functions with compact support in $\mathbb{R}_+$ is dense in $\mathfrak{D}'_{\beta,\sigma,\epsilon}$ for all $\beta \geq 0$, $\sigma > 0$ and $\epsilon \in \mathcal{E}$.*

Proposition 2.9. *Let ϵ, σ, β such that $|\epsilon| < \sigma r_b(\beta)$. For all $f, \tilde{f} \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$, we have $f * \tilde{f} \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exists a universal constant $C_2 > 0$ such that*

$$\left\|f * \tilde{f}\right\|_{\beta,\sigma,\epsilon,d} \leq C_2\|f\|_{\beta,\sigma,\epsilon,d}\left\|\tilde{f}\right\|_{\beta,\sigma,\epsilon,d} \tag{2.6}$$

for all $f, \tilde{f} \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$.

In this paper, for all integers $k \geq 1$, we will denote $\partial_r^{-k} f(r)$ the convolution $\mathscr{H}^{*k} * f$ for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$ where $\mathscr{H}^{*k}$ stands for the convolution product of $\mathscr{H}$ with itself $k-1$ times for $k \geq 2$ and with the convention that $\mathscr{H}^{*1} = \mathscr{H}$. From Propositions 2.7 and 2.9, we deduce the following.

Corollary 2.10. *Let ϵ, σ, β be such that $|\epsilon| < \sigma r_b(\beta)$ and let $k \geq 1$ be an integer. For all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$, one has $\partial_r^{-k} f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exists a universal constant $C_3 > 0$ such that*

$$\left\| \partial_r^{-k} f(r) \right\|_{\beta,\sigma,\epsilon,d} \leq C_3 \left(\frac{|\epsilon|}{\sigma r_b(\beta)} \right)^k \| f(r) \|_{\beta,\sigma,\epsilon,d} \tag{2.7}$$

for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$.

In the next proposition, we study norm estimates for the multiplication by bounded analytic functions.

Proposition 2.11. *Let σ and $\beta \geq 0$ such that*

$$\frac{3}{2} \frac{\sigma}{|\epsilon|} r_b(\beta) e^{1-(\sigma/|\epsilon|) r_b(\beta)} < 1, \quad |\epsilon| < \sigma r_b(\beta), \tag{2.8}$$

and let h be a C^∞-function on $\mathbb{R}_+$ such that there exist constants $C_h > 0, \mu > 0$ and $\rho > |\epsilon|/(\sigma r_b(\beta))$ such that

$$\left| h^{(q)}(r) \right| \leq C_h \frac{q!}{(\rho(r+\mu))^{(q+1)}} \tag{2.9}$$

for all $r \in \mathbb{R}_+$. Then, for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$, we have $h(r) f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exists a constant $C_4 > 0$ (depending on μ, ρ) such that

$$\| h(r) f(r) \|_{\beta,\sigma,\epsilon,d} \leq C_4 C_h \| f(r) \|_{\beta,\sigma,\epsilon,d} \tag{2.10}$$

for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$.

Proof. The proof can be found in [14] and is inspired from [25, Lemma 2.9.1], but for the sake of completeness, we sketch it below. Without loss of generality, we can assume that f has the following form $f(t) = \Delta_k^{(k)}(t)$, where $\Delta_k \in L^1(\mathbb{R}_+)$ with $\mathrm{supp}(\Delta_k) \in [k, k+1]$, for $k \geq 1$. Put $g_{k,j}(t) = h^{(k-j)}(t) \Delta_k(t)$. Then, $\mathrm{supp}(g_{k,j}(t)) \subset [k, k+1]$.

From the Leibniz formula, we get the identity

$$h(t) \Delta_k^{(k)}(t) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} g_{k,j}^{(j)}(t). \tag{2.11}$$

On the other hand, one can rewrite $g_{k,j}^{(j)}(t) = (\mathcal{P}^{[k-j]} g_{k,j})^{(k)}$, where $\mathrm{supp}(\mathcal{P}^{[k-j]} g_{k,j}) \in [k, +\infty)$ and $\mathcal{P}^{[q]}$ denotes the qth iteration of $\mathcal{P}$.

Due to Lemma 2.5, $g_{k,j}^{(j)}$ can be written $g_{k,j}^{(j)} = \sum_{l=k}^{+\infty}(\widetilde{\Delta}_{l,j})^{(l)}$, with $\widetilde{\Delta}_{l,j} = G_{l,j}1_{[l,l+1]}, G_{l,j} = \wp(G_{l-1,j}1_{[l,+\infty)})$ and $G_{k,j} = \wp^{[k-j]}g_{k,j}$.

Therefore, we get the following identity

$$h(t)\Delta_k^{(k)}(t) = (h(t)\Delta_k(t))^{(k)} + \sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\widetilde{\Delta}_{k,j}^{(k)} + \sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\sum_{l=k+1}^{+\infty}\widetilde{\Delta}_{l,j}^{(l)}. \tag{2.12}$$

First of all, we have

$$\begin{aligned}\left\|(h(t)\Delta_k(t))^{(k)}\right\|_{\beta,\sigma,\epsilon,d} &= \left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k \int_0^{+\infty}|h(t)\Delta_k(t)|e^{-\sigma r_b(\beta)t/|\epsilon|}dt \\ &\leq \frac{C_h}{\rho\mu}\left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k \|\Delta_k(t)\|_{\beta,\sigma,\epsilon},\end{aligned} \tag{2.13}$$

where $C_h > 0$ is given in (2.9). From Lemma 2.6, we have the estimates

$$\begin{aligned}\left\|\widetilde{\Delta}_{k+n,j}\right\|_{\beta,\sigma,\epsilon} &\leq \left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^n \left\|\wp^{[k-j]}g_{k,j}\right\|_{\beta,\sigma,\epsilon}, \\ \left\|\widetilde{\Delta}_{l,j}\right\|_{\beta,\sigma,\epsilon} &\leq e^{(2-(l-k))(\sigma/|\epsilon|)r_b(\beta)}\frac{(l-k)^{l-k-1}}{(l-k-1)!}\left\|\wp^{[k-j]}g_{k,j}\right\|_{\beta,\sigma,\epsilon},\end{aligned} \tag{2.14}$$

for $n = 0, 1, 2$ and all $l \geq k+3$. Now, we give estimates for $\|\wp^{[k-j]}g_{k,j}\|_{\beta,\sigma,\epsilon}$.

Using the Taylor formula with integral remainder and the hypothesis (2.9), we get

$$\left|\wp^{[k-j]}g_{k,j}(t)\right| \leq C_h\frac{(k-j)!}{(k-j-1)!}\int_k^t \frac{(t-s)^{k-j-1}}{(\rho(s+\mu))^{1+(k-j)}}|\Delta_k(s)|ds. \tag{2.15}$$

Hence, from the Fubini theorem and the identity

$$\int_s^{+\infty} e^{-(\sigma r_b(\beta)/|\epsilon|)t}(t-s)^{k-j-1}dt = e^{-(\sigma r_b(\beta)/|\epsilon|)s}(k-j)!\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{(k-j)}, \tag{2.16}$$

we deduce

$$\begin{aligned}&\int_k^{+\infty} e^{-(\sigma r_b(\beta)/|\epsilon|)t}\left|\wp^{[k-j]}g_{k,j}(t)\right|dt \\ &\quad\leq C_h(k-j)(k-j)!\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{(k-j)}\int_k^{+\infty} e^{-(\sigma r_b(\beta)/|\epsilon|)s}\frac{|\Delta_k(s)|}{(\rho(s+\mu))^{1+(k-j)}}ds\end{aligned} \tag{2.17}$$

and hence

$$\left\|\mathcal{D}^{[k-j]}g_{k,j}(t)\right\|_{\beta,\sigma,\epsilon} \leq \frac{C_h(k-j)(k-j)!}{\left(\rho(k+\mu)\right)^{1+k-j}}\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{(k-j)}\|\Delta_k(s)\|_{\beta,\sigma,\epsilon}. \tag{2.18}$$

From (2.14) and (2.18), we obtain

$$\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\left\|\widetilde{\Delta}_{k+n,j}^{(k+n)}(t)\right\|_{\beta,\sigma,\epsilon,d} \leq C_h A_k\left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k\|\Delta_k(s)\|_{\beta,\sigma,\epsilon}, \tag{2.19}$$

for $n = 0, 1, 2$, all $k \geq 1$, where

$$A_k = \sum_{j=0}^{k-1}\frac{k!(k-j)}{j!\left(\rho(k+\mu)\right)^{1+k-j}}\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{(k-j)}. \tag{2.20}$$

Now, we need to estimate A_k. Due to the Stirling formula, $k! \sim k^k e^{-k}(2\pi k)^{1/2}$ as k tends to infinity, there exists a universal constant $C_{4,1} > 0$ such that

$$A_k \leq C_{4,1}\frac{k^k}{(k+\mu)^k}\frac{1}{\rho(k+\mu)}(2\pi k)^{1/2}e^{-k}\sum_{j=0}^{k-1}\frac{(k-j)\left((k+\mu)(\sigma r_b(\beta)/|\epsilon|)\rho\right)^j}{j!\left((\sigma r_b(\beta)/|\epsilon|)\rho\right)^k} \tag{2.21}$$

for all $k \geq 1$. Using the hypothesis $\sigma r_b(\beta)\rho/|\epsilon| \geq 1$, we have

$$\sum_{j=0}^{k-1}(k-j)\frac{(k+\mu)^j}{j!}\frac{\left((\sigma r_b(\beta)/|\epsilon|)\rho\right)^j}{\left((\sigma r_b(\beta)/|\epsilon|)\rho\right)^k} \leq \sum_{j=0}^{k-1}(k-j)\frac{(k+\mu)^j}{j!} = k\frac{(k+\mu)^{k-1}}{(k-1)!} - \mu\sum_{j=0}^{k-2}\frac{(k+\mu)^j}{j!}. \tag{2.22}$$

Using again the Stirling formula, we get a constant $C_{4,\mu} > 0$ (depending on μ) such that

$$k\frac{(k+\mu)^{k-1}}{(k-1)!} \leq C_{4,\mu}k^{1/2}e^k \tag{2.23}$$

for all $k \geq 1$. Moreover,

$$\mu\sum_{j=0}^{k-2}\frac{(k+\mu)^j}{j!} \leq \mu\sum_{j=0}^{+\infty}\frac{(k+\mu)^j}{j!} = \mu e^{k+\mu}. \tag{2.24}$$

Hence,

$$\sum_{j=0}^{k-1}(k-j)\frac{(k+\mu)^j}{j!}\frac{\left((\sigma r_b(\beta)/|\epsilon|)\rho\right)^j}{\left((\sigma r_b(\beta)/|\epsilon|)\rho\right)^k} \leq \left(C_{4,\mu}k^{1/2} + \mu e^{\mu}\right)e^k \tag{2.25}$$

for all $k \geq 1$. Finally, we obtain a constant $C_{4,\mu,\rho} > 0$ depending only on ρ, μ such that

$$A_k \leq C_{4,\mu,\rho} \tag{2.26}$$

for all $k \geq 1$. From (2.14) and (2.18), we have

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \sum_{l=k+3}^{+\infty} \left\| \tilde{\Delta}_{l,j}^{(l)} \right\|_{\beta,\sigma,\epsilon,d} \leq C_h A_k \tilde{A}_k \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^k \|\Delta_k(s)\|_{\beta,\sigma,\epsilon}, \tag{2.27}$$

where

$$\tilde{A}_k = \sum_{l=k+3}^{+\infty} \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^{l-k} e^{(2-(l-k))(\sigma r_b(\beta)/|\epsilon|)} \frac{(l-k)^{(l-k-1)}}{(l-k-1)!} = \sum_{h=3}^{\infty} \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^h e^{(2-h)(\sigma r_b(\beta)/|\epsilon|)} \frac{h^{(h-1)}}{(h-1)!}. \tag{2.28}$$

Now, we show that $\tilde{A}_k$, $k \geq 1$, is a bounded sequence. Again, by the Stirling formula, we get a universal constant $C_{4,2} > 0$ such that

$$\begin{aligned} \tilde{A}_k &\leq C_{4,2} \exp\left(2 \frac{\sigma}{|\epsilon|} r_b(\beta) \right) \sum_{h=3}^{+\infty} \left(\frac{\sigma}{|\epsilon|} r_b(\beta) \right)^h \exp\left(h \left(1 - \frac{\sigma}{|\epsilon|} r_b(\beta) \right) \right) \left(\frac{h}{h-1} \right)^{h-1} \frac{1}{(2\pi(h-1))^{1/2}} \\ &\leq C_{4,2} \exp\left(2 \frac{\sigma}{|\epsilon|} r_b(\beta) \right) \sum_{h=3}^{+\infty} \left(\frac{3(\sigma/|\epsilon|) r_b(\beta)}{2} \exp\left(1 - \frac{\sigma}{|\epsilon|} r_b(\beta) \right) \right)^h. \end{aligned} \tag{2.29}$$

From the assumption (2.8) and the estimates that for all $m_1, m_2 > 0$ two real numbers, we have

$$\sup_{x \geq 0} x^{m_1} \exp(-m_2 x) = \left(\frac{m_1}{m_2} \right)^{m_1} e^{-m_1}, \tag{2.30}$$

we get a constant $0 < \delta < 1$ such that

$$\tilde{A}_k \leq C_{4,2} \frac{e^3}{1-\delta} \left(\frac{3(\sigma/|\epsilon|) r_b(\beta)}{2} \right)^3 \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta) \right) \leq \frac{3^6 C_{4,2}}{2^3 (1-\delta)} \tag{2.31}$$

for all $k \geq 1$.

Finally, from the equality (2.12) and estimates (2.13), (2.19), (2.26), (2.27) and (2.31), we get a constant $C_{4,\mu,\rho,1} > 0$ depending only on μ, ρ such that $||h(t)\Delta_k^{(k)}(t)||_{\beta,\sigma,\epsilon,d} \leq C_h C_{4,\mu,\rho,1} ||\Delta_k^{(k)}(t)||_{\beta,\sigma,\epsilon,d}$ for all $k \geq 1$. It remains to consider the case $k = 0$.

When $k = 0$, let $f(t) = \Delta_0(t) \in L^1(\mathbb{R}_+)$, with $\mathrm{supp}(\Delta_0) \in [0,1]$. By definition, we can write

$$\|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon,d} = \|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon}$$
$$= \int_0^1 |h(t)||\Delta_0(t)| \exp\left(-\frac{\sigma r_b(\beta)}{|\epsilon|}t\right)dt \le \frac{C_h}{\rho\mu}\|\Delta_0(t)\|_{\beta,\sigma,\epsilon} = \frac{C_h}{\rho\mu}\|\Delta_0(t)\|_{\beta,\sigma,\epsilon,d}. \tag{2.32}$$

□

In the next proposition, we study norm estimates for the multiplication by polynomials.

Proposition 2.12. *Let σ and $\beta \ge 0$ such that*

$$\frac{3}{2}\frac{\sigma}{|\epsilon|} r_b(\beta) e^{1-(\sigma/|\epsilon|)r_b(\beta)} < 1, \quad |\epsilon| < \sigma \tag{2.33}$$

and let $s_1, k_2 \ge 1$ be integers. Then, for all $f \in \mathfrak{D}'_{\beta-k_2,\sigma,\epsilon}$, one has $r^{s_1}f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exists a constant $C_5 > 0$ (depending on s_1,σ) such that

$$\left\|r^{s_1} f(r)\right\|_{\beta,\sigma,\epsilon,d} \le C_5|\epsilon|^{s_1}(\beta+1)^{bs_1}\left\|f(r)\right\|_{\beta-k_2,\sigma,\epsilon,d} \tag{2.34}$$

for all $f \in \mathfrak{D}'_{\beta-k_2,\sigma,\epsilon}$.

Proof. The proof is an adaptation of Proposition 2.11. Without loss of generality, we can assume that f has the following form $f(t) = \Delta_k^{(k)}(t)$ where $\Delta_k \in L^1(\mathbb{R}_+)$ with $\mathrm{supp}(\Delta_k) \in [k, k+1]$, for $k \ge 1$. We also put $h(t) = t^{s_1}$. Let $g_{k,j}(t) = h^{(k-j)}(t)\Delta_k(t)$. Then, $\mathrm{supp}(g_{k,j}(t)) \subset [k, k+1]$. From the Leibniz formula, we get the identity

$$h(t)\Delta_k^{(k)}(t) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} g_{k,j}^{(j)}(t). \tag{2.35}$$

On the other hand, one can rewrite $g_{k,j}^{(j)}(t) = (\mathcal{P}^{[k-j]}g_{k,j})^{(k)}$, where $\mathrm{supp}(\mathcal{P}^{[k-j]}g_{k,j}) \in [k, +\infty)$ and $\mathcal{P}^{[q]}$ denotes the qth iteration of $\mathcal{P}$.

Due to Lemma 2.5, $g_{k,j}^{(j)}$ can be written $g_{k,j}^{(j)} = \sum_{l=k}^{+\infty}(\widetilde{\Delta}_{l,j})^{(l)}$, with $\widetilde{\Delta}_{l,j} = G_{l,j}1_{[l,l+1]}$, $G_{l,j} = \mathcal{P}(G_{l-1,j}1_{[l,+\infty)})$ and $G_{k,j} = \mathcal{P}^{[k-j]}g_{k,j}$. Therefore, we get the following identity as:

$$h(t)\Delta_k^{(k)}(t) = (h(t)\Delta_k(t))^{(k)} + \sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\widetilde{\Delta}_{k,j}^{(k)} + \sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\sum_{l=k+1}^{+\infty}\widetilde{\Delta}_{l,j}^{(l)}. \tag{2.36}$$

(1) We first give estimates for $\|(h(t)\Delta_k(t))^{(k)}\|_{\beta,\sigma,\epsilon,d}$. We write

$$\begin{aligned}
\left\|(h(t)\Delta_k(t))^{(k)}\right\|_{\beta,\sigma,\epsilon,d} &= \left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k \int_0^{+\infty} \tau^{s_1}|\Delta_k(\tau)|\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)\tau\right)d\tau \\
&= \left(\frac{\sigma r_b(\beta-k_2)}{|\epsilon|}\right)^k \left(\frac{r_b(\beta)}{r_b(\beta-k_2)}\right)^k \\
&\quad\times \int_k^{k+1} \tau^{s_1}\exp\left(-\frac{\sigma}{|\epsilon|}(r_b(\beta)-r_b(\beta-k_2))\tau\right) \\
&\quad\times |\Delta_k(\tau)|\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta-k_2)\tau\right)d\tau \\
&\le A(\epsilon,\beta)\left(\frac{\sigma r_b(\beta-k_2)}{|\epsilon|}\right)^k \int_k^{k+1}|\Delta_k(\tau)|\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta-k_2)\tau\right)d\tau,
\end{aligned} \tag{2.37}$$

where

$$A(\epsilon,\beta) = \sup_{k\ge1}\left(\left(\frac{r_b(\beta)}{r_b(\beta-k_2)}\right)^k (k+1)^{s_1}\exp\left(-\frac{\sigma}{|\epsilon|}(r_b(\beta)-r_b(\beta-k_2))k\right)\right). \tag{2.38}$$

Now, we gives estimates for $A(\epsilon,\beta)$. We write

$$\begin{aligned}
&\left(\frac{r_b(\beta)}{r_b(\beta-k_2)}\right)^k (k+1)^{s_1}\exp\left(-\frac{\sigma}{|\epsilon|}(r_b(\beta)-r_b(\beta-k_2))k\right) \\
&\quad= (k+1)^{s_1}\exp\left(-k\frac{\sigma}{|\epsilon|}(\psi(r_b(\beta))-\psi(r_b(\beta-k_2)))\right) \\
&\quad\le 2^{s_1}k^{s_1}\exp\left(-k\frac{\sigma}{|\epsilon|}(\psi(r_b(\beta))-\psi(r_b(\beta-k_2)))\right),
\end{aligned} \tag{2.39}$$

where $\psi(x) = x - (|\epsilon|/\sigma)\log(x)$ for all $k \ge 1$. From the Taylor formula applied to ψ on $[r_b(\beta - k_2), r_b(\beta)]$, we get that

$$\psi(r_b(\beta)) - \psi(r_b(\beta-k_2)) \ge \left(1-\frac{|\epsilon|}{\sigma}\right)(r_b(\beta)-r_b(\beta-k_2)) \ge \left(1-\frac{|\epsilon|}{\sigma}\right)\frac{k_2}{(\beta+1)^b} \tag{2.40}$$

for all $\beta \ge k_2$. Now, we recall that for all $m_1, m_2 > 0$ two real numbers, we have

$$\sup_{x\ge0} x^{m_1}\exp(-m_2x) = \left(\frac{m_1}{m_2}\right)^{m_1} e^{-m_1}. \tag{2.41}$$

From (2.39), (2.40) and (2.41), we deduce that

$$A(\epsilon,\beta) \leq 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\sigma)k_2\sigma}\right)^{s_1}|\epsilon|^{s_1}(\beta+1)^{bs_1} \tag{2.42}$$

for all $\beta \geq k_2$. From (2.37) and (2.42), we deduce that

$$\left\|(h(t)\Delta_k(t))^{(k)}\right\|_{\beta,\sigma,\epsilon,d} \leq 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\sigma)k_2\sigma}\right)^{s_1}|\epsilon|^{s_1}(\beta+1)^{bs_1}\|f(t)\|_{\beta-k_2,\sigma,\epsilon,d}. \tag{2.43}$$

(2) We give estimates for $||\widetilde{\Delta}_{l,j}||_{\beta,\sigma,\epsilon}$ for all $0 \leq j \leq k-1$ and all $l \geq k$. From Lemma 2.6, we have the estimates

$$\begin{gathered}\left\|\widetilde{\Delta}_{k+n,j}\right\|_{\beta,\sigma,\epsilon} \leq \left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^n\left\|\mathcal{D}^{[k-j]}g_{k,j}\right\|_{\beta,\sigma,\epsilon}, \\ \left\|\widetilde{\Delta}_{l,j}\right\|_{\beta,\sigma,\epsilon} \leq e^{(2-(l-k))(\sigma/|\epsilon|)r_b(\beta)}\frac{(l-k)^{l-k-1}}{(l-k-1)!}\left\|\mathcal{D}^{[k-j]}g_{k,j}\right\|_{\beta,\sigma,\epsilon},\end{gathered} \tag{2.44}$$

for $n = 0,1,2$ and all $l \geq k+3$. Now, we give estimates for $||\mathcal{D}^{[k-j]}g_{k,j}||_{\beta,\sigma,\epsilon}$. Using the Taylor formula with integral remainder, we have that

$$\left|\mathcal{D}^{[k-j]}g_{k,j}(t)\right| \leq \frac{1}{(k-j-1)!}\int_k^t (t-s)^{k-j-1}\left|h^{(k-j)}(s)\Delta_k(s)\right|ds, \tag{2.45}$$

and from the classical identity

$$\int_s^{+\infty}\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)t\right)(t-s)^{k-j-1}dt = \exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)s\right)\frac{(k-j)!}{((\sigma/|\epsilon|)r_b(\beta))^{k-j}} \tag{2.46}$$

we get from the Fubini theorem that

$$\begin{aligned}\left\|\mathcal{D}^{[k-j]}g_{k,j}(t)\right\|_{\beta,\sigma,\epsilon} &= \int_k^{+\infty}\left|\mathcal{D}^{[k-j]}g_{k,j}(t)\right|\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)t\right)dt \\ &\leq \int_k^{+\infty}\left(\int_s^{\infty}\frac{(t-s)^{k-j-1}}{(k-j-1)!}\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)t\right)dt\right)\left|h^{(k-j)}(s)\Delta_k(s)\right|ds \\ &= \left(\frac{1}{(\sigma/|\epsilon|)r_b(\beta)}\right)^{k-j}(k-j)\int_k^{\infty}\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)s\right)\left|h^{(k-j)}(s)\Delta_k(s)\right|ds.\end{aligned} \tag{2.47}$$

Again, we write

$$\int_k^{\infty} \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta) s\right) \left| h^{(k-j)}(s) \Delta_k(s) \right| ds$$
$$= \int_k^{\infty} \left| h^{(k-j)}(s) \right| \exp\left(-\frac{\sigma}{|\epsilon|} (r_b(\beta) - r_b(\beta - k_2)) s\right) |\Delta_k(s)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta - k_2) s\right) ds. \tag{2.48}$$

From the expression of h, we have that

$$\left| h^{(k-j)}(s) \right| \leq \frac{s_1! s^{s_1}}{s^{k-j}} \leq \frac{s_1! s^{s_1}}{k^{k-j}} \tag{2.49}$$

for all $s \geq k$, if $1 \leq k - j \leq s_1$, and $h^{(k-j)}(s) = 0$, if $k - j > s_1$. Using (2.49) in the right-hand side of the equality (2.48), we deduce from (2.47) that

$$\left\| \wp^{[k-j]} g_{k,j}(t) \right\|_{\beta,\sigma,\epsilon} \leq s_1! \frac{(k-j)}{k^{k-j}} \left(\frac{|\epsilon|}{\sigma r_b(\beta)} \right)^{k-j} \times \int_k^{k+1} s^{s_1} \exp\left(-\frac{\sigma}{|\epsilon|} (r_b(\beta) - r_b(\beta - k_2)) s\right)$$
$$\times |\Delta_k(s)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta - k_2) s\right) ds \tag{2.50}$$

if $1 \leq k - j \leq s_1$, and $\|\wp^{[k-j]} g_{k,j}(t)\|_{\beta,\sigma,\epsilon} = 0$ if $k - j > s_1$.

(3) We give estimates for $\sum_{j=0}^{k-1} k! \|\widetilde{\Delta}_{k+n,j}^{(k+n)}\|_{\beta,\sigma,\epsilon,d} / (j!(k-j)!)$, for $n = 0, 1, 2$. From the estimates (2.44) and (2.50), we get that

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \left\| \widetilde{\Delta}_{k+n,j}^{(k+n)} \right\|_{\beta,\sigma,\epsilon,d} \leq \sum_{j \geq 0, j \geq k - s_1}^{k-1} \frac{k!}{j!(k-j)!} s_1! \frac{(k-j)}{k^{k-j}} \left(\frac{|\epsilon|}{\sigma r_b(\beta)} \right)^{k-j}$$
$$\times \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^k \int_k^{k+1} s^{s_1} \exp\left(-\frac{\sigma}{|\epsilon|} (r_b(\beta) - r_b(\beta - k_2)) s\right)$$
$$\times |\Delta_k(s)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta - k_2) s\right) ds. \tag{2.51}$$

From (2.37) and (2.42), we deduce from (2.51) that

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \left\| \widetilde{\Delta}_{k+n,j}^{(k+n)} \right\|_{\beta,\sigma,\epsilon,d} \leq A_k 2^{s_1} \left(\frac{s_1 e^{-1}}{(1 - |\epsilon|/\sigma) k_2 \sigma} \right)^{s_1} |\epsilon|^{s_1} (\beta + 1)^{b s_1} \| f(t) \|_{\beta - k_2, \sigma, \epsilon, d}, \tag{2.52}$$

where

$$A_k = \sum_{j=k-s_1, j\geq 0}^{k-1} \frac{k! s_1!}{j!(k-j-1)! k^{k-j}} \left(\frac{|\epsilon|}{\sigma r_b(\beta)} \right)^{k-j} \tag{2.53}$$

for all $k \geq 1$, and $n = 0, 1, 2$. Now, we show that A_k, $k \geq 1$, is a bounded sequence. We have

$$A_k \leq s_1! \frac{k!}{k^k} \left(\sum_{m=0}^{s_1-1} \frac{k^{k-s_1+m}}{(k-s_1+m)!(s_1-m-1)!} \right) \tag{2.54}$$

for all $k \geq s_1$. From the Stirling formula which asserts that $k! \sim k^k e^{-k} (2\pi k)^{1/2}$ as $k \to +\infty$, we get a universal constant $C_1 > 0$ and a constant $C_2 > 0$ (depending on s_1, m) such that

$$\begin{gathered} \frac{k!}{k^k} \leq C_1 e^{-k} (2\pi k)^{1/2}, \\ \frac{k^{k-s_1+m}}{(k-s_1+m)!} \leq C_1 \frac{k! e^k}{(k-s_1+m)!(2\pi k)^{1/2} k^{s_1-m}} \leq C_2 \frac{e^k}{(2\pi k)^{1/2}} \end{gathered} \tag{2.55}$$

for all $k \geq 1$. From (2.54), (2.55), we get a constant $C_3 > 0$ (depending on s_1) such that

$$A_k \leq C_3 \tag{2.56}$$

for all $k \geq 1$.

(4) We give estimates for $\sum_{j=0}^{k-1} (k!/j!(k-j)!) \sum_{l=k+3}^{+\infty} \|\widetilde{\Delta}_{l,j}^{(l)}\|_{\beta,\sigma,\epsilon,d}$. From the estimates (2.44) and (2.50), we get that

$$\begin{aligned} \sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \sum_{l=k+3}^{+\infty} \left\| \widetilde{\Delta}_{l,j}^{(l)} \right\|_{\beta,\sigma,\epsilon,d} \leq & \sum_{j\geq 0, j\geq k-s_1}^{k-1} \frac{k!}{j!(k-j-1)!} \frac{s_1!}{k^{k-j}} \left(\frac{|\epsilon|}{\sigma r_b(\beta)} \right)^{k-j} \\ & \times \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^k \int_k^{k+1} s^{s_1} \exp\left(-\frac{\sigma}{|\epsilon|} (r_b(\beta) - r_b(\beta - k_2)) s \right) \\ & \times |\Delta_k(s)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta - k_2) s \right) ds \\ & \times \sum_{l=k+3}^{+\infty} \left(\frac{\sigma}{|\epsilon|} r_b(\beta) \right)^{l-k} \exp\left((2-(l-k)) \frac{\sigma}{|\epsilon|} r_b(\beta) \right) \frac{(l-k)^{l-k-1}}{(l-k-1)!}. \end{aligned} \tag{2.57}$$

Again from (2.37) and (2.42), we deduce from (2.57) that

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \sum_{l=k+3}^{+\infty} \left\| \widetilde{\Delta}_{l,j}^{(l)} \right\|_{\beta,\sigma,\epsilon,d} \leq B_k 2^{s_1} \left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\sigma) k_2 \sigma} \right)^{s_1} |\epsilon|^{s_1} (\beta+1)^{b s_1} \|f(t)\|_{\beta-k_2,\sigma,\epsilon,d}, \tag{2.58}$$

where $B_k = A_k \widetilde{A}_k$ and

$$\begin{aligned}\widetilde{A}_k &= \sum_{l=k+3}^{+\infty} \left(\frac{\sigma}{|\epsilon|} r_b(\beta)\right)^{l-k} \exp\left((2-(l-k))\frac{\sigma}{|\epsilon|} r_b(\beta)\right) \frac{(l-k)^{l-k-1}}{(l-k-1)!} \\ &= \sum_{h=3}^{+\infty} \left(\frac{\sigma}{|\epsilon|} r_b(\beta)\right)^{h} \exp\left((2-h)\frac{\sigma}{|\epsilon|} r_b(\beta)\right) \frac{h^{h-1}}{(h-1)!}\end{aligned} \tag{2.59}$$

for all $k \geq 1$. Now, we remind from (2.31) that $\widetilde{A}_k$ is a bounded sequence.

Finally, from (2.31), (2.36), (2.43), (2.52), (2.56), and (2.58), we deduce a constant $C_5 > 0$ (depending on s_1, σ) such that

$$\left\| h(t)\Delta_k^{(k)}(t) \right\|_{\beta,\sigma,\epsilon,d} \leq C_5 |\epsilon|^{s_1} (\beta+1)^{bs_1} \left\| \Delta_k^{(k)} \right\|_{\beta-k_2,\sigma,\epsilon,d}, \tag{2.60}$$

which gives the result. It remains to consider the case $k = 0$.

When $k = 0$, let $f(t) = \Delta_0(t) \in L^1(\mathbb{R}_+)$, with $\mathrm{supp}(\Delta_0) \in [0,1]$. By definition, we can write

$$\begin{aligned}\|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon,d} &= \|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon} \\ &= \int_0^1 \tau^{s_1} \exp\left(-\frac{\sigma}{|\epsilon|}(r_b(\beta) - r_b(\beta-k_2))\tau\right) |\Delta_0(\tau)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta-k_2)\tau\right) d\tau.\end{aligned} \tag{2.61}$$

Using (2.41), we deduce from (2.61) that

$$\begin{aligned}\|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon,d} &\leq \left(\frac{s_1 e^{-1}}{\sigma k_2}\right)^{s_1} |\epsilon|^{s_1} (\beta+1)^{bs_1} \int_0^1 |\Delta_0(\tau)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta-k_2)\tau\right) d\tau \\ &= \left(\frac{s_1 e^{-1}}{\sigma k_2}\right)^{s_1} |\epsilon|^{s_1} (\beta+1)^{bs_1} \|f(t)\|_{\beta-k_2,\sigma,\epsilon,d}.\end{aligned} \tag{2.62}$$

Hence there exists a constant $C_{5,1} > 0$ (depending on s_1, σ) such that

$$\left\| h(t) f(t) \right\|_{\beta,\sigma,\epsilon,d} \leq C_{5,1} |\epsilon|^{s_1} (\beta+1)^{bs_1} \left\| f(t) \right\|_{\beta-k_2,\sigma,\epsilon,d}, \tag{2.63}$$

which yields the result. □

Proposition 2.13. *Let $\sigma > \tilde{\sigma} > 0$ be real numbers such that*

$$\frac{3}{2}\frac{\sigma}{|\epsilon|} r_b(\beta) e^{1-(\sigma/|\epsilon|) r_b(\beta)} < 1, \quad |\epsilon| < \tilde{\sigma}. \tag{2.64}$$

Let $s_1 \geq 0$ be a nonnegative integer. Then, for all $f \in \mathfrak{D}'_{\beta,\tilde{\sigma},\epsilon}$, one has $r^{s_1} f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exists a constant $C_6 > 0$ (depending on s_1, σ, $\tilde{\sigma}$) such that

$$\left\| r^{s_1} f(r) \right\|_{\beta,\sigma,\epsilon,d} \leq C_6 |\epsilon|^{s_1} \left\| f(r) \right\|_{\beta,\tilde{\sigma},\epsilon,d} \tag{2.65}$$

for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$.

Proof. The line of reasoning will follow the proof of Proposition 2.12. We start from the identity (2.36).

(1) We first give estimates for $\|(h(t)\Delta_k(t))^{(k)}\|_{\beta,\sigma,\epsilon,d}$. We write

$$\begin{aligned} \left\| (h(t)\Delta_k(t))^{(k)} \right\|_{\beta,\sigma,\epsilon,d} &= \left(\frac{\sigma r_b(\beta)}{|\epsilon|} \right)^k \int_0^{+\infty} \tau^{s_1} |\Delta_k(\tau)| \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta)\tau \right) d\tau \\ &= \left(\frac{\tilde{\sigma} r_b(\beta)}{|\epsilon|} \right)^k \left(\frac{\sigma}{\tilde{\sigma}} \right)^k \int_k^{k+1} \tau^{s_1} \exp\left(-\frac{(\sigma - \tilde{\sigma})}{|\epsilon|} r_b(\beta)\tau \right) \\ &\quad \times |\Delta_k(\tau)| \exp\left(-\frac{\tilde{\sigma}}{|\epsilon|} r_b(\beta)\tau \right) d\tau \\ &\leq \widetilde{A}(\epsilon,\beta) \left(\frac{\tilde{\sigma} r_b(\beta)}{|\epsilon|} \right)^k \int_k^{k+1} |\Delta_k(\tau)| \exp\left(-\frac{\tilde{\sigma}}{|\epsilon|} r_b(\beta)\tau \right) d\tau, \end{aligned} \tag{2.66}$$

where

$$\widetilde{A}(\epsilon,\beta) = \sup_{k\geq 1} \left(\left(\frac{\sigma}{\tilde{\sigma}} \right)^k (k+1)^{s_1} \exp\left(-\frac{(\sigma - \tilde{\sigma})}{|\epsilon|} r_b(\beta) k \right) \right). \tag{2.67}$$

Now, we give estimates for $\widetilde{A}(\epsilon,\beta)$. We write

$$\begin{aligned} \left(\frac{\sigma}{\tilde{\sigma}} \right)^k (k+1)^{s_1} \exp\left(-\frac{(\sigma - \tilde{\sigma})}{|\epsilon|} r_b(\beta) k \right) &= (k+1)^{s_1} \exp\left(-k \frac{r_b(\beta)}{|\epsilon|} (\varphi(\sigma) - \varphi(\tilde{\sigma})) \right) \\ &\leq 2^{s_1} k^{s_1} \exp\left(-k \frac{r_b(\beta)}{|\epsilon|} (\varphi(\sigma) - \varphi(\tilde{\sigma})) \right), \end{aligned} \tag{2.68}$$

where $\varphi(x) = x - (|\epsilon|/r_b(\beta)) \log(x)$, for all $k \geq 1$. From the Taylor formula applied to φ on $[\tilde{\sigma}, \sigma]$, we get that

$$\varphi(\sigma) - \varphi(\tilde{\sigma}) \geq \left(1 - \frac{|\epsilon|}{\tilde{\sigma}} \right) (\sigma - \tilde{\sigma}). \tag{2.69}$$

From (2.68), (2.69), and (2.41), we deduce that

$$\widetilde{A}(\epsilon,\beta) \leq 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\widetilde{\sigma})(\sigma-\widetilde{\sigma})}\right)^{s_1}|\epsilon|^{s_1}. \tag{2.70}$$

From (2.66) and (2.70), we get that

$$\left\|(h(t)\Delta_k(t))^{(k)}\right\|_{\beta,\sigma,\epsilon,d} \leq 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\widetilde{\sigma})(\sigma-\widetilde{\sigma})}\right)^{s_1}|\epsilon|^{s_1}\|f(t)\|_{\beta,\widetilde{\sigma},\epsilon,d}. \tag{2.71}$$

(2) We give estimates for $||\widetilde{\Delta}_{l,j}||_{\beta,\sigma,\epsilon}$, for all $0 \leq j \leq k-1$, all $l \geq k$. We start from the formula (2.44) and (2.47). We write

$$\begin{aligned}
&\int_k^\infty \exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)s\right)\left|h^{(k-j)}(s)\Delta_k(s)\right|ds \\
&\quad= \int_k^\infty \left|h^{(k-j)}(s)\right|\exp\left(-\frac{(\sigma-\widetilde{\sigma})}{|\epsilon|}r_b(\beta)s\right)|\Delta_k(s)|\exp\left(-\frac{\widetilde{\sigma}}{|\epsilon|}r_b(\beta)s\right)ds.
\end{aligned} \tag{2.72}$$

We get that

$$\begin{aligned}
&\left\|\mathcal{D}^{[k-j]}g_{k,j}(t)\right\|_{\beta,\sigma,\epsilon} \\
&\quad\leq s_1!\frac{(k-j)}{k^{k-j}}\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{k-j}\int_k^{k+1} s^{s_1}\exp\left(-\frac{(\sigma-\widetilde{\sigma})}{|\epsilon|}r_b(\beta)s\right)|\Delta_k(s)|\exp\left(-\frac{\widetilde{\sigma}}{|\epsilon|}r_b(\beta)s\right)ds
\end{aligned} \tag{2.73}$$

if $1 \leq k-j \leq s_1$, and $||\mathcal{D}^{[k-j]}g_{k,j}(t)||_{\beta,\sigma,\epsilon} = 0$ if $k-j > s_1$.

(3) We give estimates for $\sum_{j=0}^{k-1} k!||\widetilde{\Delta}_{k+n,j}^{(k+n)}||_{\beta,\sigma,\epsilon,d}/(j!(k-j)!)$, for $n = 0,1,2$. From the estimates (2.44) and (2.73), we get that

$$\begin{aligned}
\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\left\|\widetilde{\Delta}_{k+n,j}^{(k+n)}\right\|_{\beta,\sigma,\epsilon,d} &\leq \sum_{j\geq 0, j\geq k-s_1}^{k-1}\frac{k!}{j!(k-j)!}s_1!\frac{(k-j)}{k^{k-j}}\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{k-j} \\
&\quad\times\left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k\int_k^{k+1} s^{s_1}\exp\left(-\frac{(\sigma-\widetilde{\sigma})}{|\epsilon|}r_b(\beta)s\right) \\
&\quad\times|\Delta_k(s)|\exp\left(-\frac{\widetilde{\sigma}}{|\epsilon|}r_b(\beta)s\right)ds.
\end{aligned} \tag{2.74}$$

From (2.66) and (2.70), we deduce from (2.74) that

$$\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\left\|\widetilde{\Delta}_{k+n,j}^{(k+n)}\right\|_{\beta,\sigma,\epsilon,d}\le A_k 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\widetilde{\sigma})(\sigma-\widetilde{\sigma})}\right)^{s_1}|\epsilon|^{s_1}\|f(t)\|_{\beta,\widetilde{\sigma},\epsilon,d}, \tag{2.75}$$

where A_k is the bounded sequence given in the proof of Proposition 2.12.

We give estimates for $\sum_{j=0}^{k-1}(k!/j!(k-j)!)\sum_{l=k+3}^{+\infty}\|\widetilde{\Delta}_{l,j}^{(l)}\|_{\beta,\sigma,\epsilon,d}$. From the estimates (2.44) and (2.73), we get that

$$\begin{aligned}\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\sum_{l=k+3}^{+\infty}\left\|\widetilde{\Delta}_{l,j}^{(l)}\right\|_{\beta,\sigma,\epsilon,d}\le{}&\sum_{j\ge0,j\ge k-s_1}^{k-1}\frac{k!}{j!(k-j-1)!}\frac{s_1!}{k^{k-j}}\left(\frac{|\epsilon|}{\sigma r_b(\beta)}\right)^{k-j}\\&\times\left(\frac{\sigma r_b(\beta)}{|\epsilon|}\right)^k\int_k^{k+1}s^{s_1}\exp\left(-\frac{(\sigma-\widetilde{\sigma})}{|\epsilon|}r_b(\beta)s\right)\\&\times|\Delta_k(s)|\exp\left(-\frac{\widetilde{\sigma}}{|\epsilon|}r_b(\beta)s\right)ds\\&\times\sum_{l=k+3}^{+\infty}\left(\frac{\sigma}{|\epsilon|}r_b(\beta)\right)^{l-k}\exp\left((2-(l-k))\frac{\sigma}{|\epsilon|}r_b(\beta)\right)\frac{(l-k)^{l-k-1}}{(l-k-1)!}.\end{aligned}\tag{2.76}$$

From (2.66) and (2.70), we deduce from (2.76), that

$$\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\sum_{l=k+3}^{+\infty}\left\|\widetilde{\Delta}_{l,j}^{(l)}\right\|_{\beta,\sigma,\epsilon,d}\le B_k 2^{s_1}\left(\frac{s_1 e^{-1}}{(1-|\epsilon|/\widetilde{\sigma})(\sigma-\widetilde{\sigma})}\right)^{s_1}|\epsilon|^{s_1}\|f(t)\|_{\beta,\widetilde{\sigma},\epsilon,d}, \tag{2.77}$$

where B_k is the bounded sequence given in the proof of Proposition 2.12.

Finally, from (2.31), (2.36), (2.56), (2.71), (2.75), and (2.77), we deduce a constant $C_6>0$ (depending on $s_1,\sigma,\widetilde{\sigma}$) such that

$$\left\|h(t)\Delta_k^{(k)}(t)\right\|_{\beta,\sigma,d}\le C_6|\epsilon|^{s_1}\left\|\Delta_k^{(k)}\right\|_{\beta,\widetilde{\sigma},\epsilon,d}, \tag{2.78}$$

which gives the result. It remains to consider the case $k=0$.

When $k=0$, let $f(t)=\Delta_0(t)\in L^1(\mathbb{R}_+)$, with $\operatorname{supp}(\Delta_0)\in[0,1]$. By definition, we can write

$$\begin{aligned}&\|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon,d}\\&\quad=\|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon}=\int_0^1\tau^{s_1}\exp\left(-\frac{(\sigma-\widetilde{\sigma})}{|\epsilon|}r_b(\beta)\tau\right)|\Delta_0(\tau)|\exp\left(-\frac{\widetilde{\sigma}}{|\epsilon|}r_b(\beta)\tau\right)d\tau.\end{aligned}\tag{2.79}$$

Using (2.41), we deduce from (2.79) that

$$\begin{aligned} \|h(t)\Delta_0(t)\|_{\beta,\sigma,\epsilon,d} &\le \left(\frac{s_1 e^{-1}}{\sigma-\tilde{\sigma}}\right)^{s_1}|\epsilon|^{s_1}\int_0^1 |\Delta_0(\tau)|\exp\left(-\frac{\tilde{\sigma}}{|\epsilon|}r_b(\beta)\tau\right)d\tau \\ &= \left(\frac{s_1 e^{-1}}{\sigma-\tilde{\sigma}}\right)^{s_1}|\epsilon|^{s_1}\left\|f(t)\right\|_{\beta,\tilde{\sigma},\epsilon,d}. \end{aligned} \tag{2.80}$$

Hence, there exists a constant $C_{6,1} > 0$ (depending on s_1, σ, $\tilde{\sigma}$) such that

$$\left\|h(t)f(t)\right\|_{\beta,\sigma,\epsilon,d} \le C_{6,1}|\epsilon|^{s_1}\left\|f(t)\right\|_{\beta,\tilde{\sigma},\epsilon,d}, \tag{2.81}$$

which yields the result. □

2.2. Banach Spaces of Formal Power Series with Coefficients in Spaces of Distributions

Definition 2.14. Let $\delta > 0$ be a real number. We denote by $\mathfrak{D}'(\sigma,\epsilon,\delta)$ the vector space of formal series $v(r,z) = \sum_{\beta\ge 0} v_\beta(r)z^\beta/\beta!$ such that $v_\beta(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$ for all $\beta \ge 0$ and

$$\|v(r,z)\|_{(\sigma,\epsilon,d,\delta)} := \sum_{\beta\ge 0}\left\|v_\beta(r)\right\|_{\beta,\sigma,\epsilon,d}\frac{\delta^\beta}{\beta!} \tag{2.82}$$

is finite. One can check that the normed space $(\mathfrak{D}'(\sigma,\epsilon,\delta), \|\cdot\|_{(\sigma,\epsilon,d,\delta)})$ is a Banach space.

In the next proposition, we study some parameter depending linear operators acting on the space $\mathfrak{D}'(\sigma,\epsilon,\delta)$.

Proposition 2.15. *Let $s_1, s_2, k_1, k_2 \ge 0$ be positive integers. Assume that the condition*

$$k_2 \ge b s_1 \tag{2.83}$$

holds. Then, if

$$|\epsilon| < \sigma, \qquad \frac{3(\sigma/|\epsilon|)\zeta(b)}{2}e^{1-\sigma/|\epsilon|} < 1, \tag{2.84}$$

the operator $\tau^{s_1}\partial_\tau^{-k_1}\partial_z^{-k_2}$ is a bounded linear operator from the space $(\mathfrak{D}'(\sigma,\epsilon,\delta), \|\cdot\|_{(\sigma,\epsilon,d,\delta)})$ into itself. Moreover, there exists a constant $C_7 > 0$ (depending on b, s_1, k_2, σ) such that

$$\left\|r^{s_1}\partial_r^{-k_1}\partial_z^{-k_2}v(r,z)\right\|_{(\sigma,\epsilon,d,\delta)} \le |\epsilon|^{s_1+k_1}C_7\delta^{k_2}\|v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \tag{2.85}$$

for all $v \in \mathfrak{D}'(\sigma,\epsilon,\delta)$.

Proof. Let $v(r,z) \in \mathfrak{D}'(\sigma,\epsilon,\delta)$. By definition, we have

$$\left\| r^{s_1}\partial_r^{-k_1}\partial_z^{-k_2}v(r,z)\right\|_{(\sigma,\epsilon,d,\delta)} = \sum_{\beta\geq k_2}\left\| r^{s_1}\partial_r^{-k_1}v_{\beta-k_2}(r)\right\|_{\beta,\sigma,\epsilon,d}\frac{\delta^{\beta}}{\beta!}. \tag{2.86}$$

From Corollary 2.10 and Proposition 2.12, we get a constant $C_{3,5} > 0$ (depending on s_1,σ) such that

$$\begin{aligned}\left\| r^{s_1}\partial_r^{-k_1}\partial_z^{-k_2}v(r,z)\right\|_{(\sigma,\epsilon,d,\delta)} \leq C_{3,5}\sum_{\beta\geq k_2}|\epsilon|^{s_1+k_1}(\beta+1)^{bs_1}\frac{(\beta-k_2)!}{\beta!}\\ \times\left\| v_{\beta-k_2}(r)\right\|_{\beta-k_2,\sigma,\epsilon,d}\delta^{k_2}\frac{\delta^{\beta-k_2}}{(\beta-k_2)!}.\end{aligned} \tag{2.87}$$

From the assumptions (2.83), we get a constant $C_{b,s_1,k_2} > 0$ (depending on b, s_1, k_2) such that

$$(\beta+1)^{bs_1}\frac{(\beta-k_2)!}{\beta!} \leq C_{b,s_1,k_2} \tag{2.88}$$

for all $\beta \geq k_2$. Finally, from the estimates (2.87) and (2.88), we get the inequality (2.85). □

In the next proposition, we study linear operators of multiplication by bounded holomorphic and C^{∞} functions.

Proposition 2.16. *For all $\beta \geq 0$, let $h_\beta(\tau)$ be a C^{∞} function with respect to r on $\mathbb{R}_+$ such that there exist $A, B, \rho, \mu > 0$ with*

$$\left|h_\beta^{(q)}(r)\right| \leq AB^{-\beta}\frac{\beta!q!}{(\rho(r+\mu))^{q+1}} \tag{2.89}$$

for all $r \in \mathbb{R}_+$. One consider the series

$$h(r,z) = \sum_{\beta\geq 0}h_\beta(r)\frac{z^{\beta}}{\beta!}, \tag{2.90}$$

which is convergent for all $|z| < B$, all $r \in \mathbb{R}_+$. Let $0 < \delta < B$. Then, if

$$|\epsilon| < \sigma, \quad |\epsilon| < \rho\sigma, \quad \frac{3(\sigma/|\epsilon|)\zeta(b)}{2}e^{1-\sigma/|\epsilon|} < 1, \tag{2.91}$$

the linear operator of multiplication by $h(r,z)$ is continuous from $(\mathfrak{D}'(\sigma,\epsilon,\delta), \|\cdot\|_{(\sigma,\epsilon,\delta)})$ into itself. Moreover, there exists a constant C_8 (depending on μ, ρ, B) such that

$$\|h(r,z)v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \leq C_8 A \|v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \tag{2.92}$$

for all $v(r,z) \in \mathfrak{D}'(\sigma,\epsilon,\delta)$ satisfying (2.91).

Proof. Let $v(r,z) = \sum_{\beta\geq 0} v_\beta(r) z^\beta/\beta! \in \mathfrak{D}'(\sigma,\epsilon,\delta)$. By definition, we have that

$$\|h(\tau,z)v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \leq \sum_{\beta\geq 0}\left(\sum_{\beta_1+\beta_2=\beta} \|h_{\beta_1}(r)v_{\beta_2}(r)\|_{\beta,\sigma,\epsilon,d}\frac{\beta!}{\beta_1!\beta_2!}\right)\frac{\delta^\beta}{\beta!}. \tag{2.93}$$

From Proposition 2.11 and Remark 2.4, we deduce that there exists $C_4 > 0$ (depending on μ,ρ) such that

$$\left\|h_{\beta_1}(r)v_{\beta_2}(r)\right\|_{\beta,\sigma,\epsilon,d} \leq C_4 A B^{-\beta_1}\beta_1! \left\|v_{\beta_2}(r)\right\|_{\beta,\sigma,\epsilon,d} \leq C_4 A B^{-\beta_1}\beta_1! \left\|v_{\beta_2}(r)\right\|_{\beta_2,\sigma,\epsilon,d} \tag{2.94}$$

for all $\beta_1,\beta_2 \geq 0$ such that $\beta_1+\beta_2=\beta$. From (2.93) and (2.94), we deduce that

$$\|h(r,z)v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \leq C_4 A\left(\sum_{\beta\geq 0}\left(\frac{\delta}{B}\right)^\beta\right)\|v(r,z)\|_{(\sigma,\epsilon,d,\delta)} \tag{2.95}$$

which yields (2.92). □

2.3. Cauchy Problems in Analytic Functions Spaces with Dependence on Initial Data

In this section, we recall the well-know-Cauchy Kowalevski theorem in some spaces of analytic functions for which the dependence on the coefficients and initial data can be obtained.

The following Banach spaces were used in [26].

Definition 2.17. Let T,X be real numbers such that $T,X>0$. We define a vector space $G(T,X)$ of holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$. A formal series $U(t,x) \in \mathbb{C}[[t,x]]$,

$$U(t,x) = \sum_{l,\beta\geq 0} u_{l,\beta}\frac{t^l}{l!}\frac{x^\beta}{\beta!} \tag{2.96}$$

belongs to $G(T,X)$ if the series

$$\sum_{l,\beta\geq 0}\frac{|u_{l,\beta}|}{(l+\beta)!}T^l X^\beta \tag{2.97}$$

converges. We also define a norm on $G(T,X)$ as

$$\|U(t,x)\|_{(T,X)} = \sum_{l,\beta\geq 0} \frac{|u_{l,\beta}|}{(l+\beta)!} T^l X^\beta. \tag{2.98}$$

One can easily show that $(G(T,X), \|\cdot\|_{(T,X)})$ is a Banach space.

Remark 2.18. Let $U(t,x)$ be in $G(T_0,X_0)$ for given $T_0, X_0 > 0$. Then, $U(t,x)$ also belongs to the spaces $G(T,X)$ for all $T \leq T_0$ and $X \leq X_0$. Moreover, the maps $T \to \|U(t,x)\|_{(T,X)}$ and $X \to \|U(t,x)\|_{(T,X)}$ are increasing functions from $[0,T_0]$ (resp., $[0,X_0]$) into $\mathbb{R}_+$.

We depart from some preliminary lemma from [26]. In the following, for $u(t,x) \in \mathbb{C}[[t,x]]$, we denote by $\partial_x^{-1} u(t,x)$ the formal series $\int_0^x u(t,\tau)d\tau$.

Lemma 2.19. *Let $h_0, h_1 \in \mathbb{N}$ such that $h_0 \leq h_1$. The operator $\partial_t^{h_0}\partial_x^{-h_1}$ is a bounded linear operator from $(G(T,X), \|\cdot\|_{(T,X)})$ into itself. Moreover, there exists a universal constant $C_{10} > 0$ such that the estimates*

$$\left\|\partial_t^{h_0}\partial_x^{-h_1} U(t,x)\right\|_{(T,X)} \leq C_{10} T^{-h_0} X^{h_1} \|U(t,x)\|_{(T,X)} \tag{2.99}$$

hold for all $U(t,x) \in G(T,X)$.

Lemma 2.20. *Let $A(t,x) = \sum_{l,\beta\geq 0} a_{l,\beta} t^l x^\beta / l!\beta!$ be an analytic function on an open polydisc containing $D(0,T)\times D(0,X)$ and let $U(t,x)$ be in $G(T,X)$. Then, the product $A(t,x)U(t,x)$ belongs to $G(T,X)$. Moreover,*

$$\|A(t,x)U(t,x)\|_{(T,X)} \leq |A|(T,X)\|U(t,x)\|_{(T,X)} \tag{2.100}$$

where $|A|(T,X) = \sum_{l,\beta\geq 0} |a_{l,\beta}| T^l X^\beta / l!\beta!$.

Proof. Let

$$U(t,x) = \sum_{l,\beta\geq 0} u_{l,\beta} \frac{t^l}{l!} \frac{x^\beta}{\beta!}. \tag{2.101}$$

We have

$$A(t,x)U(t,x) = \sum_{l,\beta\geq 0} v_{l,\beta} \frac{t^l}{l!} \frac{x^\beta}{\beta!}, \tag{2.102}$$

where

$$v_{l,\beta} = \sum_{l_1+l_2=l} \sum_{\beta_1+\beta_2=\beta} \left(\frac{a_{l_1,\beta_1}}{l_1!\beta_1!} \frac{u_{l_2,\beta_2}}{l_2!\beta_2!} \beta! l! \right) \tag{2.103}$$

for all $l,\beta \geq 0$. By definition, we have

$$|A|(T,X)\|U_2(t,x)\|_{(T,X)} = \sum_{l,\beta\geq 0}\left(\sum_{l_1+l_2=l}\sum_{\beta_1+\beta_2=\beta}\frac{|a_{l_1,\beta_1}||u_{l_2,\beta_2}|}{l_1!\beta_1!(l_2+\beta_2)!}\right)T^lX^\beta,$$
$$\|A(t,x)U(t,x)\|_{(T,X)} = \sum_{l,\beta\geq 0}\left|\sum_{l_1+l_2=l}\sum_{\beta_1+\beta_2=\beta}\frac{a_{l_1,\beta_1}u_{l_2,\beta_2}l!\beta!}{l_1!\beta_1!l_2!\beta_2!}\right|\frac{T^lX^\beta}{(l+\beta)!}. \tag{2.104}$$

On the other side, the next inequalities are well known:

$$\frac{l!\beta!}{l_1!\beta_1!l_2!\beta_2!} \leq \frac{(l+\beta)!}{(l_1+\beta_1)!(l_2+\beta_2)!} \leq \frac{(l+\beta)!}{l_1!\beta_1!(l_2+\beta_2)!} \tag{2.105}$$

for all $l_1,l_2 \geq 0$ such that $l_1+l_2=l$ and $\beta_1,\beta_2 \geq 0$ such that $\beta_1+\beta_2=\beta$.

Finally, from (2.105), we deduce that $\|A(t,x)U(t,x)\|_{(T,X)}$ converges and that the estimates (2.100) hold. □

Lemma 2.21. *Let $h_1,h_2 \in \mathbb{N}$ and let $U(t,x)$ be in $G(T_0,X_0)$ for given $T_0,X_0>0$. Then, there exist $T,X>0$ small enough (depending on T_0,X_0) such that the formal series $\partial_t^{h_1}\partial_x^{h_2}U(t,x)$ belongs to $G(T,X)$. Moreover, there exists a constant $C_{11}>0$ (depending on h_1,h_2) such that*

$$\left\|(\partial_t^{h_1}\partial_x^{h_2}U)(t,x)\right\|_{(T,X)} \leq C_{11}T^{-h_1}X^{-h_2}\|U(t,x)\|_{(T_0,X_0)} \tag{2.106}$$

for all $U(t,x) \in G(T_0,X_0)$.

Let $\mathcal{C}_1$ be a finite subset of $\mathbb{N}^2$. For all $(l_0,l_1) \in \mathcal{C}_1$, let $c_{l_0,l_1}(t,x) = \sum_{l,\beta\geq 0} c_{l_0,l_1,l,\beta}t^lx^\beta/l!\beta!$ be analytic functions on some polydisc containing the closed polydisc $\overline{D}(0,T_0)\times\overline{D}(0,X_0)$ for some $T_0,X_0>0$. As in Lemma 2.20, we define

$$|c_{l_0,l_1}|(t,x) = \sum_{l,\beta\geq 0}\frac{|c_{l_0,l_1,l,\beta}|t^lx^\beta}{l!\beta!}, \tag{2.107}$$

which converges on $\overline{D}(0,T_0)\times\overline{D}(0,X_0)$. We also consider $d(t,x) \in G(T_d,X_d)$, for some $T_d,X_d>0$. The following proposition holds.

Proposition 2.22. *Let $S \geq 1$ be an integer. One make the following assumptions. For all $(l_0,l_1) \in \mathcal{C}_1$, one has*

$$S > l_1, \qquad S \geq l_0 + l_1. \tag{2.108}$$

One consider the following Cauchy problem:

$$\partial_x^S U(t,x) = \sum_{(l_0,l_1)\in\mathcal{C}_1} c_{l_0,l_1}(t,x)\partial_t^{l_0}\partial_x^{l_1}U(t,x) + d(t,x) \tag{2.109}$$

for the given initial conditions

$$\left(\partial_x^j U\right)(t,0) = U_j(t), \quad 0 \le j \le S-1, \tag{2.110}$$

which are analytic functions on some disc *containing the closed* disc$\overline{D}(0,T_0)$. *If* $U_j(t) = \sum_{l\ge 0} U_{j,l} t^l/l!$, *we define* $|U_j|(t) = \sum_{l\ge 0} |U_{j,l}| t^l/l!$, *which converges for all* $t \in \overline{D}(0,T_0)$.

Then, there exist $T_1 > 0$ *with* $0 < T_1 < \min(T_0, T_d)$ *(depending on* T_0,T_d,$\mathcal{C}_1$*) and* $X_1 > 0$ *with* $0 < X_1 < \min(X_0, X_d)$ *(depending on* S, T_0, $\mathcal{C}_1$, $\max_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_0,X_0)$*) such that the problem* (2.109), (2.110) *has a unique formal solution* $U(t,x) \in G(T_1,X_1)$. *Moreover, there exist constants* $C_{12,1}, C_{12,2}, C_{12,3} > 0$ *(depending on* S,T_0,X_0,$\mathcal{C}_1$*) such that*

$$\|U(t,x)\|_{(T_1,X_1)} \le \max_{0\le j\le S-1} \left|U_j\right|(T_0)\left(C_{12,1} \max_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_0,X_0) + C_{12,2}\right) + C_{12,3}\|d(t,x)\|_{(T_d,X_d)}. \tag{2.111}$$

Proof. We denote by $\mathcal{D}$ the linear operator from $\mathbb{C}[[t,x]]$ into itself defined by

$$\mathcal{D}(H(t,x)) := \partial_x^S H(t,x) - \sum_{(l_0,l_1)\in\mathcal{C}_1} c_{l_0,l_1}(t,x)\partial_t^{l_0}\partial_x^{l_1} H(t,x), \tag{2.112}$$

and $\mathcal{A}$ denotes the linear map from $\mathbb{C}[[t,x]]$ into itself:

$$\mathcal{A}(H(t,x)) := \sum_{(l_0,l_1)\in\mathcal{C}_1} c_{l_0,l_1}(t,x)\partial_t^{l_0}\partial_x^{l_1-S} H(t,x) \tag{2.113}$$

for all $H(t,x) \in \mathbb{C}[[t,x]]$. By construction, we have that $\mathcal{D} \circ \partial_x^{-S} = \mathrm{id} - \mathcal{A}$, where id represents the identity map $H \mapsto H$ from $\mathbb{C}[[t,x]]$ into itself.

Now, we show that for any given $T_1 > 0$ such that $0 < T_1 \le T_0$, there exists $X_{\mathcal{A},T_1} > 0$ with $0 < X_{\mathcal{A},T_1} \le X_0$ (depending on S, T_1, $\mathcal{C}_1$, $\max_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_0,X_0)$) such that $\mathrm{id} - \mathcal{A}$ is an invertible map from $G(T_1,X)$ into itself for all $0 < X \le X_{\mathcal{A},T_1}$. Moreover, the following inequality

$$\left\|(\mathrm{id} - \mathcal{A})^{-1} C(t,x)\right\|_{(T_1,X)} \le 2\|C(t,x)\|_{(T_1,X)} \tag{2.114}$$

holds for all $C(t,x) \in G(T_1,X)$, for any $0 < X \le X_{\mathcal{A},T_1}$. Indeed, from the assumption (2.108) and Lemmas 2.19 and 2.20, we get a universal constant $C_{10,1} > 0$ such that

$$\begin{aligned}
\|\mathcal{A}(C(t,x))\|_{(T_1,X)} &\le C_{10,1}\left(\sum_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_1,X) T_1^{-l_0} X^{S-l_1}\right)\|C(t,x)\|_{(T_1,X)} \\
&\le C_{10,1} \max_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_0,X_0)\left(\sum_{(l_0,l_1)\in\mathcal{C}_1} T_1^{-l_0} X_{\mathcal{A},T_1}^{S-l_1}\right)\|C(t,x)\|_{(T_1,X)} \\
&:= N_{T_1,X_{\mathcal{A},T_1}}\|C(t,x)\|_{(T_1,X)}
\end{aligned} \tag{2.115}$$

for all $C(t,x) \in G(T_1, X)$. Since $S > l_1$, for all $(l_0, l_1) \in \mathcal{C}_1$, for the given $T_1 > 0$ one can choose $X_{\mathcal{A},T_1}$ small enough such that $N_{T_1,X_{\mathcal{A},T_1}} \leq 1/2$. Therefore, the inequality (2.114) holds.

Let $w(t,x) = \sum_{j=0}^{S-1} U_j(t)x^j/j!$. From the hypothesis (2.110), we deduce that $\mathcal{D}(w(t,x))$ and $w(t,x)$ belong to $G(T_1, X_0)$, for some $0 < T_1 < T_0$ (depending on $\mathcal{C}_1$, T_0). Indeed, from Lemmas 2.20 and 2.21 we get constants $C_{11,1} > 0$, $0 < T_1 < T_0$ (depending on $\mathcal{C}_1$, T_0) such that

$$
\begin{aligned}
\|\mathcal{D}(w(t,x))\|_{(T_1,X_0)} &\leq \sum_{(l_0,l_1)\in\mathcal{C}_1} |c_{l_0,l_1}|(T_1,X_0)\left(\sum_{j=0}^{S-1-l_1}\left\|\partial_t^{l_0}U_{j+l_1}(t)\right\|_{(T_1,X_0)}\frac{X_0^j}{j!}\right)\\
&\leq C_{11,1}\sum_{(l_0,l_1)\in\mathcal{C}_1}|c_{l_0,l_1}|(T_1,X_0)T_1^{-l_0}\left(\sum_{j=0}^{S-1-l_1}\|U_{j+l_1}(t)\|_{(T_0,X_0)}\frac{X_0^j}{j!}\right)\\
&\leq C_{11,1}\sum_{(l_0,l_1)\in\mathcal{C}_1}|c_{l_0,l_1}|(T_1,X_0)T_1^{-l_0}\left(\sum_{j=0}^{S-1-l_1}|U_{j+l_1}|(T_0)\frac{X_0^j}{j!}\right)\\
&\leq C_{11,1}\max_{(l_0,l_1)\in\mathcal{C}_1}|c_{l_0,l_1}|(T_0,X_0)\max_{0\leq j\leq S-1}|U_j|(T_0)\\
&\quad\times\sum_{(l_0,l_1)\in\mathcal{C}_1}T_1^{-l_0}\left(\sum_{j=0}^{S-1-l_1}\frac{X_0^j}{j!}\right),
\end{aligned}
\tag{2.116}
$$

$$
\begin{aligned}
\|(w(t,x))\|_{(T_1,X_0)} &\leq \sum_{j=0}^{S-1}\|U_j(t)\|_{(T_1,X_0)}\frac{X_0^j}{j!} \leq \sum_{j=0}^{S-1}|U_j|(T_1)\frac{X_0^j}{j!}\\
&\leq \max_{0\leq j\leq S-1}|U_j|(T_0)\sum_{j=0}^{S-1}\frac{X_0^j}{j!}.
\end{aligned}
\tag{2.117}
$$

Now, for this constructed $T_1 > 0$ satisfying (2.116) that we choose in such a way that $T_1 < T_d$ also holds, we select $X_1 > 0$ such that $0 < X_1 < \min(X_{\mathcal{A},T_1}, X_d)$. From the estimates (2.116) and Remark 2.4, we deduce that $\mathcal{D}(w(t,x))$, $w(t,x)$, and $d(t,x)$ belong to $G(T_1, X_1)$. From (2.114), we deduce the existence of a unique $H(t,x) \in G(T_1, X_1)$ such that

$$
\left(\mathcal{D}\circ\partial_x^{-S}\right)H(t,x) = -\mathcal{D}(w(t,x)) + d(t,x). \tag{2.118}
$$

Now, we put $U(t,x) = \partial_x^{-S}H(t,x) + w(t,x)$. By Lemma 2.19, we deduce that $U(t,x) \in G(T_1, X_1)$ and solves the problem (2.109), (2.110). Moreover, from (2.114) and (2.116), we get constants $C_{12,1}, C_{12,2}, C_{12,3} > 0$ (depending on S, T_0, X_0, $\mathcal{C}_1$) such that (2.111) holds, which yields the result. □

3. Laplace Transform on the Spaces $\mathfrak{D}'(\sigma, \epsilon, \delta)$

We first introduce the definition of Laplace transform of a staircase distribution.

Proposition 3.1. (1) *Let $\beta \geq 0$ be an integer, $\sigma > 0$ a real number, and $\epsilon \in \mathcal{E}$. Let*

$$f(r) = \sum_{k=0}^{+\infty} (\Delta_k(r))^{(k)} \in \mathfrak{D}'_{\beta,\sigma,\epsilon} \tag{3.1}$$

and choose $\theta \in [-\pi, \pi)$. Then, there exist $\rho_\theta > 0$, $\rho > 0$ such that the function

$$\mathcal{L}_\theta(f)(t) = \sum_{k=0}^{+\infty} \left(\frac{e^{i\theta}}{t}\right)^{k+1} \int_0^\infty \Delta_k(f)(r) \exp\left(-\frac{re^{i\theta}}{t}\right) dr \tag{3.2}$$

is holomorphic on the sector $S_{\theta,\rho_\theta,|\epsilon|\rho} = \{t \in \mathbb{C}^* / |\theta - \arg(t)| < \rho_\theta, |t| < |\epsilon|\rho\}$ for all $\epsilon \in \mathcal{E}$. Moreover, for all compacts $K \subset S_{\theta,\rho_\theta,|\epsilon|\rho}$, there exists $C_K > 0$ (depending on K and σ) such that

$$\left|\mathcal{L}_\theta(f)(t)\right| \leq C_K \|f\|_{\beta,\sigma,\epsilon,d} \tag{3.3}$$

for all $t \in K$.

(2) *Let $\delta > 0$ and let $f(r,z) = \sum_{\beta \geq 0} f_\beta(r) z^\beta / \beta! \in \mathfrak{D}'(\sigma, \epsilon, \delta)$. We define the Laplace transform of $f(r,z)$ in direction $\theta \in [-\pi, \pi)$ to be the function*

$$\mathcal{L}_\theta(f(r,z))(t) = \sum_{\beta \geq 0} \frac{\mathcal{L}_\theta(f_\beta)(t) z^\beta}{\beta!}, \tag{3.4}$$

which defines a holomorphic function on $S_{\theta,\rho_\theta,|\epsilon|\rho} \times D(0,\delta)$, for some $\rho_\theta > 0$, $\rho > 0$, for all $\epsilon \in \mathcal{E}$. Moreover, for all compacts $K \subset S_{\theta,\rho_\theta,|\epsilon|\rho}$, there exists $C_K > 0$ (depending on K and σ) such that

$$\left|\mathcal{L}_\theta(f(r,z))(t)\right| \leq C_K \|f(r,z)\|_{(\sigma,\epsilon,d,\delta)} \tag{3.5}$$

for all $(t,z) \in K \times D(0,\delta)$.

Proof. We prove part (1). The second part (2) is a direct application of (1). We have

$$\begin{aligned} \left|\mathcal{L}_\theta(f)(t)\right| \leq \sum_{k=0}^{+\infty} \frac{1}{|t|^{k+1}} \int_0^{+\infty} &\left|\Delta_k(f)(r)\right| \exp\left(-\frac{\sigma r_b(\beta)}{|\epsilon|} r\right) \\ &\times \exp\left(-r\left(\frac{\cos(\theta - \arg(t))}{|t|} - \frac{\sigma}{|\epsilon|} r_b(\beta)\right)\right) dr. \end{aligned} \tag{3.6}$$

We choose $\delta_1 > 0$ and $\rho_\theta > 0$ such that $\cos(\theta - \arg(t)) > \delta_1$ for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. Moreover, we choose $0 < \delta_2 < \delta_1$ and $\rho > 0$ such that

$$|t| < |\epsilon| \frac{\delta_1 - \delta_2}{\sigma r_b(\beta)}, \qquad \frac{|\epsilon| e^{-\delta_2/|t|}}{|t| \sigma r_b(\beta)} < 1 \tag{3.7}$$

for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. Let $k \geq 0$ an integer, for $r \in [k, k+1]$, we get that

$$\exp\left(-r\left(\frac{\cos(\theta - \arg(t))}{|t|} - \frac{\sigma}{|\epsilon|}r_b(\beta)\right)\right) \leq \exp\left(-\frac{k\delta_2}{|t|}\right). \tag{3.8}$$

We deduce that for $k = 0$,

$$\begin{aligned}
&\frac{1}{|t|}\int_0^{+\infty} |\Delta_0(f)(r)| \exp\left(-\frac{\sigma r_b(\beta)}{|\epsilon|}r\right) \times \exp\left(-r\left(\frac{\cos(\theta - \arg(t))}{|t|} - \frac{\sigma}{|\epsilon|}r_b(\beta)\right)\right)dr \\
&\quad \leq \frac{1}{|t|}\|\Delta_0(f)(r)\|_{\beta,\sigma,\epsilon}
\end{aligned} \tag{3.9}$$

and for $k \geq 1$,

$$\begin{aligned}
&\frac{1}{|t|^{k+1}}\int_0^{+\infty} |\Delta_k(f)(r)| \exp\left(-\frac{\sigma r_b(\beta)}{|\epsilon|}r\right) \times \exp\left(-r\left(\frac{\cos(\theta - \arg(t))}{|t|} - \frac{\sigma}{|\epsilon|}r_b(\beta)\right)\right)dr \\
&\quad \leq \frac{1}{|t|}\left(\frac{|\epsilon|e^{-\delta_2/|t|}}{|t|\sigma r_b(\beta)}\right)^k \left(\frac{\sigma}{|\epsilon|}r_b(\beta)\right)^k \|\Delta_k(f)(r)\|_{\beta,\sigma,\epsilon} \\
&\quad \leq \frac{|\epsilon|e^{-\delta_2/|t|}}{|t|^2\sigma r_b(\beta)}\left(\frac{\sigma}{|\epsilon|}r_b(\beta)\right)^k \|\Delta_k(f)(r)\|_{\beta,\sigma,\epsilon}.
\end{aligned} \tag{3.10}$$

From the estimates (3.9) and (3.10) we get the inequality (3.3). □

In the next proposition, we show that if f is a function, then the Laplace transform of f introduced in Proposition 3.1 coincides with the classical one.

Proposition 3.2. *Let $f(r) \in L_{\beta,\sigma/2,\epsilon}$. Then, from Proposition 2.7, one knows that $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. The Laplace transform $\mathcal{L}_\theta(f)(t)$ coincides with the classical Laplace transform of f in the direction θ defined by*

$$T_\theta(f)(t) = \frac{e^{i\theta}}{t}\int_0^{+\infty} f(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr \tag{3.11}$$

for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$.

Proof. From Proposition 2.7, the staircase decomposition of $f = \sum_{k\geq 0}(\Delta_k(f))^{(k)}$ has the following form $\Delta_k(r) = G_k(r)1_{[k,k+1]}$, with $G_k = \mathcal{P}(G_{k-1}1_{[k,+\infty)})$ and $G_0(r) = f(r)$ for all $k \geq 0$. We have to compute the integrals

$$A_k = \frac{(e^{i\theta})^{k+1}}{t^{k+1}}\int_k^{k+1} \Delta_k(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr \tag{3.12}$$

for all $k \geq 0$. For $k = 0$, we have that

$$A_0 = \frac{e^{i\theta}}{t} \int_0^1 f(r) \exp\left(-\frac{re^{i\theta}}{t}\right) dr. \tag{3.13}$$

For $k = 1$, by one integration by parts, we get that

$$A_1 = -\frac{e^{i\theta}}{t}\left[G_1(r)\exp\left(-\frac{re^{i\theta}}{t}\right)\right]_1^2 + \frac{e^{i\theta}}{t}\int_1^2 f(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr, \tag{3.14}$$

and using successive integrations by parts, we get that

$$A_k = \sum_{m=1}^{k} -\left(\frac{e^{i\theta}}{t}\right)^m \left[G_m(r)\exp(-\frac{re^{i\theta}}{t})\right]_k^{k+1} + \frac{e^{i\theta}}{t}\int_k^{k+1} f(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr \tag{3.15}$$

for all $k \geq 1$. On the other hand, from the hypothesis that $f(r) \in L_{\beta,\sigma/2,\epsilon}$ and from the fact that $G_m(r) = 0$ for all $r \leq m$, we have that the next telescopic sum

$$\sum_{k=1}^{+\infty} -\left(\frac{e^{i\theta}}{t}\right)^m \left[G_m(r)\exp(-\frac{re^{i\theta}}{t})\right]_k^{k+1} \tag{3.16}$$

is convergent and equal to zero for all $m \geq 1$. Finally, we deduce that $\sum_{k\geq 0} A_k = T_\theta(f)(t)$. □

In the next proposition, we describe the action of multiplication by a polynomial and derivation on the Laplace transform.

Proposition 3.3. *Let $f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$. Then, the following relations*

$$\mathcal{L}_\theta\left(e^{i\theta}\partial_r^{-1} f\right)(t) = t\mathcal{L}_\theta(f)(t), \qquad \mathcal{L}_\theta\left(e^{i\theta} r f(r)\right)(t) = \left(t^2\partial_t + t\right)\mathcal{L}_\theta(f)(t) \tag{3.17}$$

hold for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. Let $s, k_0 \geq 0$ be two integers such that $s \geq 2k_0$. Then, there exist a finite subset $\mathcal{O}_{s,k_0} \subset \mathbb{N}^2$ such that for all $(q,p) \in \mathcal{O}_{s,k_0}$, $q + p = s - k_0$ and integers $\alpha_{q,p}^{s,k_0} \in \mathbb{Z}$, for $(q,p) \in \mathcal{O}_{s,k_0}$ (depending on s, k_0) such that

$$t^s\partial_t^{k_0}\mathcal{L}_\theta(f)(t) = \mathcal{L}_\theta\left(e^{i(s-k_0)\theta}\sum_{(q,p)\in\mathcal{O}_{s,k_0}} \alpha_{q,p}^{s,k_0} r^q \partial_r^{-p} f(r)\right)(t) \tag{3.18}$$

for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$.

Proof. First of all, we have to check that the relations (3.17) and (3.18) hold when $f \in \mathfrak{D}(\mathbb{R}_+)$. Since $\mathfrak{D}(\mathbb{R}_+)$ is dense in $(\mathfrak{D}'_{\beta,\sigma,\epsilon}, \|\cdot\|_{\beta,\sigma,\epsilon,d})$, from the inequality (3.3) and with the help of Corollary 2.10 and Proposition 2.12, we will get that (3.17) and (3.18) hold for all $f \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$.

Now, let $f \in \mathfrak{D}(\mathbb{R}_+)$. The first relation of (3.17) is obtained by integrating once by parts and the second formula of (3.17) is a consequence of the equality

$$\partial_t\left(\frac{e^{i\theta}}{t}\int_0^{+\infty} f(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr\right) = -\frac{e^{i\theta}}{t^2}\int_0^{+\infty} f(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr + \frac{e^{2i\theta}}{t^3}\int_0^{+\infty} rf(r)\exp\left(-\frac{re^{i\theta}}{t}\right)dr \tag{3.19}$$

for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. To get the formula (3.18), we first show the following relation:

$$\partial_t(\mathcal{L}_\theta(f(r))(t) = \mathcal{L}_\theta\left(e^{-i\theta}\left(r\partial_r^2 + \partial_r\right)f(r)\right)(t) \tag{3.20}$$

for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. Indeed, using one integration by parts, we get that

$$\mathcal{L}_\theta\left(e^{-i\theta}\left(r\partial_r^2 + \partial_r\right)f(r)\right)(t) = \frac{e^{i\theta}}{t^2}\int_0^{+\infty} \partial_r f(r) r\exp\left(-\frac{re^{i\theta}}{t}\right)dr. \tag{3.21}$$

By a second integration by parts on the right-hand side of (3.21) and by comparison with (3.19), we get (3.20). Now, let $s, k_0 \in \mathbb{N}$ be such that $s \geq 2k_0$. Applying the first relation of (3.17) and (3.20), we get that

$$t^s\partial_t^{k_0}\mathcal{L}_\theta(f)(t) = \mathcal{L}_\theta\left(e^{i(s-k_0)\theta}\partial_r^{-s}\left(r\partial_r^2 + \partial_r\right)^{(k_0)}f(r)\right)(t). \tag{3.22}$$

Now, we recall a variant of Lemmas 5 and 6 in [2].

Lemma 3.4. *For all $k_0 \geq 1$, there exist constants $a_{k,k_0} \in \mathbb{N}$, $k_0 \leq k \leq 2k_0$ such that*

$$\left(r\partial_r^2 + \partial_r\right)^{k_0}u(r) = \sum_{k=k_0}^{2k_0} a_{k,k_0}r^{k-k_0}\partial_r^k u(r) \tag{3.23}$$

for all C^∞ functions $u : \mathbb{R}_+ \to \mathbb{C}$.

Lemma 3.5. *Let $a, b, c \geq 0$ be positive integers such that $a \geq b$ and $a \geq c$. We put $\delta = a + b - c$. Then, for all C^∞ function $u : \mathbb{R}_+ \to \mathbb{C}$, the function $\partial_r^{-a}(r^b\partial_r^c u(r))$ can be written in the form*

$$\partial_r^{-a}\left(r^b\partial_r^c u(r)\right) = \sum_{(b',c')\in\mathcal{O}_\delta} \alpha_{b',c'}r^{b'}\partial_r^{c'}u(r), \tag{3.24}$$

where $\mathcal{O}_\delta$ is a finite subset of $\mathbb{Z}^2$ such that for all $(b',c') \in \mathcal{O}_\delta$, $b'-c' = \delta$, $b' \geq 0$, $c' \leq 0$, and $\alpha_{b',c'} \in \mathbb{Z}$.

Finally, we observe that the relation (3.18) follows from (3.22) and Lemmas 3.4 and 3.5. □

The next proposition can be found in [25, Appendix A], see also [8].

Proposition 3.6. *Let $\alpha \geq 1$ and $f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$ with $|\epsilon| < \sigma r_b(\beta)$. Then, for every $l \geq 0$, the expression $(f(t-\alpha l)1_{[\alpha l,+\infty)})^{(l)}$ belongs to $\mathfrak{D}'_{\beta,\sigma,\epsilon}$. Moreover, there exist a universal constant $A > 0$ and $B(\sigma, b, \epsilon) > 0$ (depending on σ, b, ϵ) such that*

$$\left\| \left(f(t-\alpha l)1_{[\alpha l,+\infty)} \right)^{(l)} \right\|_{\beta,\sigma,\epsilon,d} \leq A(B(\sigma,b,\epsilon))^{l} \| f(r) \|_{\beta,\sigma,\epsilon,d}, \tag{3.25}$$

with $B(\sigma, b, \epsilon) \to 0$ when $\epsilon \to 0$.

In the forthcoming proposition, we explain the action of multiplication by an exponential function on the Laplace transform.

Proposition 3.7. *Let $\alpha \geq 1$ and $f(r) \in \mathfrak{D}'_{\beta,\sigma,\epsilon}$ with $|\epsilon| < \sigma r_b(\beta)$. From the latter proposition, one knows that $F_l(r) = (f(r-\alpha l)1_{[\alpha l,+\infty)})^{(l)}$ belongs to $\mathfrak{D}'_{\beta,\sigma,\epsilon}$. The following formula*

$$\mathcal{L}_{\theta}(F_l)(t) = \left(\frac{e^{i\theta}}{t} \right)^{l} \exp\left(-\frac{\alpha l e^{i\theta}}{t} \right) \mathcal{L}_{\theta}(f)(t) \tag{3.26}$$

holds for all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$.

Proof. Since $\mathfrak{D}(\mathbb{R}_+)$ is dense in $\mathfrak{D}'_{\beta,\sigma,\epsilon}$, it is sufficient to prove that

$$\mathcal{L}_{\theta}(F_l)(t) = \left(\frac{e^{i\theta}}{t} \right)^{l} \exp\left(-\frac{\alpha l e^{i\theta}}{t} \right) T_{\theta}(f)(t) \tag{3.27}$$

for all $f \in \mathfrak{D}(\mathbb{R}_+)$, all $t \in S_{\theta,\rho_\theta,|\epsilon|\rho}$. Then, we get the inequality (3.26) by using (3.3) and Proposition 3.6. Now, let $f \in \mathfrak{D}(\mathbb{R}_+)$. We write

$$\left(f(\tau-\alpha l)1_{[\alpha l,+\infty)} \right)^{(l)} = \partial_{\tau}^{-r} \left(f(\tau-\alpha l)1_{[\alpha l,+\infty)} \right)^{(l+r)}, \tag{3.28}$$

where $r \geq 0$ is an integer chosen such that $\alpha l \in [l+r, l+r+1]$. From our assumption, we have that $\tau \mapsto f(\tau-\alpha l)1_{[\alpha l,+\infty)}$ belongs to $L^1(\mathbb{R}_+)$ and that $\operatorname{supp}(f(\tau-\alpha l)1_{[\alpha l,+\infty)}) \subset [l+r,+\infty)$. By Lemma 2.5, we deduce that $(f(\tau-\alpha l)1_{[\alpha l,+\infty)})^{(l+r)}$ is a staircase distribution $\sum_{h\geq 0} \widetilde{\Delta}_{h,l}^{(h)}(\tau)$ where the functions $\widetilde{\Delta}_{h,l}(\tau)$ are constructed as follows:

$$\widetilde{\Delta}_{j,l}(\tau) = 0, \quad \text{for } 0 \leq j \leq l+r-1, \quad \widetilde{\Delta}_{l+r,l}(\tau) = f(\tau-\alpha l)1_{[\alpha l,+\infty)}1_{[l+r,l+r+1]}, \tag{3.29}$$

and for all $n \geq 1$, we have $\widetilde{\Delta}_{l+r+n,l}(\tau) = G_n(\tau)1_{[l+r+n,l+r+n+1]}$ where

$$G_n(\tau) = \partial_{\tau}^{-1}\left(G_{n-1}(\tau)1_{[l+r+n,+\infty)} \right), \qquad G_0 = f(\tau-\alpha l)1_{[\alpha l,+\infty)}. \tag{3.30}$$

By definition, we have

$$\mathcal{L}_\theta\Big(\big(f(\tau-\alpha l)1_{[\alpha l,+\infty)}\big)^{(l+r)}\Big)(t)=\sum_{h=0}^{\infty}\left(\frac{e^{i\theta}}{t}\right)^{h+1}\int_0^{\infty}\widetilde{\Delta}_{h,l}(\tau)\exp\left(-\frac{\tau e^{i\theta}}{t}\right)d\tau. \tag{3.31}$$

Now, we will compute the integrals $A_{h,l}=(e^{i\theta}/t)^{h+1}\int_0^{+\infty}\widetilde{\Delta}_{h,l}(\tau)\exp(-\tau e^{i\theta}/t)d\tau$ for all $h\geq 0$. By construction, we have that $A_{h,l}=0$ for all $0\leq h\leq l+r-1$. For $h=l+r$, we get

$$\begin{aligned}A_{l+r,l}&=\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\int_{\alpha l}^{l+r+1}f(\tau-\alpha l)\exp\left(-\frac{\tau e^{i\theta}}{t}\right)d\tau\\&=\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\exp\left(-\frac{\alpha l e^{i\theta}}{t}\right)\int_0^{(1-\alpha)l+r+1}f(s)\exp\left(-\frac{se^{i\theta}}{t}\right)ds.\end{aligned}\tag{3.32}$$

For $h=l+r+1$, by one integration by parts, we get that

$$\begin{aligned}A_{l+r+1,l}&=\left[-\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\exp\left(-\frac{\tau e^{i\theta}}{t}\right)G_1(\tau)\right]_{l+r+1}^{l+r+2}\\&\quad+\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\int_{l+r+1}^{l+r+2}f(\tau-\alpha l)\exp\left(-\frac{\tau e^{i\theta}}{t}\right)d\tau\\&=\left[-\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\exp\left(-\frac{\tau e^{i\theta}}{t}\right)G_1(\tau)\right]_{l+r+1}^{l+r+2}\\&\quad+\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\exp\left(-\frac{\alpha l e^{i\theta}}{t}\right)\int_{(1-\alpha)l+r+1}^{(1-\alpha)l+r+2}f(s)\exp\left(-\frac{se^{i\theta}}{t}\right)ds.\end{aligned}\tag{3.33}$$

For $h=l+r+n$, with $n\geq 1$, by successive integrations by parts, we get that

$$\begin{aligned}A_{l+r+n,l}&=\sum_{q=1}^{n}\left[-\left(\frac{e^{i\theta}}{t}\right)^{l+r+q}\exp\left(-\frac{\tau e^{i\theta}}{t}\right)G_q(\tau)\right]_{l+r+n}^{l+r+n+1}\\&\quad+\left(\frac{e^{i\theta}}{t}\right)^{l+r+1}\exp\left(-\frac{\alpha l e^{i\theta}}{t}\right)\int_{(1-\alpha)l+r+n}^{(1-\alpha)l+r+n+1}f(s)\exp\left(-\frac{se^{i\theta}}{t}\right)ds.\end{aligned}\tag{3.34}$$

Since $G_q(l+r+q)=0$, for all $q\geq 1$, we deduce that the telescopic sum

$$\sum_{n=q}^{\infty}\left[\left(\frac{e^{i\theta}}{t}\right)^{l+r+q}\exp\left(-\frac{\tau e^{i\theta}}{t}\right)G_q(\tau)\right]_{l+r+n}^{l+r+n+1}\tag{3.35}$$

is equal to 0. From the formula (3.32), (3.33), (3.34), and (3.35), we get that

$$\begin{aligned}\mathcal{L}_{\theta}\Big(\big(f(\tau-\alpha l)1_{[\alpha l,+\infty)}\big)^{(l+r)}\Big)(t) &= \sum_{h=0}^{+\infty} A_{h,l}\\ &= \left(\frac{e^{i\theta}}{t}\right)^{l+r}\exp\left(-\frac{\alpha l e^{i\theta}}{t}\right)\frac{e^{i\theta}}{t}\int_0^{+\infty} f(s)\exp\left(-\frac{s e^{i\theta}}{t}\right)ds.\end{aligned}\tag{3.36}$$

From Proposition 3.3, we have that

$$\mathcal{L}_{\theta}(F_l)(t) = t^r\left(e^{i\theta}\right)^{-r}\mathcal{L}_{\theta}\Big(\big(f(\tau-\alpha l)1_{[\alpha l,+\infty)}\big)^{(l+r)}\Big)(t).\tag{3.37}$$

Finally, from (3.36) and (3.37), we get the equality (3.27). □

4. Formal and Analytic Transseries Solutions for a Singularly Perturbed Cauchy Problem

4.1. Laplace Transform and Asymptotic Expansions

We recall the definition of Borel summability of formal series with coefficients in a Banach space, see [27].

Definition 4.1. A formal series

$$\hat{X}(t) = \sum_{j=0}^{\infty}\frac{a_j}{j!}t^j \in \mathbb{E}[[t]]\tag{4.1}$$

with coefficients in a Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ is said to be 1-summable with respect to t in the direction $d \in [0, 2\pi)$ if

(i) there exists $\rho \in \mathbb{R}_+$ such that the following formal series, called formal Borel transform of $\hat{X}$ of order 1,

$$\mathcal{B}\left(\hat{X}\right)(\tau) = \sum_{j=0}^{\infty}\frac{a_j\tau^j}{(j!)^2} \in \mathbb{E}[[\tau]]\tag{4.2}$$

is absolutely convergent for $|\tau| < \rho$;

(ii) there exists $\delta > 0$ such that the series $\mathcal{B}(\hat{X})(\tau)$ can be analytically continued with respect to τ in a sector $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$. Moreover, there exist $C > 0$ and $K > 0$ such that

$$\left\|\mathcal{B}(\hat{X})(\tau)\right\|_{\mathbb{E}} \le Ce^{K|\tau|}\tag{4.3}$$

for all $\tau \in S_{d,\delta}$. We say that $\mathcal{B}(\hat{X})(\tau)$ has exponential growth of order 1 on $S_{d,\delta}$.

If this is so, the vector valued Laplace transform of order 1 of $\mathcal{B}(\hat{X})(\tau)$ in the direction d is defined by

$$\mathcal{L}^d\left(\mathcal{B}\left(\hat{X}\right)\right)(t) = t^{-1}\int_{L_\gamma} \mathcal{B}\left(\hat{X}\right)(\tau)e^{-(\tau/t)}d\tau \tag{4.4}$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on t and is chosen in such a way that $\cos(\gamma - \arg(t)) \geq \delta_1 > 0$, for some fixed δ_1, for all t in a sector

$$S_{d,\theta,R} = \left\{ t \in \mathbb{C}^* : |t| < R, \left|d - \arg(t)\right| < \frac{\theta}{2} \right\}, \tag{4.5}$$

where $\pi < \theta < \pi + 2\delta$ and $0 < R < \delta_1/K$. The function $\mathcal{L}^d(\mathcal{B}(\hat{X}))(t)$ is called the 1-sum of the formal series $\hat{X}(t)$ in the direction d. The function $\mathcal{L}^d(\mathcal{B}(\hat{X}))(t)$ is a holomorphic and a bounded function on the sector $S_{d,\theta,R}$. Moreover, the function $\mathcal{L}^d(\mathcal{B}(\hat{X}))(t)$ has the formal series $\hat{X}(t)$ as Gevrey asymptotic expansion of order 1 with respect to t on $S_{d,\theta,R}$. This means that for all $0 < \theta_1 < \theta$, there exist $C, M > 0$ such that

$$\left\| \mathcal{L}^d\left(\mathcal{B}\left(\hat{X}\right)\right)(t) - \sum_{p=0}^{n-1} \frac{a_p}{p!} t^p \right\|_{\mathbb{E}} \leq CM^n n! |t|^n \tag{4.6}$$

for all $n \geq 1$, all $t \in S_{d,\theta_1,R}$.

In the next proposition, we recall some well-known identities for the Borel transform that will be useful in the sequel.

Proposition 4.2. *Let $\hat{X}(t) = \sum_{n\geq 0} a_n t^n/n!$ and $\hat{G}(t) = \sum_{n\geq 0} b_n t^n/n!$ be formal series in $\mathbb{E}[[t]]$. One has the following equalities as formal series in $\mathbb{E}[[\tau]]$:*

$$\begin{gathered}\left(\tau\partial_\tau^2 + \partial_\tau\right)\left(\mathcal{B}\left(\hat{X}\right)(\tau)\right) = \mathcal{B}\left(\partial_t \hat{X}(t)\right)(\tau), \qquad \partial_\tau^{-1}\left(\mathcal{B}\left(\hat{X}\right)\right)(\tau) = \mathcal{B}\left(t\hat{X}(t)\right)(\tau), \\ \tau\mathcal{B}\left(\hat{X}\right)(\tau) = \mathcal{B}\left(\left(t^2\partial_t + t\right)\hat{X}(t)\right)(\tau).\end{gathered} \tag{4.7}$$

4.2. Formal Transseries Solutions for an Auxiliary Singular Cauchy Problem

Let $S \geq 1$ be an integer. Let $\mathcal{S}$ be a finite subset of $\mathbb{N}^3$ and let

$$b_{s,k_0,k_1}(z,\epsilon) = \sum_{\beta\geq 0} \frac{b_{s,k_0,k_1,\beta}(\epsilon) z^\beta}{\beta!} \tag{4.8}$$

be holomorphic and bounded functions on a polydisc $D(0,\rho) \times D(0,\epsilon_0)$, for some $\rho, \epsilon_0 > 0$, with $\epsilon_0 < 1$, for all $(s,k_0,k_1) \in \mathcal{S}$. We consider the following singular Cauchy problems:

$$T^2\partial_T\partial_z^S\widehat{Y}(T,z,\epsilon) + (T+1)\partial_z^S\widehat{Y}(T,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)\epsilon^{k_0-s}T^s\left(\partial_T^{k_0}\partial_z^{k_1}\widehat{Y}\right)(T,z,\epsilon), \quad (4.9)$$

for given formal transseries initial conditions

$$\left(\partial_z^j\widehat{Y}\right)(T,0,\epsilon) = \sum_{h\geq 0}\frac{\exp(-h\lambda/T)}{h!}\widehat{\varphi}_{h,j}(T,\epsilon), \quad 0 \leq j \leq S-1, \quad (4.10)$$

where $\widehat{\varphi}_{h,j}(T,\epsilon) = \sum_{m\geq 0}\varphi_{h,j,m}(\epsilon)T^m/m! \in \mathbb{C}[[T]]$ for all $\epsilon \in \mathcal{E}$ and $\lambda \in \mathbb{C}^*$.

Proposition 4.3. *The problem* (4.9), (4.10) *has a formal transseries solutions*

$$\widehat{Y}(T,z,\epsilon) = \sum_{h\geq 0}\frac{\exp(-h\lambda/T)}{h!}\widehat{Y}_h(T,z,\epsilon), \quad (4.11)$$

where the formal series $\widehat{Y}_h(T,z,\epsilon) \in \mathbb{C}[[T,z]]$, for all $\epsilon \in \mathcal{E}$, all $h \geq 0$, satisfy the following singular Cauchy problems:

$$\begin{aligned}
&T^2\partial_T\partial_z^S\widehat{Y}_h(T,z,\epsilon) + (T+1+\lambda h)\partial_z^S\widehat{Y}_h(T,z,\epsilon) \\
&\quad = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)\Bigg(\epsilon^{k_0-s}T^s\left(\partial_T^{k_0}\partial_z^{k_1}\widehat{Y}_h\right)(T,z,\epsilon) \\
&\qquad + \sum_{k_0^1+k_0^2=k_0,\, k_0^1\geq 1}\frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}c_q^{k_0^1}(h\lambda)^q\epsilon^{k_0-s}T^{s-(k_0^1+q)}\partial_T^{k_0^2}\partial_z^{k_1}\widehat{Y}_h(T,z,\epsilon)\Bigg)
\end{aligned} \quad (4.12)$$

with initial conditions

$$\left(\partial_z^j\widehat{Y}_h\right)(T,0,\epsilon) = \widehat{\varphi}_{h,j}(T,\epsilon), \quad 0 \leq j \leq S-1, \quad (4.13)$$

for some real numbers $c_q^{k_0^1}$, for $1 \leq q \leq k_0^1$ and $1 \leq k_0^1 \leq k_0$.

Proof. We have that

$$\partial_T\left(\exp\left(-\frac{h\lambda}{T}\right)\widehat{Y}_h(T,z,\epsilon)\right) = \exp\left(-\frac{h\lambda}{T}\right)\left(\frac{h\lambda}{T^2}\widehat{Y}_h(T,z,\epsilon) + \partial_T\widehat{Y}_h(T,z,\epsilon)\right), \quad (4.14)$$

and from the Leibniz rule we also have

$$\partial_T^{k_0}\left(\exp\left(-\frac{h\lambda}{T}\right)\widehat{Y}_h(T,z,\epsilon)\right) = \sum_{k_0^1+k_0^2=k_0}\frac{k_0!}{k_0^1!k_0^2!}\partial_T^{k_0^1}\left(\exp\left(-\frac{h\lambda}{T}\right)\right)\partial_T^{k_0^2}\widehat{Y}_h(T,z,\epsilon). \tag{4.15}$$

On the other hand, by the Faa Di Bruno formula we have, for all $k_0^1 \geq 1$, that

$$\begin{aligned}\partial_T^{k_0^1}\left(\exp\left(-\frac{h\lambda}{T}\right)\right) &= \sum_{q=1}^{k_0^1}\exp\left(-\frac{\lambda h}{T}\right)\sum_{(\lambda_1,\ldots,\lambda_{k_0^1})\in A_{q,k_0^1}} k_0^1!\prod_{i=1}^{k_0^1}\frac{\left((-1)^{i+1}\left(h\lambda/T^{i+1}\right)\right)^{\lambda_i}}{\lambda_i!} \\ &= \exp\left(-\frac{h\lambda}{T}\right)\left(\sum_{q=1}^{k_0^1}c_q^{k_0^1}\frac{(h\lambda)^q}{T^{k_0^1+q}}\right),\end{aligned} \tag{4.16}$$

where $A_{q,k_0^1} = \{(\lambda_1,\ldots,\lambda_{k_0^1}) \in \mathbb{N}^{k_0^1} / \sum_{i=1}^{k_0^1}\lambda_i = q, \sum_{i=1}^{k_0^1} i\lambda_i = k_0^1\}$ and $c_q^{k_0^1} \in \mathbb{R}$, for all $q = 1,\ldots,k_0^1$.

Using the expressions (4.14), (4.15), (4.16), by plugging the formal expansion $\widehat{Y}(T,z,\epsilon)$ into the problem (4.9), (4.10) and by identification of the coefficients of $\exp(-h\lambda/T)$ we get that $\widehat{Y}_h$ satisfies the problem (4.12), (4.13). □

4.3. Formal Solutions to a Sequence of Regular Cauchy Problems

Proposition 4.4. *One makes the assumption that*

$$S > k_1, \qquad s \geq 2k_0 \tag{4.17}$$

for all $(s,k_0,k_1) \in \mathcal{S}$*. Then, the problem* (4.12), (4.13) *has a unique formal solution* $\widehat{Y}_h(T,z,\epsilon) \in \mathbb{C}[[T,z]]$ *for all* $\epsilon \in \mathcal{E}$*. Let*

$$\widehat{Y}_h(T,z,\epsilon) = \sum_{m\geq 0}\frac{Y_{h,m}(z,\epsilon)T^m}{m!}, \tag{4.18}$$

where $Y_{h,m}(z,\epsilon) \in \mathbb{C}[[z]]$*, be the formal solution of* (4.12), (4.13) *for all* $\epsilon \in \mathcal{E}$*. One denotes by*

$$V_h(\tau,z,\epsilon) = \sum_{m\geq 0}Y_{h,m}(z,\epsilon)\frac{\tau^m}{(m!)^2} \tag{4.19}$$

the formal Borel transform of $\widehat{Y}_h$ *with respect to* T. *Then, for all* $h \geq 0$, $V_h(\tau, z, \epsilon)$ *satisfies the problem*

$$
\begin{aligned}
(\tau + 1 + \lambda h)\partial_z^S V_h(\tau, z, \epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon) \Bigg(& \epsilon^{k_0-s} \sum_{(r,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{r,p}\tau^r\partial_\tau^{-p}\partial_z^{k_1}V_h(\tau,z,\epsilon) \\
& + \sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1} c_q^{k_0^1}(h\lambda)^q \epsilon^{k_0-s} \\
& \times \sum_{(r,p)\in\mathcal{O}^2_{s-k_0-q}} \alpha^{2,q}_{r,p}\tau^r\partial_\tau^{-p}\partial_z^{k_1}V_h(\tau,z,\epsilon) \Bigg)
\end{aligned}
\tag{4.20}
$$

with initial data

$$
\left(\partial_z^j V_h\right)(\tau,0,\epsilon) = v_{h,j}(\tau,\epsilon) = \sum_{m\geq 0}\varphi_{h,j,m}(\epsilon)\frac{\tau^m}{(m!)^2} \in \mathbb{C}[[\tau]], \quad 0 \leq j \leq S-1, \tag{4.21}
$$

where $\mathcal{O}^1_{s-k_0}$ *is a finite subset of* $\mathbb{N}^2$ *such that* $(r,p) \in \mathcal{O}^1_{s-k_0}$ *implies* $r+p = s-k_0$ *and* $\mathcal{O}^2_{s-k_0-q}$ *is a finite subset of* $\mathbb{N}^2$ *such that* $(r,p) \in \mathcal{O}^2_{s-k_0-q}$ *implies* $r+p = s-k_0-q$, *and* $\alpha^1_{r,p}$, $\alpha^{2,q}_{r,p}$ *are integers.*

Proof. The proof follows by direct computation on the problems (4.12) and (4.13), using Proposition 4.2 and the following two lemmas from [2].

Lemma 4.5. *For all* $k_0 \geq 1$, *there exist constants* $a_{k,k_0} \in \mathbb{N}$, $k_0 \leq k \leq 2k_0$, *such that*

$$
\left(\tau\partial_\tau^2 + \partial_\tau\right)^{k_0} u(\tau) = \sum_{k=k_0}^{2k_0} a_{k,k_0}\tau^{k-k_0}\partial_\tau^k u(\tau) \tag{4.22}
$$

for all holomorphic functions $u : \Omega \to \mathbb{C}$ *on an open set* $\Omega \subset \mathbb{C}$.

Lemma 4.6. *Let* $a, b, c \geq 0$ *be positive integers such that* $a \geq b$ *and* $a \geq c$. *We put* $\delta = a+b-c$. *Then, for all holomorphic functions* $u : \Omega \to \mathbb{C}$, *the function* $\partial_\tau^{-a}(\tau^b\partial_\tau^c u(\tau))$ *can be written in the form*

$$
\partial_\tau^{-a}\left(\tau^b\partial_\tau^c u(\tau)\right) = \sum_{(b',c')\in\mathcal{O}_\delta} \alpha_{b',c'}\tau^{b'}\partial_\tau^{c'}u(\tau), \tag{4.23}
$$

where $\mathcal{O}_\delta$ *is a finite subset of* $\mathbb{Z}^2$ *such that for all* $(b',c') \in \mathcal{O}_\delta$, $b'-c' = \delta$, $b' \geq 0$, $c' \leq 0$, *and* $\alpha_{b',c'} \in \mathbb{Z}$.

□

4.4. An Auxiliary Cauchy Problem

We denote by Ω_1 an open star-shaped domain in $\mathbb{C}$ (meaning that Ω_1 is an open subset of $\mathbb{C}$ such that for all $x \in \Omega_1$, the segment $[0,x]$ belongs to Ω_1). Let Ω_2 be an open set in $\mathbb{C}^*$

contained in the disc $D(0,\epsilon_0)$. We denote by $\Omega = \Omega_1 \times \Omega_2$. For any open set $\mathfrak{D} \subset \mathbb{C}$, we denote by $\mathcal{O}(\mathfrak{D})$ the vector space of holomorphic functions on $\mathfrak{D}$.

Definition 4.7. Let $b > 1$ a real number and let $r_b(\beta) = \sum_{n=0}^{\beta} 1/(n+1)^b$ for all integers $\beta \geq 0$. Let $\epsilon \in \Omega_2$ and $\sigma > 0$ be a real number. We denote by $E_{\beta,\epsilon,\sigma,\Omega}$ the vector space of all functions $v \in \mathcal{O}(\Omega_1)$ such that

$$\|v(\tau)\|_{\beta,\epsilon,\sigma,\Omega} := \sup_{\tau\in\Omega_1}|v(\tau)|\left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)\exp\left(-\frac{\sigma}{2|\epsilon|}r_b(\beta)|\tau|\right) \tag{4.24}$$

is finite.

Proposition 4.8. *One makes the assumption that*

$$S > k_1, \qquad s \geq 2k_0 \tag{4.25}$$

for all $(s,k_0,k_1) \in \mathcal{S}$*. Moreover, one makes the assumption that there exists* $c', \delta' > 0$ *such that*

$$|\tau + 1 + h\lambda| \geq c'|\tau+1| > \delta', \quad \forall \tau \in \Omega_1,\ \forall h \in \mathbb{N}. \tag{4.26}$$

For all $h \geq 0$*, all* $\epsilon \in \Omega_2$*, the problem* (4.20) *with initial conditions*

$$\left(\partial_z^j V_h\right)(\tau,0,\epsilon) = v_{h,j}(\tau,\epsilon) \in \mathcal{O}(\Omega_1), \quad 0 \leq j \leq S-1 \tag{4.27}$$

has a unique formal series

$$V_h(\tau,z,\epsilon) = \sum_{\beta\geq 0} v_{h,\beta}(\tau,\epsilon)\frac{z^\beta}{\beta!} \in \mathcal{O}(\Omega_1)[[z]], \tag{4.28}$$

where $v_{h,\beta}(\tau,\epsilon)$ *satisfies the following recursion:*

$$\begin{aligned}
&(\tau+1+h\lambda)v_{h,\beta+S}(\tau,\epsilon)\\
&\quad= \sum_{(s,k_0,k_1)\in\mathcal{S}}\ \sum_{\beta_1+\beta_2=\beta} \beta!\frac{b_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!}\epsilon^{k_0-s}\left(\sum_{(r,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{r,p}\tau^r\partial_\tau^{-p}\frac{v_{h,\beta_2+k_1}(\tau,\epsilon)}{\beta_2!}\right)\\
&\qquad+ \sum_{k_0^1+k_0^2=k_0,\,k_0^1\geq 1}\frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\ \sum_{\beta_1+\beta_2=\beta}\beta!\frac{b_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!}c_q^{k_0^1}(h\lambda)^q\\
&\qquad\times\epsilon^{k_0-s}\left(\sum_{(r,p)\in\mathcal{O}^2_{s-k_0-q}}\alpha^{2,q}_{r,p}\tau^r\partial_\tau^{-p}\frac{v_{h,\beta_2+k_1}(\tau,\epsilon)}{\beta_2!}\right)
\end{aligned} \tag{4.29}$$

for all $\tau \in \Omega_1$*, all* $\epsilon \in \Omega_2$.

Proposition 4.9. *One makes the assumption that*

$$S > k_1, \qquad s \geq 2k_0 \tag{4.30}$$

for all $(s,k_0,k_1) \in \mathcal{S}$. *Let also the assumption* (4.26) *holds. Let us assume that*

$$v_{h,j}(\tau,\epsilon) \in E_{j,\epsilon,\sigma,\Omega}, \quad \forall h \geq 0,\ \forall 0 \leq j \leq S-1,\ \forall \epsilon \in \Omega_2. \tag{4.31}$$

Then, one has that $v_{h,\beta}(\tau,\epsilon) \in E_{\beta,\epsilon,\sigma,\Omega}$ *for all* $\beta \geq 0$, *all* $h \geq 0$, *all* $\epsilon \in \Omega_2$. *We put* $v_{h,\beta}(\epsilon) = ||v_{h,\beta}(\tau,\epsilon)||_{\beta,\epsilon,\sigma,\Omega}$, *for all* $h \geq 0$, *all* $\beta \geq 0$, *and all* $\epsilon \in \Omega_2$. *Then, the following inequalities hold: there exist two constants* $C_{18}^1, C_{18}^2 > 0$ *(depending on* $\mathcal{S}$,σ,S*) such that*

$$\begin{aligned} v_{h,\beta+S}(\epsilon) \leq & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\left|b_{s,k_0,k_1,\beta_1}(\epsilon)\right|}{\beta_1!} \\ & \times C_{18}^1\Big((\beta+S+1)^{b(s-k_0)} + (\beta+S+1)^{b(s-k_0+2)}\Big)\frac{v_{h,\beta_2+k_1}(\epsilon)}{\beta_2!} \\ & + \sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\sum_{\beta_1+\beta_2=\beta}\beta!\frac{\left|b_{s,k_0,k_1,\beta_1}(\epsilon)\right|}{\beta_1!}\left|c_q^{k_0^1}\right|h^q|\lambda|^q \\ & \times |\epsilon|^{-q}C_{18}^2\Big((\beta+S+1)^{b(s-k_0-q)} + (\beta+S+1)^{b(s-k_0-q+2)}\Big)\frac{v_{h,\beta_2+k_1}(\epsilon)}{\beta_2!} \end{aligned} \tag{4.32}$$

for all $h \geq 0$, *all* $\beta \geq 0$.

Proof. The proof follows by direct computation using the recursion (4.29) and the next lemma. We keep the notations of Proposition 4.8.

Lemma 4.10. *There exists a constant* $C_{18} > 0$ *(depending on* s, σ, S, k_0, k_1*) such that*

$$\begin{aligned} & \left\|\tau^r\partial_\tau^{-p}v_{h,\beta_2+k_1}(\tau,\epsilon)\right\|_{\beta+S,\epsilon,\sigma,\Omega} \\ & \quad \leq |\epsilon|^{r+p}C_{18}\Big((\beta+S+1)^{b(r+p)} + (\beta+S+1)^{b(r+p+2)}\Big)\left\|v_{h,\beta_2+k_1}(\tau,\epsilon)\right\|_{\beta_2+k_1,\epsilon,\sigma,\Omega} \end{aligned} \tag{4.33}$$

for all $h \geq 0$, *and all* $\beta \geq 0$, $0 \leq \beta_2 \leq \beta$, *all* $(r,p) \in \mathbb{N}^2$ *with* $r+p \leq s-k_0$.

Proof. We follow the proof of Lemma 1 from [2]. By definition, we have that $\partial_\tau^{-1}v_{h,\beta_2+k_1}(\tau,\epsilon) = \int_0^\tau v_{h,\beta_2+k_1}(\tau_1,\epsilon)d\tau_1$ for all $\tau \in \Omega_1$. Using the parametrization $\tau_1 = h_1\tau$ with $0 \leq h_1 \leq 1$, we get that

$$\partial_\tau^{-1}v_{h,\beta_2+k_1}(\tau,\epsilon) = \tau\int_0^1 v_{h,\beta_2+k_1}(h_1\tau,\epsilon)M_1(h_1)dh_1, \tag{4.34}$$

where $M_1(h_1) = 1$. More generally, for all $p \geq 2$, we have by definition:

$$\partial_\tau^{-p} v_{h,\beta_2+k_1}(\tau,\epsilon) = \int_0^\tau \int_0^{\tau_1} \cdots \int_0^{\tau_{p-1}} v_{h,\beta_2+k_1}(\tau_p,\epsilon)\, d\tau_p d\tau_{p-1} \cdots d\tau_1 \tag{4.35}$$

for all $\tau \in \Omega_1$. Using the parametrization $\tau_j = h_j \tau_{j-1}$, $\tau_1 = h_1\tau$, with $0 \leq h_j \leq 1$, for $2 \leq j \leq p$, we can write

$$\partial_\tau^{-p} v_{h,\beta_2+k_1}(\tau,\epsilon) = \tau^p \int_0^1 \cdots \int_0^1 v_{h,\beta_2+k_1}(h_p \cdots h_1\tau,\epsilon) M_p(h_1,\ldots,h_p)\, dh_p dh_{p-1} \cdots dh_1, \tag{4.36}$$

where $M_p(h_1,\ldots,h_p)$ is a monomial in $h_1,\ldots,h_p$ whose coefficient is equal to 1. Using these latter expressions, we now write

$$\begin{aligned}
&\left|\tau^r \partial_\tau^{-p} v_{h,\beta_2+k_1}(\tau,\epsilon)\right| \\
&\quad = \left| \tau^{r+p} \int_0^1 \cdots \int_0^1 v_{h,\beta_2+k_1}(h_p\cdots h_1\tau,\epsilon) \left(1 + \frac{\left|h_p\cdots h_1\tau\right|^2}{|\epsilon|^2}\right) \exp\left(-\frac{\sigma}{2|\epsilon|} r_b(\beta_2+k_1)\left|h_p\cdots h_1\tau\right|\right)\right. \\
&\qquad \left. \times \frac{\exp\left((\sigma/2|\epsilon|) r_b(\beta_2+k_1)\left|h_p\cdots h_1\tau\right|\right)}{1 + \left|h_p\cdots h_1\tau\right|^2/|\epsilon|^2} M_p(h_1,\ldots,h_p)\, dh_p \ldots dh_1 \right|.
\end{aligned} \tag{4.37}$$

Therefore,

$$\begin{aligned}
&\left|\tau^r \partial_\tau^{-p} v_{h,\beta_2+k_1}(\tau,\epsilon)\right| \left(1 + \frac{|\tau|^2}{|\epsilon|^2}\right) \exp\left(-\frac{\sigma}{2|\epsilon|} r_b(\beta+S)|\tau|\right) \\
&\quad \leq \left\| v_{h,\beta_2+k_1}(\tau,\epsilon)\right\|_{\beta_2+k_1,\epsilon,\sigma,\Omega} |\tau|^{r+p} \left(1 + \frac{|\tau|^2}{|\epsilon|^2}\right) \exp\left(-\frac{\sigma}{2|\epsilon|}\left(r_b(\beta+S) - r_b(\beta_2+k_1)\right)|\tau|\right).
\end{aligned} \tag{4.38}$$

By construction of $r_b(\beta)$, we have

$$r_b(\beta+S) - r_b(\beta_2+k_1) = \sum_{n=\beta_2+k_1+1}^{\beta+S} \frac{1}{(n+1)^b} \geq \frac{\beta - \beta_2 + S - k_1}{(\beta+S+1)^b} \geq \frac{S-k_1}{(\beta+S+1)^b} \tag{4.39}$$

for all $\beta \geq 0$. From (4.38) and (4.39), we get that

$$\begin{aligned}
&\left|\tau^r \partial_\tau^{-p} v_{h,\beta_2+k_1}(\tau,\epsilon)\right| \left(1 + \frac{|\tau|^2}{|\epsilon|^2}\right) \exp\left(-\frac{\sigma}{2|\epsilon|} r_b(\beta+S)|\tau|\right) \\
&\quad \leq \left\| v_{h,\beta_2+k_1}(\tau,\epsilon)\right\|_{\beta_2+k_1,\epsilon,\sigma,\Omega} |\tau|^{r+p} \left(1 + \frac{|\tau|^2}{|\epsilon|^2}\right) \exp\left(-\frac{\sigma}{2|\epsilon|} \frac{S-k_1}{(\beta+S+1)^b}|\tau|\right)
\end{aligned} \tag{4.40}$$

for all $\beta \geq 0$. From (2.41), we deduce that

$$\begin{aligned} &|\tau|^{r+p}\left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)\exp\left(-\frac{\sigma}{2|\epsilon|}\frac{S-k_1}{(\beta+S+1)^b}|\tau|\right) \\ &\quad \leq |\epsilon|^{r+p}\left(\left(\frac{2(r+p)e^{-1}}{\sigma(S-k_1)}\right)^{r+p}(\beta+S+1)^{b(r+p)} \right. \\ &\qquad \left. +\left(\frac{2(r+p+2)e^{-1}}{\sigma(S-k_1)}\right)^{r+p+2}(\beta+S+1)^{b(r+p+2)}\right) \end{aligned} \tag{4.41}$$

for all $\tau \in \Omega_1$. From the estimates (4.40) and (4.41), we deduce the inequality (4.33). □

Proposition 4.11. *Assume that the conditions* (4.26) *and* (4.31) *hold. Assume moreover, that*

$$S \geq b(s-k_0+2)+k_1, \qquad s \geq 2k_0 \tag{4.42}$$

for all $(s,k_0,k_1) \in \mathcal{S}$ *and that the following sums converge near the origin in* $\mathbb{C}$,

$$W_j(u) := \sum_{h\geq 0}\sup_{\epsilon\in\Omega_2}\left\|v_{h,j}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,\Omega}\frac{u^h}{h!} \in \mathbb{C}\{u\}, \quad 0 \leq j \leq S-1. \tag{4.43}$$

One make also the hypothesis that for all $(s,k_0,k_1) \in \mathcal{S}$, *one can write*

$$b_{s,k_0,k_1}(z,\epsilon) = \epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon), \tag{4.44}$$

where $\widetilde{b}_{s,k_0,k_1}(z,\epsilon) = \sum_{\beta\geq 0}\widetilde{b}_{s,k_0,k_1,\beta}(\epsilon)z^\beta/\beta!$ *is holomorphic for all* $\epsilon \in D(0,\epsilon_0)$ *on* $D(0,\rho)$. *Then, the problem* (4.20) *with initial data*

$$\left(\partial_z^j V_h\right)(\tau,0,\epsilon) = v_{h,j}(\tau,\epsilon), \quad 0 \leq j \leq S-1 \tag{4.45}$$

has a unique solution $V_h(\tau,z,\epsilon)$ *which is holomorphic with respect to* $(\tau,z) \in \Omega_1 \times D(0,x_1/2)$ *for all* $\epsilon \in \Omega_2$.

The constant x_1 *is such that* $0 < x_1 < \rho$ *and depends on* S, u_0 *(which denotes a common radius of absolute convergence of the series* (4.43)*),* $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, *where* $x_0 < \rho$ *and* $|b|_{s,k_0,k_1}$, $|\widetilde{b}|_{s,k_0,k_1}$ *are defined below.*

Moreover, the following estimates hold: there exists a constant u_1 such that $0 < u_1 < u_0$ (depending on u_0, $\mathcal{S}$, and b, σ) and a constant $C_{19} > 0$ (depending on $\max_{0\leq j\leq S-1} W_j(u_0)$ (where W_j are defined above), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\tilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b) such that

$$|V_h(\tau,z,\epsilon)| \leq \frac{C_{19}}{1-2|z|/x_1} h!\left(\frac{2}{u_1}\right)^h \left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1} \exp\left(\frac{\sigma}{2|\epsilon|}\zeta(b)|\tau|\right) \tag{4.46}$$

for all $(\tau,z) \in \Omega_1 \times D(0,x_1/2)$, all $\epsilon \in \Omega_2$, and all $h \geq 0$.

Proof. We consider the following Cauchy problem

$$\begin{aligned}
\partial_x^S W(u,x) = &\sum_{(s,k_0,k_1)\in\mathcal{S}} C_{18}^1\Big((x\partial_x+S+1)^{b(s-k_0)} \\
&\qquad\qquad + (x\partial_x+S+1)^{b(s-k_0+2)}\Big)\Big(|b|_{s,k_0,k_1}(x)\partial_x^{k_1}W(u,x)\Big) \\
&+ \sum_{k_0^1+k_0^2=k_0,\, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1} C_{18}^2\left|c_q^{k_0^1}\right||\lambda|^q \\
&\times\Big((x\partial_x+S+1)^{b(s-k_0-q)} + (x\partial_x+S+1)^{b(s-k_0-q+2)}\Big) \\
&\times\Big(\left|\tilde{b}\right|_{s,k_0,k_1}(x)(u\partial_u)^q\partial_x^{k_1}W(u,x)\Big)
\end{aligned} \tag{4.47}$$

for given initial data

$$\left(\partial_x^j W\right)(u,0) = W_j(u) = \sum_{h\geq 0}\sup_{\epsilon\in\Omega_2}\left|v_{h,j}(\epsilon)\right|\frac{u^h}{h!} \in \mathbb{C}\{u\}, \quad 0 \leq j \leq S-1, \tag{4.48}$$

where

$$|b|_{s,k_0,k_1}(x) = \sum_{\beta\geq 0}\sup_{\epsilon\in D(0,\epsilon_0)}\left|b_{s,k_0,k_1,\beta}(\epsilon)\right|\frac{x^\beta}{\beta!}, \qquad \left|\tilde{b}\right|_{s,k_0,k_1}(x) = \sum_{\beta\geq 0}\sup_{\epsilon\in D(0,\epsilon_0)}\left|\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)\right|\frac{x^\beta}{\beta!} \tag{4.49}$$

are convergent series near the origin in $\mathbb{C}$ with respect to x. From the assumption (4.42) and the fact that $b > 1$, we also deduce that

$$S \geq b(s-k_0-q+2)+q+k_1 \tag{4.50}$$

for all $(s,k_0,k_1) \in \mathcal{S}$ and all $0 \leq q \leq k_0$. Since the initial data (4.48) and the coefficients (4.47) are analytic near the origin, we get that all the hypotheses of the classical Cauchy Kowalevski theorem from Proposition 2.22 are fulfilled. We deduce the existence of U_1 with $0 < U_1 < U_0$, where U_0 denotes a common radius of absolute convergence for the series (4.48), which depends on U_0, $\mathcal{S}$ and b, and X_1 with $0 < X_1 < \rho$ (depending on S, U_0, $\mathcal{S}$, b,

σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(X_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\tilde{b}|_{s,k_0,k_1}(X_0)$, where $X_0 < \rho$) such that there exist a unique formal series $W(u,x) \in G(U_1, X_1)$ which solves the problem (4.47), (4.48).

Now, let $W(u,x) = \sum_{h,\beta\geq 0} w_{h,\beta}(u^h/h!)(x^\beta/\beta!)$ be its Taylor expansion at $(0,0)$. Then, by construction the sequence $w_{h,\beta}$ satisfies the following equalities:

$$\begin{aligned}
w_{h,\beta+S} = & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\sup_{\epsilon\in D(0,\epsilon_0)}\left|b_{s,k_0,k_1,\beta_1}(\epsilon)\right|}{\beta_1!} C_{18}^1 \\
& \times \left((\beta+S+1)^{b(s-k_0)} + (\beta+S+1)^{b(s-k_0+2)} \right) \frac{w_{h,\beta_2+k_1}}{\beta_2!} \\
& + \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\sup_{\epsilon\in D(0,\epsilon_0)}\left|\tilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)\right|}{\beta_1!} \left| c_q^{k_0^1} \right| h^q |\lambda|^q \\
& \times C_{18}^2 \left((\beta+S+1)^{b(s-k_0-q)} + (\beta+S+1)^{b(s-k_0-q+2)} \right) \frac{w_{h,\beta_2+k_1}}{\beta_2!}
\end{aligned} \tag{4.51}$$

for all $h \geq 0$ and all $\beta \geq 0$, with

$$w_{h,j} = \sup_{\epsilon\in\Omega_2}\left|v_{h,j}(\epsilon)\right|, \quad \forall h \geq 0, \ \forall 0 \leq j \leq S-1. \tag{4.52}$$

Using the inequality (4.32) and the equality (4.51), with the initial conditions (4.52), one gets that

$$\sup_{\epsilon\in\Omega_2}\left|v_{h,\beta}(\epsilon)\right| \leq w_{h,\beta} \tag{4.53}$$

for all $h \geq 0$, all $\beta \geq 0$. Using the fact that $W(u,x) \in G(U_1, X_1)$ and the estimates (2.111), we deduce from (4.53) that there exist a constant $C_{19} > 0$ (depending on $\max_{0\leq j\leq S-1} W_j(U_0)$,$|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(X_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\tilde{b}|_{s,k_0,k_1}(X_0)$, S, U_0, X_0, $\mathcal{S}$, b, σ) such that

$$\begin{aligned}
\left|v_{h,\beta}(\tau,\epsilon)\right| & \leq C_{19}(h+\beta)! \left(\frac{1}{U_1}\right)^h \left(\frac{1}{X_1}\right)^\beta \left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1} \exp\left(\frac{\sigma}{2|\epsilon|} r_b(\beta)|\tau|\right) \\
& \leq C_{19} h! \beta! \left(\frac{2}{U_1}\right)^h \left(\frac{2}{X_1}\right)^\beta \left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1} \exp\left(\frac{\sigma}{2|\epsilon|} r_b(\beta)|\tau|\right)
\end{aligned} \tag{4.54}$$

for all $\tau \in \Omega_1$, all $\epsilon \in \Omega_2$, all $h \geq 0$, and all $\beta \geq 0$. □

4.5. Analytic Solutions for a Sequence of Singular Cauchy Problems

Assume that the conditions (4.42) and (4.44) hold. We consider the following problem:

$$T^2\partial_T\partial_z^S Y_{h,S_d,\mathcal{E}}(T,z,\epsilon) + (T+1+\lambda h)\partial_z^S Y_{h,S_d,\mathcal{E}}(T,z,\epsilon)$$
$$= \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)\Bigg(\epsilon^{k_0-s}T^s\left(\partial_T^{k_0}\partial_z^{k_1}Y_{h,S_d,\mathcal{E}}\right)(T,z,\epsilon)$$
$$+\sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1}\frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}c_q^{k_0^1}(h\lambda)^q\epsilon^{k_0-s}T^{s-(k_0^1+q)}\partial_T^{k_0^2}\partial_z^{k_1}Y_{h,S_d,\mathcal{E}}(T,z,\epsilon)\Bigg) \tag{4.55}$$

with initial conditions

$$\left(\partial_z^j Y_{h,S_d,\mathcal{E}}\right)(T,0,\epsilon) = \varphi_{h,j,S_d,\mathcal{E}}(T,\epsilon), \quad 0\leq j\leq S-1. \tag{4.56}$$

The initial conditions $\varphi_{h,j,S_d,\mathcal{E}}(T,\epsilon)$, $0\leq j\leq S-1$ are defined as follows. Let S_d be an open sector centered at 0, with infinite radius and bisecting direction $d\in[0,2\pi)$, $D(0,\tau_0)$ an open disc centered at 0 with radius $\tau_0>0$, and $\mathcal{E}$ an open sector centered at 0 contained in the disc $D(0,\epsilon_0)$. We make the assumption that the condition (4.26) holds for the set $\Omega_1 = (S_d\cup D(0,\tau_0))$. We consider a set of functions $v_{h,j}(\tau,\epsilon)\in E_{j,\epsilon,\sigma,D(0,\tau_0)\times(D(0,\epsilon_0)\setminus\{0\})}$ for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$ such that

$$W_{j,\tau_0,\epsilon_0}(u) := \sum_{h\geq 0}\sup_{\epsilon\in D(0,\epsilon_0)\setminus\{0\}}\left\|v_{h,j}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,D(0,\tau_0)\times(D(0,\epsilon_0)\setminus\{0\})}\frac{u^h}{h!}\in\mathbb{C}\{u\}, \quad 0\leq j\leq S-1. \tag{4.57}$$

We also assume that for all $h\geq 0$ and all $0\leq j\leq S-1$, $v_{h,j}(\tau,\epsilon)$ has an analytic continuation denoted by $v_{h,j,S_d,\mathcal{E}}(\tau,\epsilon)\in E_{j,\epsilon,\sigma,(S_d\cup D(0,\tau_0))\times\mathcal{E}}$ for all $\epsilon\in\mathcal{E}$ such that

$$W_{j,S_d,\mathcal{E}}(u) := \sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}}\left\|v_{h,j,S_d,\mathcal{E}}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,(S_d\cup D(0,\tau_0))\times\mathcal{E}}\frac{u^h}{h!}\in\mathbb{C}\{u\}, \quad 0\leq j\leq S-1. \tag{4.58}$$

Let

$$v_{h,j}(\tau,\epsilon) = \sum_{m\geq 0}\varphi_{h,j,m}(\epsilon)\frac{\tau^m}{(m!)^2} \tag{4.59}$$

be the convergent Taylor expansion of $v_{h,j}$ with respect to τ on $D(0,\tau_0)$ for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$. We consider the formal series

$$\widehat{\varphi}_{h,j}(T,\epsilon) = \sum_{m\geq 0}\varphi_{h,j,m}(\epsilon)\frac{T^m}{m!} \tag{4.60}$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. We define $\varphi_{h,j,S_d,\mathcal{E}}(T,\epsilon)$ as the 1-sum (in the sense of Definition 4.1) of $\widehat{\varphi}_{j,h}(T,\epsilon)$ in the direction d. From the hypotheses, we deduce that $T \mapsto \varphi_{h,j,S_d,\mathcal{E}}(T,\epsilon)$ defines a holomorphic function for all $T \in U_{d,\theta,\iota|\epsilon|}$, for all $\epsilon \in \mathcal{E}$, where

$$U_{d,\theta,\iota|\epsilon|} = \left\{ T \in \mathbb{C}^* : |T| < \iota|\epsilon|, \left|d - \arg(T)\right| < \frac{\theta}{2} \right\} \tag{4.61}$$

for some $\theta > \pi$ and some constant $\iota > 0$ (independent of ϵ) for all $0 \leq j \leq S-1$.

Proposition 4.12. *Assume that the conditions* (4.26), (4.31), (4.42), *and* (4.44) *hold.*

Then, the problem (4.55), (4.56) *has a solution $(T,z) \mapsto Y_{h,S_d,\mathcal{E}}(T,z,\epsilon)$ which is holomorphic and bounded on the set $U_{d,\theta,\iota'|\epsilon|} \times D(0,x_1/4)$, for some $\iota' > 0$ (independent of ϵ), for all $\epsilon \in \mathcal{E}$, where $0 < x_1 < \rho$ depends on S, u_0 (which denotes a common radius of absolute convergence of the series* (4.57), (4.58)*), $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$.*

The function $Y_{h,S_d,\mathcal{E}}(T,z,\epsilon)$ can be written as the Laplace transform of order 1 in the direction d (in the sense of Definition 4.1) of a function $V_{h,S_d,\mathcal{E}}(\tau,z,\epsilon)$, which is holomorphic on the domain $(S_d \cup D(0,\tau_0)) \times D(0,x_1/2) \times \mathcal{E}$ and satisfies the following estimates.

There exists a constant u_1 such that $0 < u_1 < u_0$ (depending on u_0, $\mathcal{S}$ and b,σ) and a constant $C_{\Omega(d,\mathcal{E})} > 0$ (depending on $\max_{0\leq j\leq S-1} W_{j,S_d,\mathcal{E}}(u_0)$ (where $W_{j,S_d,\mathcal{E}}$ are defined above), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b) such that

$$|V_{h,S_d,\mathcal{E}}(\tau,z,\epsilon)| \leq \frac{C_{\Omega(d,\mathcal{E})}}{1-2|z|/x_1} h! \left(\frac{2}{u_1}\right)^h \left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1} \exp\left(\frac{\sigma}{2|\epsilon|}\zeta(b)|\tau|\right) \tag{4.62}$$

for all $(\tau,z,\epsilon) \in (S_d \cup D(0,\tau_0)) \times D(0,x_1/2) \times \mathcal{E}$, all $h \geq 0$.

Moreover, the function $V_{h,S_d,\mathcal{E}}(\tau,z,\epsilon)$ is the analytic continuation of a function $V_h(\tau,z,\epsilon)$ which is holomorphic on the punctured polydisc $D(0,\tau_0) \times D(0,x_1/2) \times (D(0,\epsilon_0) \setminus \{0\})$ and verifies the following estimates.

There exists a constant $C_{\Omega_{\tau_0,\epsilon_0}} > 0$ (depending on $\max_{0\leq j\leq S-1} W_{j,\tau_0,\epsilon_0}(u_0)$ (where W_{j,τ_0,ϵ_0} are defined above), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b) such that

$$|V_h(\tau,z,\epsilon)| \leq \frac{C_{\Omega_{\tau_0,\epsilon_0}}}{1-2|z|/x_1} h! \left(\frac{2}{u_1}\right)^h \left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1} \exp\left(\frac{\sigma}{2|\epsilon|}\zeta(b)|\tau|\right) \tag{4.63}$$

for all $\tau \in D(0,\tau_0)$, all $z \in D(0,x_1/2)$, all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, and all $h \geq 0$.

Proof. From the hypotheses of Proposition 4.12, we deduce from Proposition 4.11 applied to the situation $\Omega = D(0,\tau_0) \times (D(0,\epsilon_0) \setminus \{0\})$ the existence of a holomorphic function $V_h(\tau,z,\epsilon)$ satisfying the estimates (4.63), which is the solution of the problem (4.20) with initial conditions $(\partial_z^j V_h)(\tau,0,\epsilon) = v_{h,j}(\tau,\epsilon)$, $0 \leq j \leq S-1$, on the domain $D(0,\tau_0) \times D(0,x_1/2) \times (D(0,\epsilon_0) \setminus \{0\})$. Likewise, from Proposition 4.11 applied to the situation $\Omega = (S_d \cup D(0,\tau_0)) \times \mathcal{E}$, we get the existence of a holomorphic function $V_{h,S_d,\mathcal{E}}(\tau,z,\epsilon)$ satisfying (4.62), which is the solution of the problem (4.20) with initial conditions $(\partial_z^j V_h)(\tau,0,\epsilon) = v_{h,j,S_d,\mathcal{E}}(\tau,\epsilon)$, $0 \leq j \leq S-1$ on the domain $(S_d \cup D(0,\tau_0)) \times D(0,x_1/2) \times \mathcal{E}$.

With Proposition 4.3, we deduce that the formal solution $\widehat{Y}_h(T,z,\epsilon)$ of the problem (4.12), (4.13) is 1-summable with respect to T in the direction d as series in the Banach space

$\mathcal{O}(D(0, x_1/4))$, for all $\epsilon \in \mathcal{E}$. We denote by $Y_{h,S_d,\mathcal{E}}(T, z, \epsilon)$ its 1-sum which is holomorphic with respect to T on a domain $U_{d,\theta,\nu|\epsilon|}$ due to Definition 4.1 and the estimates (4.62). Moreover, from the algebraic properties of the κ-summability procedure, see [27, Section 6.3], we deduce that $Y_{h,S_d,\mathcal{E}}(T, z, \epsilon)$ is a solution of the problem (4.55), (4.56). □

4.6. Summability in a Complex Parameter

We recall the definition of a good covering.

Definition 4.13. Let $\nu \geq 2$ be an integer. For all $0 \leq i \leq \nu - 1$, we consider open sectors $\mathcal{E}_i$ centered at 0, with radius ϵ_0, bisecting direction $\kappa_i \in [0, 2\pi)$ and opening $\pi + \delta_i$, with $\delta_i > 0$, such that $\mathcal{E}_i \cap \mathcal{E}_{i+1} \neq \emptyset$ for all $0 \leq i \leq \nu - 1$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$) and such that $\cup_{i=0}^{\nu-1} \mathcal{E}_i = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is some neighborhood of 0 in $\mathbb{C}$. Such a set of sectors $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ is called a good covering in $\mathbb{C}^*$.

Definition 4.14. Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in $\mathbb{C}^*$. Let $\mathcal{T}$ be an open sector centered at 0 with radius $r_{\mathcal{T}}$ and consider a family of open sectors

$$U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}} := \left\{ t \in \mathbb{C} : |t| < \epsilon_0 r_{\mathcal{T}}, \left| d_i - \arg(t) \right| < \frac{\theta}{2} \right\}, \tag{4.64}$$

where $d_i \in [0, 2\pi)$, for $0 \leq i \leq \nu - 1$, where $\theta > \pi$, which satisfy the following properties:

(1) For all $0 \leq i \leq \nu - 1$, all $h \in \mathbb{N}$, $\arg(d_i) \neq \arg(-1 - \lambda h)$.

(2) For all $0 \leq i \leq \nu - 1$, for all $t \in \mathcal{T}$, and all $\epsilon \in \mathcal{E}_i$, we have that $\epsilon t \in U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}}$.

(3) (3.1) We assume that $d_0 < \arg(\lambda) < d_1$. We consider the two closed sectors

$$\mathcal{M}_{d_0} = \left\{ \tau \in \frac{\mathbb{C}^*}{\arg(\tau)} \in \left[d_0, \arg(\lambda) \right] \right\}, \quad \mathcal{M}_{d_1} = \left\{ \tau \in \frac{\mathbb{C}^*}{\arg(\tau)} \in \left[\arg(\lambda), d_1 \right] \right\}. \tag{4.65}$$

We make the assumption that there exist two constants $c', \delta' > 0$ with

$$|\tau + 1 + \lambda h| \geq c' |\tau + 1| > \delta' \tag{4.66}$$

for all $\tau \in \mathcal{M}_{d_0} \cup \mathcal{M}_{d_1} \cup D(0, \tau_0)$ and all $h \geq 0$.

(3.2) There exists $0 < \delta_{\mathcal{T}} < \pi/2$ such that $\arg(\lambda/(\epsilon t)) \in (-\pi/2 + \delta_{\mathcal{T}}, \pi/2 - \delta_{\mathcal{T}})$ for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$ and all $t \in \mathcal{T}$.

We say that the family $\{\{U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0 \leq i \leq \nu-1}, \mathcal{T}, \lambda\}$ is associated to the good covering $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$.

Now, we consider a set of functions $\varphi_{h,i,j}(T, \epsilon)$ for $0 \leq i \leq \nu - 1$, $0 \leq j \leq S - 1$, $h \geq 0$, constructed as follows. For all $0 \leq i \leq \nu - 1$, let S_{d_i} be an open sector of infinite radius centered at 0, with bisecting direction d_i and with opening $n_i > \theta - \pi$. The numbers $\theta > \pi$ and $n_i > 0$ are chosen in such a way that $-1 - \lambda h \notin S_{d_i}$ for all $0 \leq i \leq \nu - 1$ and all $h \geq 0$. Now, we put

$$\varphi_{h,i,j}(T, \epsilon) := \varphi_{h,j,S_{d_i},\mathcal{E}_i}(T, \epsilon) \tag{4.67}$$

for all $T \in U_{d_i,\theta,\iota|\epsilon|}$ and all $\epsilon \in \mathcal{E}_i$, where $\varphi_{h,j,S_{d_i},\mathcal{E}_i}(T,\epsilon)$ is given by the formula (4.56). Recalling how these functions are constructed, we consider a set of functions

$$v_{h,j}(\tau,\epsilon) \in E_{j,\epsilon,\sigma,D(0,\tau_0)\times(D(0,\epsilon_0)\setminus\{0\})} \tag{4.68}$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$ such that

$$\begin{gathered} W_{j,\tau_0,\epsilon_0}(u) := \sum_{h\geq 0} \sup_{\epsilon\in D(0,\epsilon_0)\setminus\{0\}} \left\| v_{h,j}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,D(0,\tau_0)\times(D(0,\epsilon_0)\setminus\{0\})} \frac{u^h}{h!} \in \mathbb{C}\{u\}, \\ 0 \leq j \leq S-1. \end{gathered} \tag{4.69}$$

We also assume that for all $h \geq 0$, all $0 \leq j \leq S-1$, $v_{h,j}(\tau,\epsilon)$ has an analytic continuation denoted by $v_{h,j,S_{d_i},\mathcal{E}_i}(\tau,\epsilon) \in E_{j,\epsilon,\sigma,(S_{d_i}\cup D(0,\tau_0))\times\mathcal{E}_i}$ for all $\epsilon \in \mathcal{E}_i$ such that

$$W_{j,S_{d_i},\mathcal{E}_i}(u) := \sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}_i}\left\| v_{h,j,S_{d_i},\mathcal{E}_i}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,(S_{d_i}\cup D(0,\tau_0))\times\mathcal{E}_i} \frac{u^h}{h!} \in \mathbb{C}\{u\}, \quad 0 \leq j \leq S-1. \tag{4.70}$$

Let

$$v_{h,j}(\tau,\epsilon) = \sum_{m\geq 0} \varphi_{h,j,m}(\epsilon)\frac{\tau^m}{(m!)^2} \tag{4.71}$$

be the convergent Taylor expansion of $v_{h,j}$ with respect to τ on $D(0,\tau_0)$ for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. We consider the formal series

$$\widehat{\varphi}_{h,j}(T,\epsilon) = \sum_{m\geq 0} \varphi_{h,j,m}(\epsilon)\frac{T^m}{m!} \tag{4.72}$$

for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. We define $\varphi_{h,j,S_{d_i},\mathcal{E}_i}(T,\epsilon)$ as the 1-sum (in the sense of Definition 4.1) of $\widehat{\varphi}_{j,h}(T,\epsilon)$ in the direction d_i. We deduce that $T \mapsto \varphi_{h,j,S_{d_i},\mathcal{E}_i}(T,\epsilon)$ defines a holomorphic function for all $T \in U_{d_i,\theta,\iota|\epsilon|}$ and for all $\epsilon \in \mathcal{E}_i$, where

$$U_{d_i,\theta,\iota|\epsilon|} = \left\{ T \in \mathbb{C}^* : |T| < \iota|\epsilon|, \left|d_i - \arg(T)\right| < \frac{\theta}{2}\right\} \tag{4.73}$$

for some $\theta > \pi$ and some constant $\iota > 0$ (independent of ϵ) for all $0 \leq j \leq S-1$.

From Proposition 4.12, for all $0 \leq i \leq \nu-1$, we consider the solution $Y_{h,S_{d_i},\mathcal{E}_i}(T,z,\epsilon)$ of the problem (4.55) with the initial conditions

$$\left(\partial_z^j Y_{h,S_{d_i},\mathcal{E}_i}\right)(T,0,\epsilon) = \varphi_{h,i,j}(T,\epsilon), \quad 0 \leq j \leq S-1,\ h \geq 0, \tag{4.74}$$

which defines a bounded and holomorphic function on $U_{d_i,\theta,\upsilon|\epsilon|} \times D(0,x_1/4) \times \mathcal{E}_i$.

Proposition 4.15. *The function defined by*

$$X_{h,i}(t,z,\epsilon) = Y_{h,S_{d_i},\mathcal{E}_i}(\epsilon t,z,\epsilon) \tag{4.75}$$

is holomorphic and bounded on $(\mathcal{T} \cap D(0,\iota'')) \times D(0,x_1/4) \times \mathcal{E}_i$, for all $h \geq 0$, all $0 \leq i \leq \nu - 1$, and for some $0 < \iota'' < \iota'$.

Moreover, the functions $G_{h,i} : \epsilon \mapsto X_{h,i}(t,z,\epsilon)$ from $\mathcal{E}_i$ into the Banach space $\mathcal{O}((\mathcal{T} \cap D(0,\iota'')) \times D(0,x_1/4))$ are the 1-sums on $\mathcal{E}_i$ of a formal series $\widehat{G}_h(\epsilon) \in \mathcal{O}((\mathcal{T} \cap D(0,\iota'')) \times D(0,x_1/4))[[\epsilon]]$. In other words, for all $h \geq 0$, there exists a function $g_h(s,t,z)$ which is holomorphic on $D(0,s_h) \times (\mathcal{T} \cap D(0,\iota'')) \times D(0,x_1/4)$ which admits for all $0 \leq i \leq \nu-1$, an analytic continuation $g_{h,i}(s,t,z)$ which is holomorphic on $(\mathcal{G}_{\kappa_i} \cup D(0,s_h)) \times (\mathcal{T} \cap D(0,\iota'')) \times D(0,x_1/4)$, where $\mathcal{G}_{\kappa_i}$ is an open sector centered at 0 with infinite radius and bisecting direction κ_i such that

$$X_{h,i}(t,z,\epsilon) = \epsilon^{-1} \int_{L_{\kappa_i}} g_{h,i}(s,t,z) e^{-s/\epsilon} ds \tag{4.76}$$

along a half-line $L_{\kappa_i} = \mathbb{R}_+ e^{i\kappa_i} \subset \mathcal{G}_{\kappa_i} \cup \{0\}$.

Proof. The proof is based on a cohomological criterion for summability of formal series with coefficients in a Banach space, see [27, page 121], which is known as the Ramis-Sibuya theorem in the literature.

Theorem (RS). *Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a Banach space over $\mathbb{C}$ and $\{\mathcal{E}_i\}_{0\leq i\leq \nu-1}$ a good covering in $\mathbb{C}^*$. For all $0 \leq i \leq \nu-1$, let G_i be a holomorphic function from $\mathcal{E}_i$ into the Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and let the cocycle $\Delta_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon)$ be a holomorphic function from the sector $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ into $\mathbb{E}$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$ and $G_\nu = G_0$). We make the following assumptions.*

(1) *The functions $G_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to the origin in $\mathbb{C}$ for all $0 \leq i \leq \nu-1$.*

(2) *The functions $\Delta_i(\epsilon)$ are exponentially flat of order 1 on Z_i for all $0 \leq i \leq \nu-1$. This means that there exist constants $C_i, A_i > 0$ such that*

$$\|\Delta_i(\epsilon)\|_{\mathbb{E}} \leq C_i e^{-A_i/|\epsilon|} \tag{4.77}$$

for all $\epsilon \in Z_i$ all $0 \leq i \leq \nu-1$.

Then, for all $0 \leq i \leq \nu-1$, the functions $G_i(\epsilon)$ are the 1 -sums on $\mathcal{E}_i$ of a 1 -summable formal series $\widehat{G}(\epsilon)$ in ϵ with coefficients in the Banach space $\mathbb{E}$.

By Definition 4.14 and the construction of $Y_{h,S_{d_i},\mathcal{E}_i}(T,z,\epsilon)$ in Proposition 4.12, we get that the function $X_{h,i}(t,z,\epsilon) = Y_{h,S_{d_i},\mathcal{E}_i}(\epsilon t,z,\epsilon)$ defines a bounded and holomorphic function on the domain $(\mathcal{T} \cap D(0,\iota\prime)) \times D(0,x_1/4) \times \mathcal{E}_i$ for all $h \geq 0$ all $0 \leq i \leq \nu-1$, where $0 < x_1 < \rho$ depends on S, $u_0 > 0$ (which denotes a common radius of absolute convergence of the series (4.69), (4.70)), $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$. More precisely, we have the following.

Lemma 4.16. *Consider the following:*

(1) *There exist a constant $0 < \iota'' < \iota'$, a constant u_1 such that $0 < u_1 < u_0$ (depending on u_0, $\mathcal{S}$ and b, σ), a constant x_1 such that $0 < x_1 < \rho$ (depending on S, u_0, $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$), and a constant*

$\widetilde{C}_i > 0$ *(depending on* $\max_{0\le j\le S-1} W_{j,S_{d_i},\mathcal{E}_i}(u_0)$ *(where* $W_{j,S_{d_i},\mathcal{E}_i}$ *are defined above),* $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b*) such that*

$$\sup_{t\in\mathcal{T}\cap D(0,\iota''),z\in D(0,x_1/4)} |X_{h,i}(t,z,\epsilon)| \le 2\widetilde{C}_i h!\left(\frac{2}{u_1}\right)^h \tag{4.78}$$

for all $\epsilon\in\mathcal{E}_i$*, for all* $0\le i\le \nu-1$*, and all* $h\ge 0$.

(2) *There exist a constant* $0<\iota''\le\iota'$*, a constant* u_1 *such that* $0<u_1<u_0$ *(depending on* u_0*,* $\mathcal{S}$ *and* b*,* σ*), a constant* x_1 *such that* $0<x_1<\rho$ *(depending on* S*,* u_0*,* $\mathcal{S}$*,* b*,* σ*,* $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$*, where* $x_0<\rho$*), a constant* $M_i>0$*, a constant* $K_i>0$ *(depending on* $\max_{0\le j\le S-1}W_{j,S_{d_q},\mathcal{E}_q}(u_0)$*, for* $q=i,i+1$ *(where* $W_{j,S_{d_q},\mathcal{E}_q}$ *are defined above),* $\max_{0\le j\le S-1}W_{j,\tau_0,\epsilon_0}(u_0)$, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$*,* S*,* u_0*,* x_0*,* $\mathcal{S}$*,* b*) such that*

$$\sup_{t\in\mathcal{T}\cap D(0,\iota''),z\in D(0,x_1/4)} |X_{h,i+1}(t,z,\epsilon)-X_{h,i}(t,z,\epsilon)| \le h!\left(\frac{2}{u_1}\right)^h 2K_i e^{-M_i/|\epsilon|} \tag{4.79}$$

for all $\epsilon\in\mathcal{E}_i\cap\mathcal{E}_{i+1}$*, for all* $0\le i\le\nu-1$*, and all* $h\ge 0$ *(where by convention* $X_{h,\nu}=X_{h,0}$*).*

Proof. (1) Let i be an integer such that $0\le i\le \nu-1$. From Proposition 4.12, we can write

$$X_{h,i}(t,z,\epsilon) = (\epsilon t)^{-1}\int_{L_{\gamma_i}} V_{h,S_{d_i},\mathcal{E}_i}(\tau,z,\epsilon)e^{-\tau/\epsilon t}d\tau, \tag{4.80}$$

where $L_{\gamma_i}=\mathbb{R}_+e^{\sqrt{-1}\gamma_i}\subset S_{d_i}\cup\{0\}$ and $V_{h,S_{d_i},\mathcal{E}_i}$ is a holomorphic function on $(S_{d_i}\cup D(0,\tau_0))\times D(0,x_1/4)\times\mathcal{E}_i$ for which the estimates (4.62) hold. By construction, the direction γ_i (which depends on ϵt) is chosen in such a way that $\cos(\gamma_i-\arg(\epsilon t))\ge\delta_1$, for all $\epsilon\in\mathcal{E}_i$, all $t\in\mathcal{T}\cap D(0,\iota')$, and for some fixed $\delta_1>0$. From the estimates (4.62), we get

$$\begin{aligned}
|X_{h,i}(t,z,\epsilon)| &\le |\epsilon t|^{-1}\int_0^{+\infty}\frac{C_{\Omega(d_i,\mathcal{E}_i)}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\left(1+\frac{r^2}{|\epsilon|^2}\right)^{-1}e^{\sigma\zeta(b)r/2|\epsilon|}e^{-r/|\epsilon||t|\cos(\gamma_i-\arg(\epsilon t))}dr\\
&\le |\epsilon t|^{-1}\int_0^{+\infty}\frac{C_{\Omega(d_i,\mathcal{E}_i)}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h e^{(\sigma\zeta(b)/2-\delta_1/|t|)(r/|\epsilon|)}dr\\
&= \frac{C_{\Omega(d_i,\mathcal{E}_i)}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\frac{1}{\delta_1-(\sigma\zeta(b)/2)|t|}\le\frac{C_{\Omega(d_i,\mathcal{E}_i)}}{\delta_2(1-2|z|/x_1)}h!\left(\frac{2}{u_1}\right)^h
\end{aligned} \tag{4.81}$$

for all $t\in\mathcal{T}\cap D(0,\iota')$, with $|t|<2(\delta_1-\delta_2)/(\sigma\zeta(b))$, for some $0<\delta_2<\delta_1$, and for all $\epsilon\in\mathcal{E}_{i+1}\cap\mathcal{E}_i$.

(2) Let i an integer such that $0 \leq i \leq \nu - 1$. From Proposition 4.12, we can write again

$$X_{h,i}(t,z,\epsilon) = (\epsilon t)^{-1} \int_{L_{\gamma_i}} V_{h,S_{d_i},\mathcal{E}_i}(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau,$$
$$X_{i+1}(t,z,\epsilon) = (\epsilon t)^{-1} \int_{L_{\gamma_{i+1}}} V_{h,S_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau, \tag{4.82}$$

where $L_{\gamma_i} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_i} \subset S_{d_i} \cup \{0\}$, $L_{\gamma_{i+1}} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_{i+1}} \subset S_{d_{i+1}} \cup \{0\}$, and $V_{h,S_{d_i},\mathcal{E}_i}$ (resp., $V_{h,S_{d_{i+1}},\mathcal{E}_{i+1}}$) is a holomorphic function on $(S_{d_i} \cup D(0,\tau_0)) \times D(0,x_1/4) \times \mathcal{E}_i$ (resp., on $(S_{d_{i+1}} \cup D(0,\tau_0)) \times D(0,x_1/4) \times \mathcal{E}_{i+1}$) for which the estimates (4.62) hold and which is moreover an analytic continuation of a function $V_h(\tau,z,\epsilon)$ which satisfies the estimates (4.63).

From the fact that $\tau \mapsto V_h(\tau,z,\epsilon)$ is holomorphic on $D(0,\tau_0)$ for all $(z,\epsilon) \in D(0,x_1/4) \times (D(0,\epsilon_0) \setminus \{0\})$, the integral of $\tau \mapsto V_h(\tau,z,\epsilon)$ along the union of a segment starting from 0 to $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$, an arc of circle with radius $\tau_0/2$ connecting $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$ and $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ and a segment starting from $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ to 0 is equal to zero. Therefore, we can rewrite the difference $X_{h,i+1} - X_{h,i}$ as a sum of three integrals:

$$\begin{aligned} X_{h,i+1}(t,z,\epsilon) - X_{h,i}(t,z,\epsilon) = (\epsilon t)^{-1} \Bigg(&\int_{L_{\tau_0/2,\gamma_{i+1}}} V_{h,S_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau \\ &- \int_{L_{\tau_0/2,\gamma_i}} V_{h,S_{d_i},\mathcal{E}_i}(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau \\ &+ \int_{C(\tau_0/2,\gamma_i,\gamma_{i+1})} V_h(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau \Bigg), \end{aligned} \tag{4.83}$$

where $L_{\tau_0/2,\gamma_i} = [\tau_0/2,+\infty)e^{\sqrt{-1}\gamma_i}$, $L_{\tau_0/2,\gamma_{i+1}} = [\tau_0/2,+\infty)e^{\sqrt{-1}\gamma_{i+1}}$, and $C(\tau_0/2,\gamma_i,\gamma_{i+1})$ is an arc of circle with radius $\tau_0/2$ connecting $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ with $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$ with a well-chosen orientation.

We give estimates for $I_1 = |(\epsilon t)^{-1} \int_{L_{\tau_0/2,\gamma_{i+1}}} V_{h,S_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\tau/\epsilon t} d\tau|$. By construction, the direction γ_{i+1} (which depends on ϵt) is chosen in such a way that $\cos(\gamma_{i+1} - \arg(\epsilon t)) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$, all $t \in \mathcal{T} \cap D(0,\iota)$, and for some fixed $\delta_1 > 0$. From the estimates (4.62), we get

$$\begin{aligned} I_1 &\leq |\epsilon t|^{-1} \int_{\tau_0/2}^{+\infty} \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{1-2|z|/x_1} h! \left(\frac{2}{u_1}\right)^h \left(1 + \frac{r^2}{|\epsilon|^2}\right)^{-1} e^{\sigma\zeta(b)r/2|\epsilon|} e^{-(r/|\epsilon||t|)\cos(\gamma_{i+1}-\arg(\epsilon t))} dr \\ &\leq |\epsilon t|^{-1} \int_{\tau_0/2}^{+\infty} \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{1-2|z|/x_1} h! \left(\frac{2}{u_1}\right)^h e^{(\sigma\zeta(b)/2-\delta_1/|t|)(r/|\epsilon|)} dr \\ &= \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{1-2|z|/x_1} h! \left(\frac{2}{u_1}\right)^h \frac{e^{-((\delta_1/|t|-\sigma\zeta(b)/2)(\tau_0/2))(1/|\epsilon|)}}{\delta_1 - (\sigma\zeta(b)/2)|t|} \\ &\leq \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{\delta_2(1-2|z|/x_1)} h! \left(\frac{2}{u_1}\right)^h e^{-(\delta_2\tau_0/2)/|\epsilon|\iota'} \end{aligned} \tag{4.84}$$

for all $t \in \mathcal{T} \cap D(0,\iota')$, with $|t| < 2(\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$.

We give estimates for $I_2 = |(\epsilon t)^{-1}\int_{L_{\tau_0/2,\gamma_i}} V_{h,S_{d_i},\mathcal{E}_i}(\tau,z,\epsilon)e^{-\tau/\epsilon t}d\tau|$. By construction, the direction γ_i (which depends on ϵt) is chosen in such a way that there exists a fixed $\delta_1 > 0$ with $\cos(\gamma_i - \arg(\epsilon t)) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$ and all $t \in \mathcal{T} \cap D(0,\iota')$. From the estimates (4.62), we deduce as before that

$$I_2 \leq \frac{C_{\Omega(d_i,\mathcal{E}_i)}}{\delta_2(1-2|z|/x_1)}h!\left(\frac{2}{u_1}\right)^h e^{-(\delta_2\tau_0/2)/|\epsilon|\iota'} \tag{4.85}$$

for all $t \in \mathcal{T}\cap D(0,\iota')$, with $|t| < 2(\delta_1-\delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1}\cap\mathcal{E}_i$.

Finally, we get estimates for $I_3 = |\epsilon t|^{-1}|\int_{C(\tau_0/2,\gamma_i,\gamma_{i+1})} V_h(\tau,z,\epsilon)e^{-\tau/\epsilon t}d\tau|$. From the estimates (4.63), we have

$$I_3 \leq |\epsilon t|^{-1}\left|\int_{\gamma_i}^{\gamma_{i+1}} \frac{C_{\Omega_{\tau_0,\epsilon_0}}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\left(1+\frac{(\tau_0/2)^2}{|\epsilon|^2}\right)^{-1} e^{\sigma\zeta(b)\tau_0/4|\epsilon|}e^{-(\tau_0/2|\epsilon||t|)\cos(\theta-\arg(\epsilon t))}\frac{\tau_0}{2}d\theta\right|. \tag{4.86}$$

By construction, the arc of circle $C(\tau_0/2,\gamma_i,\gamma_{i+1})$ is chosen in such a way that that $\cos(\theta - \arg(\epsilon t)) \geq \delta_1$ for all $\theta \in [\gamma_i,\gamma_{i+1}]$ (if $\gamma_i < \gamma_{i+1}$), $\theta \in [\gamma_{i+1},\gamma_i]$ (if $\gamma_{i+1} < \gamma_i$) for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$. From (4.86), we deduce that

$$\begin{aligned} I_3 &\leq |\gamma_{i+1}-\gamma_i|\frac{C_{\Omega_{\tau_0,\epsilon_0}}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\frac{\tau_0}{2}\frac{1}{|\epsilon t|}e^{-((\delta_1/|t|-\sigma\zeta(b)/2)(\tau_0/2))(1/|\epsilon|)} \\ &\leq |\gamma_{i+1}-\gamma_i|\frac{C_{\Omega_{\tau_0,\epsilon_0}}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\frac{\tau_0}{2}\frac{1}{|\epsilon t|}e^{-(\delta_2\tau_0/4)/|\epsilon t|}e^{-(\delta_2\tau_0/4)/|\epsilon|\iota'} \end{aligned} \tag{4.87}$$

for all $t \in \mathcal{T}\cap D(0,\iota')$, with $|t| < 2(\delta_1-\delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1}\cap\mathcal{E}_i$. Using the inequality (4.87) and the estimates (2.41), we deduce that

$$I_3 \leq |\gamma_{i+1}-\gamma_i|\frac{C_{\Omega_{\tau_0,\epsilon_0}}}{1-2|z|/x_1}h!\left(\frac{2}{u_1}\right)^h\frac{2e^{-1}}{\delta_2}e^{-(\delta_2\tau_0/4)/|\epsilon|\iota'} \tag{4.88}$$

for all $t \in \mathcal{T} \cap D(0,\iota')$, with $|t| < 2(\delta_1 - \delta_2)/(\sigma\zeta(b))$, and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$.

Finally, collecting the inequalities (4.84), (4.85), and (4.88), we deduce from (4.83), that

$$\begin{aligned} &|X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \\ &\quad\leq \frac{h!(2/u_1)^h}{1-2|z|/x_1}\left(\frac{C_{\Omega(d_{i+1},\mathcal{E}_i)}+C_{\Omega(d_i,\mathcal{E}_i)}}{\delta_2}e^{-(\delta_2\tau_0/2)/|\epsilon|\iota'} + |\gamma_{i+1}-\gamma_i|C_{\Omega_{\tau_0,\epsilon_0}}\frac{2e^{-1}}{\delta_2}e^{-(\delta_2\tau_0/4)/|\epsilon|\iota'}\right) \end{aligned} \tag{4.89}$$

for all $t \in \mathcal{T} \cap D(0,\iota')$, with $|t| < 2(\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$, and for all $0 \leq i \leq \nu - 1$. Hence, the estimates (4.79) hold.

Now, let us fix $h \geq 0$. For all $0 \leq i \leq \nu - 1$, we define $G_{h,i}(\epsilon) := (t,z) \mapsto X_{h,i}(t,z,\epsilon)$, which is, by Lemma 4.16, a holomorphic and bounded function from $\mathcal{E}_i$ into the Banach space

$\mathbb{E} = \mathcal{O}((\mathcal{T} \cap D(0, \iota'')) \times D(0, x_1/4))$ of holomorphic and bounded functions on the set $(\mathcal{T} \cap D(0, \iota'')) \times D(0, x_1/4)$ equipped with the supremum norm. Therefore, the property (1) of Theorem (RS) is satisfied for the functions $G_{h,i}$, $0 \leq i \leq \nu - 1$. From the estimates (4.79), we get that the cocycle $\Delta_i = G_{h,i+1}(\epsilon) - G_{h,i}(\epsilon)$ is exponentially flat of order 1 on $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ for all $0 \leq i \leq \nu - 1$. We deduce that the property (2) of Theorem (RS) is fulfilled for the functions $G_{h,i}$, $0 \leq i \leq \nu - 1$. From Theorem (RS), we get that $G_{h,i}(\epsilon)$ are the 1-sums of a formal series $\widehat{G}_h(\epsilon)$ with coefficients in $\mathbb{E}$. In particular, from Definition 4.1, we deduce the existence of the functions $g_{h,i}(s, t, z)$ which satisfy the expression (4.76). □

4.7. Analytic Transseries Solutions for a Singularly Perturbed Cauchy Problem

We keep the notations of the previous section.

Proposition 4.17. *The following singularly perturbed Cauchy problem*

$$\epsilon t^2 \partial_t \partial_z^S Z_0(t, z, \epsilon) + (\epsilon t + 1)\partial_z^S Z_0(t, z, \epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z, \epsilon) t^s \left(\partial_t^{k_0} \partial_z^{k_1} Z_0\right)(t, z, \epsilon) \tag{4.90}$$

for given initial data

$$\left(\partial_z^j Z_0\right)(t, 0, \epsilon) = \gamma_{0,j}(t, \epsilon) = \sum_{h\geq 0} \frac{\exp(-h\lambda/\epsilon t)}{h!} \varphi_{h,0,j}(\epsilon t, \epsilon), \quad 0 \leq j \leq S - 1, \tag{4.91}$$

which are holomorphic and bounded functions on $(\mathcal{T} \cap D(0, \iota'')) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$*, has a solution*

$$Z_0(t, z, \epsilon) = \sum_{h\geq 0} \frac{\exp(-h\lambda/\epsilon t)}{h!} X_{h,0}(t, z, \epsilon), \tag{4.92}$$

which defines a holomorphic and bounded function on $(\mathcal{T} \cap D(0, \iota'')) \times D(0, \delta_{Z_0}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$*, for some* $\iota'', \delta_{Z_0} > 0$.

Proof. Let $h \geq 0$ and $0 \leq j \leq S - 1$. By construction, we have that $\varphi_{h,0,j}(\epsilon t, \epsilon) = (\partial_z^j X_{h,0})(t, 0, \epsilon)$ for all $t \in \mathcal{T}$ and all $\epsilon \in \mathcal{E}_0$. From Lemma 4.16, (1), we get that there exist a constant $\iota'' > 0$, a constant u_1 such that $0 < u_1 < u_0$ (depending on u_0, $\mathcal{S}$ and b, σ), and a constant $\check{C}_0 > 0$ (depending on $\max_{0\leq j\leq S-1} W_{j,S_{d_0},\mathcal{E}_0}(u_0)$ (where $W_{j,S_{d_0},\mathcal{E}_0}$ are defined above), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}} |\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b) such that

$$\sup_{t\in\mathcal{T}\cap D(0,\iota'')} \left|\varphi_{h,0,j}(\epsilon t, \epsilon)\right| \leq h! \left(\frac{2}{u_1}\right)^h \check{C}_0 \tag{4.93}$$

for all $\epsilon \in \mathcal{E}_0$, all $0 \leq j \leq S-1$, all $h \geq 0$. From (4.93) and from the property (3) of Definition 4.14, we deduce the estimates

$$\sup_{t \in \mathcal{T} \cap D(0,\iota'')} \left|\gamma_{0,j}(t,\epsilon)\right| \leq \check{C}_0 \sum_{h \geq 0} \left(\frac{2\exp(-(|\lambda|/\epsilon_0 \iota'')\cos(\pi/2 - \delta_{\mathcal{T}}))}{u_1} \right)^h, \tag{4.94}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. This latter sum converges provided that ϵ_0 is small enough. We deduce that $\gamma_{0,j}(t,\epsilon)$ defines a holomorphic and bounded function on $(\mathcal{T} \cap D(0,\iota'')) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$.

Likewise, from (4.78) and from the property (3) of Definition 4.14, we deduce that there exist a constant $\iota'' > 0$, a constant u_1 such that $0 < u_1 < u_0$ (depending on u_0, $\mathcal{S}$ and b, σ), a constant x_1 such that $0 < x_1 < \rho$ (depending on S, u_0, $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1) \in \mathcal{S}} |b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1) \in \mathcal{S}} |\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$) and a constant $\widetilde{C}_0 > 0$ (depending on $\max_{0 \leq j \leq S-1} W_{j,S_{d_0},\mathcal{E}_0}(u_0)$ (where $W_{j,S_{d_0},\mathcal{E}_0}$ are defined above), $|\lambda|$, $\max_{(s,k_0,k_1) \in \mathcal{S}} |b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1) \in \mathcal{S}} |\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, u_0, x_0, $\mathcal{S}$, b) such that

$$\sup_{t \in \mathcal{T} \cap D(0,\iota''), z \in D(0,\delta_{Z_0})} |Z_0(t,z,\epsilon)| \leq \frac{\widetilde{C}_0}{1 - 2\delta_{Z_0}/x_1} \sum_{h \geq 0} \left(\frac{2\exp(-(|\lambda|/\epsilon_0 \iota'')\cos(\pi/2 - \delta_{\mathcal{T}}))}{u_1} \right)^h \tag{4.95}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. Again, this latter sum converges if ϵ_0 is small enough and if $0 < \delta_{Z_0} \leq x_1/4$. We get that $Z_0(t,z,\epsilon)$ defines a holomorphic and bounded function on $(\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{Z_0}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$. By construction, we have that $(\partial_z^j Z_0)(t,0,\epsilon) = \gamma_{0,j}(t,\epsilon)$, for $0 \leq j \leq S-1$. Finally, from Proposition 4.3, we deduce that $Z_0(t,z,\epsilon)$ solves (4.90). □

5. Parametric Stokes Relations and Analytic Continuation of the Borel Transform in the Perturbation Parameter

5.1. Assumptions on the Initial Data

We keep the notations of the previous section. Now, we make the following additional assumption that there exists a sequence of unbounded open sectors S_{d_0,ϑ_n} such that

$$S_{d_0} \subset S_{d_0,\vartheta_n} \subset \mathcal{M}_{d_0} \cup S_{d_0} \tag{5.1}$$

for all $n \geq 0$ and a sequence of real numbers ζ_n, $n \geq 0$ such that

$$e^{i\zeta_n} \in S_{d_0,\vartheta_n}, \qquad \lim_{n \to +\infty} \zeta_n = \arg(\lambda) \tag{5.2}$$

with the property that $\arg(e^{i\zeta_n}/\epsilon t) \in (-\pi/2 + \delta_{\mathcal{T}}, \pi/2 - \delta_{\mathcal{T}})$ for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T}$, and for all $n \geq 0$ (where $\mathcal{T}$ and $\delta_{\mathcal{T}}$ were introduced in Definition 4.14). We also make the assumption that for all $n \geq 0$, the function $v_{h,j,S_{d_0},\mathcal{E}_0}(\tau,\epsilon)$ can be analytically continued to a holomorphic function $\tau \mapsto v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)$ on S_{d_0,ϑ_n} for all $\epsilon \in \mathcal{E}_0$ such that

$$v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon) \in E_{j,\epsilon,\sigma,(S_{d_0,\vartheta_n} \cup D(0,\tau_0)) \times \mathcal{E}_0} \tag{5.3}$$

with the property that

$$W_{j,S_{d_0,\vartheta_n},\mathcal{E}_0}(u) := \sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}_0}\left\|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,(S_{d_0,\vartheta_n}\cup D(0,\tau_0))\times\mathcal{E}_0}\frac{u^h}{h!}\in\mathbb{C}\{u\},\quad 0\leq j\leq S-1 \tag{5.4}$$

and has a common radius of absolute convergence (denoted by $u_{\mathcal{E}_0}>0$) for all $n\geq 0$. From the assumption (5.4), we get a constant $u_{0,j}>0$ (depending on $j\in\{0,\dots,S-1\}$) and a constant $C_{n,j}>0$ (depending on n and $j\in\{0,\dots,S-1\}$) such that

$$\sup_{\epsilon\in\mathcal{E}_0}\left\|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)\right\|_{j,\epsilon,\sigma,(S_{d_0,\vartheta_n}\cup D(0,\tau_0))\times\mathcal{E}_0}\leq C_{n,j}\left(\frac{1}{u_{0,j}}\right)^h h! \tag{5.5}$$

for all $h\geq 0$. We deduce that

$$\left|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right|\leq C_{n,j}\left(\frac{1}{u_{0,j}}\right)^h h!\exp\left(\frac{\sigma}{2|\epsilon|}r_b(j)r\right) \tag{5.6}$$

for all $r\geq 0$, all $\epsilon\in\mathcal{E}_0$, all $0\leq j\leq S-1$, and all $h\geq 0$. In particular, we have that $r\mapsto v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)$ belongs to the space $L_{0,\tilde{\sigma}/2,\epsilon}$ for $\tilde{\sigma}>\sigma r_b(S-1)$. Moreover, from Proposition 2.7, we deduce that $r\mapsto v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)$ belongs to the space $\mathfrak{D}'_{0,\tilde{\sigma},\epsilon}$ and that there exists a universal constant $C_1>0$ such that

$$\begin{aligned}\left\|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)\right\|_{0,\tilde{\sigma},\epsilon,d}&\leq C_1\left\|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)\right\|_{0,\tilde{\sigma}/2,\epsilon}\\&\leq\frac{2|\epsilon|}{\tilde{\sigma}-\sigma r_b(S-1)}C_1C_{n,j}\left(\frac{1}{u_{0,j}}\right)^h h!\end{aligned} \tag{5.7}$$

for all $0\leq j\leq S-1$, all $h\geq 0$, and all $n\geq 0$, all $\epsilon\in\mathcal{E}_0$.

We make the crucial assumption that for all $0\leq j\leq S-1$, there exists a sequence of distributions $v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\in\mathfrak{D}'_{0,\tilde{\sigma},\epsilon}$, for $h\geq 0$, a constant $u_j>0$ and a sequence $I_{n,j}>0$ with $\lim_{n\to+\infty}I_{n,j}=0$ such that

$$\sup_{\epsilon\in\mathcal{E}_0}\left\|v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)-v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{0,\tilde{\sigma},\epsilon,d}\leq I_{n,j}h!\left(\frac{1}{u_j}\right)^h \tag{5.8}$$

for all $n\geq 0$ and all $h\geq 0$. From the estimates (5.7) and (5.8), we deduce that

$$\sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}_0}\left\|v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{0,\tilde{\sigma},\epsilon,d}\frac{u^h}{h!}\in\mathbb{C}\{u\},\quad 0\leq j\leq S-1. \tag{5.9}$$

Lemma 5.1. *Let $\tilde{\sigma} > \sigma r_b(S-1)$. One can write the initial data $\gamma_{0,j}(t,\epsilon)$ in the form of a Laplace transform in direction* $\arg(\lambda)$*,*

$$\gamma_{0,j}(t,\epsilon) = \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)\right)(\epsilon t), \tag{5.10}$$

where $\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon) \in \mathfrak{D}'_{0,\tilde{\sigma},\epsilon}$ and for all $0 \leq j \leq S-1$, and all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T} \cap D(0,\iota')$.

Proof. For $0 \leq j \leq S-1$, from the definition of the initial data, we can write

$$\begin{aligned}
\gamma_{0,j}(t,\epsilon) &= \sum_{h\geq 0} \frac{\exp(-h\lambda/\epsilon t)}{h!}\frac{1}{\epsilon t}\int_{L_{\zeta_n}} v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)\exp\left(-\frac{\tau}{\epsilon t}\right)d\tau \\
&= \sum_{h\geq 0} \frac{\exp\left(-\left(h|\lambda|e^{i\arg(\lambda)}\right)/\epsilon t\right)}{h!}\frac{e^{i\zeta_n}}{\epsilon t}\int_0^{+\infty} v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\exp\left(-r\frac{e^{i\zeta_n}}{\epsilon t}\right)dr
\end{aligned} \tag{5.11}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T} \cap D(0,\iota')$, and all $n \geq 0$. Now, we can write

$$\mathcal{L}_{\zeta_n}\left(v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right)(\epsilon t) = \mathcal{L}_{\arg(\lambda)}\left(v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right) \tag{5.12}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T} \cap D(0,\iota\prime)$, all $n \geq 0$. From the continuity estimates (3.3) for the Laplace transform, we deduce that for given $t \in \mathcal{T} \cap D(0,\iota\prime)$, $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, there exists a constant $C_{\epsilon,t}$ (depending on ϵ, t) such that

$$\begin{aligned}
&\left|\mathcal{L}_{\arg(\lambda)}\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)(\epsilon t) - \mathcal{L}_{\arg(\lambda)}\left(v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right)\right| \\
&\quad\leq C_{\epsilon,t}\left\|\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) - v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)\right)\right\|_{0,\tilde{\sigma},\epsilon,d} \\
&\qquad+\left|\mathcal{L}_{\arg(\lambda)}\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right) - \mathcal{L}_{\arg(\lambda)}\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)(\epsilon t)\right|
\end{aligned} \tag{5.13}$$

for all $n \geq 0$. By letting n tend to $+\infty$ in this latter inequality and using the hypothesis (5.8), we get that

$$\mathcal{L}_{\zeta_n}\left(v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right)(\epsilon t) = \mathcal{L}_{\arg(\lambda)}\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)(\epsilon t) \tag{5.14}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T} \cap D(0,\iota')$, and all $n \geq 0$.

On the other hand, from Corollary 2.10, we have that for all $h \geq 0$, the distribution $\partial_r^{-h}(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon))$ belongs to $\mathfrak{D}'_{0,\tilde{\sigma},\epsilon}$ and that there exists a universal constant $C_3 > 0$ such that

$$\left\|\partial_r^{-h}(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon))\right\|_{0,\tilde{\sigma},\epsilon,d} \leq C_3\left(\frac{|\epsilon|}{\tilde{\sigma}}\right)^h\left\|v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{0,\tilde{\sigma},\epsilon,d} \tag{5.15}$$

for all $h \geq 0$, all $0 \leq j \leq S-1$. From (5.14) and using Propositions 3.3 and 3.7, we can write

$$\begin{aligned}
&\frac{\exp\left(-h|\lambda|e^{i\arg(\lambda)}/\epsilon t\right)}{h!}\frac{e^{i\zeta_n}}{\epsilon t}\int_0^{+\infty} v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\exp\left(-r\frac{e^{i\zeta_n}}{\epsilon t}\right)dr \\
&= \left(\frac{e^{i\arg(\lambda)}}{\epsilon t}\right)^h \frac{\exp\left(-h|\lambda|e^{i\arg(\lambda)}/\epsilon t\right)}{h!}\mathcal{L}_{\arg(\lambda)}\left(\partial_r^{-h}\left(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)\right)(\epsilon t) \\
&= \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right)(\epsilon t),
\end{aligned} \tag{5.16}$$

where

$$\mathbb{V}_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) = \frac{\left(f_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r-|\lambda|h,\epsilon)1_{[|\lambda|h,+\infty)}(r)\right)^{(h)}}{h!} \in \mathfrak{D}'_{0,\widetilde{\sigma},\epsilon} \tag{5.17}$$

with $f_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) = \partial_r^{-h}(v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)) \in \mathfrak{D}'_{0,\widetilde{\sigma},\epsilon}$ for all $h \geq 0$ and all $0 \leq j \leq S-1$. From Proposition 3.6, we have a universal constant $A > 0$ and a constant $B(\widetilde{\sigma},b,\epsilon)$ (depending on $\widetilde{\sigma}$, b, and ϵ, which tend to zero as $\epsilon \to 0$) such that

$$\left\|\mathbb{V}_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{0,\widetilde{\sigma},\epsilon,d} \leq A\frac{(B(\widetilde{\sigma},b,\epsilon))^h}{h!}\left\|f_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{0,\widetilde{\sigma},\epsilon,d}. \tag{5.18}$$

From the estimates (5.9) and using (5.15), (5.18), we deduce that the distribution

$$\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon) = \sum_{h\geq 0}\mathbb{V}_{h,j,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) \in \mathfrak{D}'_{0,\widetilde{\sigma},\epsilon,d} \tag{5.19}$$

for all $0 \leq j \leq S-1$, if $\epsilon_0 > 0$ is chosen small enough. Finally, by the continuity estimates (3.3) for the Laplace transform $\mathcal{L}_{\arg(\lambda)}$ and the formula (5.11), (5.16), we get the expression (5.10). □

On the other hand, we assume the existence of a sequence of unbounded open sectors S_{d_1,δ_n} with

$$S_{d_1} \subset S_{d_1,\delta_n} \subset \mathcal{M}_{d_1} \cup S_{d_1} \tag{5.20}$$

for all $n \geq 0$ and a sequence of real numbers ξ_n, $n \geq 0$ such that

$$e^{i\xi_n} \in S_{d_1,\delta_n}, \qquad \lim_{n\to+\infty}\xi_n = \arg(\lambda) \tag{5.21}$$

with the property that $\arg(e^{i\xi_n}/\epsilon t) \in (-\pi/2+\delta_{\mathcal{T}}, \pi/2-\delta_{\mathcal{T}})$ for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $t \in \mathcal{T}$, and all $n \geq 0$ (where $\mathcal{T}$ and $\delta_{\mathcal{T}}$ are introduced in Definition 4.14). We make the assumption that for all $n \geq 0$, the function $v_{h,j,S_{d_1},\mathcal{E}_1}(\tau,\epsilon)$ can be analytically continued to a holomorphic function $\tau \mapsto v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon)$ on S_{d_1,δ_n} for all $\epsilon \in \mathcal{E}_1$ such that

$$v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon) \in E_{j,\epsilon,\sigma,(S_{d_1,\delta_n} \cup D(0,\tau_0)) \times \mathcal{E}_1} \tag{5.22}$$

with the property that

$$W_{j,S_{d_1,\delta_n},\mathcal{E}_1}(u) := \sum_{h \geq 0} \sup_{\epsilon \in \mathcal{E}_1} \left\| v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon) \right\|_{j,\epsilon,\sigma,(S_{d_1,\delta_n} \cup D(0,\tau_0)) \times \mathcal{E}_1} \frac{u^h}{h!} \in \mathbb{C}\{u\}, \quad 0 \leq j \leq S-1 \tag{5.23}$$

and has a common radius of absolute convergence (defined by $u_{\mathcal{E}_1} > 0$) for all $n \geq 0$. From the assumption (5.23), we get a constant $u_{1,j} > 0$ (depending on $j \in \{0,\ldots,S-1\}$) and a constant $C_{n,1,j} > 0$ (depending on n and $j \in \{0,\ldots,S-1\}$) such that

$$\sup_{\epsilon \in \mathcal{E}_1} \left\| v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon) \right\|_{j,\epsilon,\sigma,(S_{d_1,\delta_n} \cup D(0,\tau_0)) \times \mathcal{E}_1} \leq C_{n,1,j} \left(\frac{1}{u_{1,j}} \right)^h h! \tag{5.24}$$

for all $h \geq 0$. We deduce that

$$\left| v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},\epsilon\right) \right| \leq C_{n,1,j} \left(\frac{1}{u_{1,j}} \right)^h h! \exp\left(\frac{\sigma}{2|\epsilon|} r_b(j) r \right) \tag{5.25}$$

for all $r \geq 0$, all $\epsilon \in \mathcal{E}_1$, all $0 \leq j \leq S-1$, and all $h \geq 0$. In particular, we have that $r \mapsto v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)$ belongs to the space $L_{0,\tilde{\sigma}/2,\epsilon}$ for $\tilde{\sigma} > \sigma r_b(S-1)$. Moreover, from Proposition 2.7, we deduce that $r \mapsto v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)$ belongs to the space $\mathfrak{D}'_{0,\tilde{\sigma},\epsilon}$ and that there exists a universal constant $C_1 > 0$ such that

$$\begin{aligned} \left\| v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon) \right\|_{0,\tilde{\sigma},\epsilon,d} &\leq C_1 \left\| v_{h,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon) \right\|_{0,\tilde{\sigma}/2,\epsilon} \\ &\leq \frac{2|\epsilon|}{\tilde{\sigma} - \sigma r_b(S-1)} C_1 C_{n,1,j} \left(\frac{1}{u_{1,j}} \right)^h h! \end{aligned} \tag{5.26}$$

for all $0 \leq j \leq S-1$, all $h \geq 0$, all $n \geq 0$, and all $\epsilon \in \mathcal{E}_1$.

Now, we make the crucial assumption that for all $0 \leq j \leq S-1$, there exists a sequence $J_{n,j} > 0$ with $\lim_{n \to +\infty} J_{n,j} = 0$ such that

$$\sup_{\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1} \left\| v_{0,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon) - \mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon) \right\|_{0,\tilde{\sigma},\epsilon,d} \leq J_{n,j} \tag{5.27}$$

for all $n \geq 0$, where $\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)$ are the distributions defined in Lemma 5.1.

5.2. The Stokes Relation and the Main Result

In the next proposition, we establish a connection formula for the two holomorphic solutions $X_{0,0}(t,z,\epsilon)$ and $X_{0,1}(t,z,\epsilon)$ of (4.90) constructed in Proposition 4.15.

Proposition 5.2. *Let the assumptions* (5.1), (5.4), (5.8), (5.20), (5.23), *and* (5.27) *hold for the initial data. Then, there exists* $0 < \delta_{D_{0,1}} < \delta_{Z_0}$ *such that one can write the following connection formula:*

$$X_{0,1}(t,z,\epsilon) = Z_0(t,z,\epsilon) = X_{0,0}(t,z,\epsilon) + \sum_{h\geq 1}\frac{\exp(-h\lambda/\epsilon t)}{h!}X_{h,0}(t,z,\epsilon) \tag{5.28}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, *all* $t \in \mathcal{T} \cap D(0,\iota'')$, *and all* $z \in D(0,\delta_{D_{0,1}})$.

The proof of this proposition will need two long steps and will be the consequence of the formula (5.79) and (5.124) from Lemmas 5.5 and 5.7.

Step 1. In this step, we show that the function $Z_0(t,z,\epsilon)$ can be expressed as a Laplace transform of some staircase distribution in direction $\arg(\lambda)$ satisfying the problem (5.80), (5.81).

From the assumption (5.4), we deduce from Proposition 4.12 that the function $V_{h,S_{d_0},\mathcal{E}_0}(\tau,z,\epsilon)$ constructed in (4.80) has an analytic continuation denoted by $V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,z,\epsilon)$ on the domain $(S_{d_0,\vartheta_n} \cup D(0,\tau_0)) \times D(0,\delta_{\mathcal{E}_0}) \times \mathcal{E}_0$ which satisfies estimates of the form (4.62) for all $n \geq 0$, where $\delta_{\mathcal{E}_0} > 0$ depends on S, $u_{\mathcal{E}_0}$ (which denotes a common radius of convergence of the series (5.4)), $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$. This constant $\delta_{\mathcal{E}_0}$ is, therefore, independent of n and h. Now, one defines the functions

$$\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon) := V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right) \tag{5.29}$$

for all $r \geq 0$, all $z \in D(0,\delta_{\mathcal{E}_0})$, all $\epsilon \in \mathcal{E}_0$, and all $n \geq 0$.

Lemma 5.3. *Let* $\check{\sigma} > \widetilde{\sigma} > \sigma r_b(S-1)$. *Then, there exists* $0 < \delta_D < \delta_{\mathcal{E}_0}$ *(depending on S, b,* $\check{\sigma}$, $|\lambda|$, u_j, $0 \leq j \leq S-1$ *(introduced in* (5.8)*),* $\mathcal{S}$, $u_{\mathcal{E}_0}$, ρ, μ, *A, B (introduced in Lemma 5.4)), there exist* $M_1 > 0$ *(depending on* $\mathcal{S}$, *S,* $\check{\sigma}$, $|\lambda|$, u_j, *for* $0 \leq j \leq S-1$, ρ, μ, *A, B),* $M_1' > 0$ *(depending on* $\mathcal{S}$, *S,* $\check{\sigma}$, $|\lambda|$, ρ, μ, ρ', μ' *(introduced in Lemma 5.4), A, B,* u_j *for* $0 \leq j \leq S-1$*) and a constant* U_1 *(depending on* $\mathcal{S}$, *S,* $\check{\sigma}$, $|\lambda|$, ρ, μ, *A, B,* $u_{\mathcal{E}_0}$, u_j *for* $0 \leq j \leq S-1$*) such that for all* $h \geq 0$ *all* $n \geq 0$, *there exists a staircase distribution* $\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon) \in \mathfrak{D}'(\check{\sigma},\epsilon,\delta_D)$ *with*

$$\sup_{\epsilon\in\mathcal{E}_0}\left\|\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon) - \mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_D)} \leq \left(M_1\max_{0\leq j\leq S-1}I_{n,j} + M_1'D_n\right)h!\left(\frac{2}{U_1}\right)^h, \tag{5.30}$$

where $I_{n,j}$ is a positive sequence (converging to 0 as n tends to ∞) introduced in the assumption (5.8) *and D_n is the positive sequence (tending to 0 as $n \to +\infty$) introduced in Lemma 5.4. Moreover, one has*

$$\sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}_0}\left\|\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_D)}\frac{u^h}{h!}\in\mathbb{C}\{u\}. \tag{5.31}$$

Proof. From the estimates (4.54), we can write

$$V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,z,\epsilon)=\sum_{\beta\geq 0}V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)\frac{z^\beta}{\beta!}, \tag{5.32}$$

where $V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)$ are holomorphic functions such that there exists a constant u_1 such that $0<u_1<u_{\mathcal{E}_0}$ (depending on $u_{\mathcal{E}_0}$, $\mathcal{S}$, and b, σ), a constant x_1 such that $0<x_1<\rho$ (depending on S, $u_{\mathcal{E}_0}$, $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0<\rho$), and a constant $C_{\Omega(d_0,\mathcal{E}_0),n}>0$ (depending on $\max_{0\leq j\leq S-1}W_{j,S_{d_0,\vartheta_n},\mathcal{E}_0}(u_{\mathcal{E}_0})$ (where $W_{j,S_{d_0,\vartheta_n},\mathcal{E}_0}$ are defined in (5.4)), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, $u_{\mathcal{E}_0}$, x_0, $\mathcal{S}$, b) with

$$\left|V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,\epsilon)\right|\leq C_{\Omega(d_0,\mathcal{E}_0),n}h!\beta!\left(\frac{2}{u_1}\right)^h\left(\frac{2}{x_1}\right)^\beta\left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1}\exp\left(\frac{\sigma}{2|\epsilon|}r_b(\beta)|\tau|\right) \tag{5.33}$$

for all $\tau\in S_{d_0,\vartheta_n}\cup D(0,\tau_0)$, $\epsilon\in\mathcal{E}_0$, all $h\geq 0$, all $\beta\geq 0$, and all $n\geq 0$. We deduce that

$$\left|V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\right|\leq C_{\Omega(d_0,\mathcal{E}_0),n}\left(\frac{2}{u_1}\right)^h\left(\frac{2}{x_1}\right)^\beta h!\beta!\exp\left(\frac{\sigma}{2|\epsilon|}r_b(\beta)r\right) \tag{5.34}$$

for all $r\geq 0$, all $\epsilon\in\mathcal{E}_0$, all $\beta\geq 0$, all $h\geq 0$, and all $n\geq 0$. In particular, $r\mapsto V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)$ belongs to $L_{\beta,\check{\sigma}/2,\epsilon}$. From Proposition 2.7, we deduce that $r\mapsto V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)$ belongs to $\mathfrak{D}'_{\beta,\check{\sigma},\epsilon}$. From Proposition 2.7 and (5.34), we get a universal constant $C_1>0$ such that

$$\begin{aligned}\left\|V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d}&\leq C_1\left\|V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon)\right\|_{\beta,\check{\sigma}/2,\epsilon}\\&\leq C_1C_{\Omega(d_0,\mathcal{E}_0),n}\frac{2|\epsilon|}{\check{\sigma}-\sigma}\left(\frac{2}{u_1}\right)^h\left(\frac{2}{x_1}\right)^\beta h!\beta!\end{aligned} \tag{5.35}$$

for all $\beta\geq 0$, all $h\geq 0$, and all $n\geq 0$. From (5.35), we deduce that the distribution

$$\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon)=\sum_{\beta\geq 0}V_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right)\frac{z^\beta}{\beta!}\in\mathfrak{D}'\left(\check{\sigma},\epsilon,\check{\delta}\right) \tag{5.36}$$

for all $\epsilon\in\mathcal{E}_0$, all $\check{\delta}<x_1/2$, all $h\geq 0$, and all $n\geq 0$.

One gets from (4.20), (4.21) and the assumption (4.44) that the following problem holds:

$$\begin{aligned}
&\left(re^{i\zeta_n}+1+\lambda h\right)\partial_z^S \mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon)\\
&\quad=\sum_{(s,k_0,k_1)\in\mathcal{S}}\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon)\epsilon^{k_0-s}e^{i(s-k_0)\zeta_n}\left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}}\alpha^1_{m,p}r^m\partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon)\right)\\
&\qquad+\sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1}\frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon)c_q^{k_0^1}(h\lambda)^q\epsilon^{k_0-s}e^{i(s-k_0-q)\zeta_n}\\
&\qquad\times\left(\sum_{(m,p)\in\mathcal{O}^2_{s-k_0-q}}\alpha^{2,q}_{m,p}r^m\partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon)\right)
\end{aligned}\tag{5.37}$$

with initial data

$$\left(\partial_z^j\mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\right)(r,0,\epsilon)=v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},\epsilon\right),\quad 0\leq j\leq S-1.\tag{5.38}$$

On the other hand, we consider the problem

$$\begin{aligned}
&\left(re^{i\arg(\lambda)}+1+\lambda h\right)\partial_z^S \mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\\
&\quad=\sum_{(s,k_0,k_1)\in\mathcal{S}}\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon)\epsilon^{k_0-s}e^{i(s-k_0)\arg(\lambda)}\left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}}\alpha^1_{m,p}r^m\partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)\\
&\qquad+\sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1}\frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1}(z,\epsilon)c_q^{k_0^1}(h\lambda)^q\epsilon^{k_0-s}e^{i(s-k_0-q)\arg(\lambda)}\\
&\qquad\times\left(\sum_{(m,p)\in\mathcal{O}^2_{s-k_0-q}}\alpha^{2,q}_{m,p}r^m\partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)
\end{aligned}\tag{5.39}$$

with initial data

$$\left(\partial_z^j\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}\right)(r,0,\epsilon)=v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon),\quad 0\leq j\leq S-1.\tag{5.40}$$

In the next lemma, we give estimates for the coefficients of (5.37) and (5.39).

Lemma 5.4. *Let*

$$\widetilde{b}_{s,k_0,k_1}(z,\epsilon)=\sum_{\beta\geq 0}\widetilde{b}_{s,k_0,k_1,\beta}(\epsilon)\frac{z^\beta}{\beta!}\tag{5.41}$$

the convergent Taylor expansion of $\tilde{b}_{s,k_0,k_1}$ *with respect to* z *near 0. Let* $\alpha \in \mathbb{R}$ *be a real number. Then, there exist positive constants* A, B, ρ, ρ', μ, μ' *and a sequence* $D_n > 0$ *such that* $\lim_{n\to+\infty} D_n = 0$ *with*

$$\left|\partial_r^q\left(\frac{\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)e^{i\alpha\arg(\lambda)}}{re^{i\arg(\lambda)}+1+\lambda h}\right)\right| \leq AB^{-\beta}\frac{\beta!q!}{(\rho(r+\mu))^{q+1}},$$
$$\left|\partial_r^q\left(\frac{\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)e^{i\alpha\zeta_n}}{re^{i\zeta_n}+1+\lambda h}\right)\right| \leq AB^{-\beta}\frac{\beta!q!}{(\rho(r+\mu))^{q+1}}, \tag{5.42}$$

$$\left|\partial_r^q\left(\frac{\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)e^{i\alpha\arg(\lambda)}}{re^{i\arg(\lambda)}+1+\lambda h}\right)-\partial_r^q\left(\frac{\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)e^{i\alpha\zeta_n}}{re^{i\zeta_n}+1+\lambda h}\right)\right| \leq D_nB^{-\beta}\frac{\beta!q!}{(\rho'(r+\mu'))^{q+1}} \tag{5.43}$$

for all $q \geq 0$, *all* $\beta \geq 0$, *all* $n \geq 0$, *all* $h \geq 0$, *all* $r \geq 0$, *and all* $\epsilon \in \mathcal{E}_0$.

Proof. We first show (5.42). From the fact that $\tilde{b}_{s,k_0,k_1}(z,\epsilon)$ is holomorphic near $z = 0$, we get from the Cauchy formula that there exist $A, B > 0$ such that

$$\left|\tilde{b}_{s,k_0,k_1,\beta}(\epsilon)\right| \leq AB^{-\beta}\beta! \tag{5.44}$$

for all $\beta \geq 0$, and all $\epsilon \in \mathcal{E}_0$. On the other hand, from Definition 4.14(3.1), there exist $\rho, \mu > 0$ such that $|re^{i\zeta_n} + 1 + \lambda h| \geq \rho(r+\mu)$ for all $r \geq 0$, all $h \geq 0$, and all $n \geq 0$. Hence,

$$\left|\partial_r^q\left(\frac{e^{i\alpha\zeta_n}}{re^{i\zeta_n}+1+\lambda h}\right)\right| \leq \frac{q!}{\left|re^{i\zeta_n}+1+\lambda h\right|^{q+1}} \leq \frac{q!}{(\rho(r+\mu))^{q+1}} \tag{5.45}$$

for all $r \geq 0$, and all $h \geq 0$, all $q \geq 0$, all $n \geq 0$. We deduce (5.42) from (5.44) and (5.45).

Now, we show (5.43). Using the classical identities $ab - cd = (a-c)b + c(b-d)$ and $b^{q+1} - a^{q+1} = (b-a) \times \sum_{s=0}^{q} a^s b^{q-s}$, we get the estimates

$$\begin{aligned}
&\left|\partial_r^q\left(\frac{e^{i\alpha\arg(\lambda)}}{re^{i\arg(\lambda)}+1+\lambda h}\right)-\partial_r^q\left(\frac{e^{i\alpha\zeta_n}}{re^{i\zeta_n}+1+\lambda h}\right)\right|\\
&\quad\leq q!\left|\frac{e^{i\alpha\arg(\lambda)}e^{iq\arg(\lambda)}}{(re^{i\arg(\lambda)}+1+\lambda h)^{q+1}}-\frac{e^{i\alpha\zeta_n}e^{iq\zeta_n}}{(re^{i\zeta_n}+1+\lambda h)^{q+1}}\right|\\
&\quad\leq q!\left(\left|e^{i\zeta_n}-e^{i\arg(\lambda)}\right|\times\left(\sum_{s=1}^{q+1}\frac{r}{\left|re^{i\arg(\lambda)}+1+\lambda h\right|^{q+2-s}\left|re^{i\zeta_n}+1+\lambda h\right|^{s}}\right)\right.\\
&\qquad\left.+\frac{\left|e^{i\alpha\arg(\lambda)}-e^{i\alpha\zeta_n}\right|+\left|e^{i\arg(\lambda)}-e^{i\zeta_n}\right|(q+1)}{\left|re^{i\arg(\lambda)}+1+\lambda h\right|^{q+1}}\right).
\end{aligned} \tag{5.46}$$

On the other hand, again from Definition 4.14 (3.1), there exist $\rho_1, \mu_1 > 0$ such that

$$\left|re^{i\arg(\lambda)}+1+\lambda h\right| \geq \rho_1(r+\mu_1), \qquad \left|re^{i\zeta_n}+1+\lambda h\right| \geq \rho_1(r+\mu_1) \tag{5.47}$$

for all $r \geq 0$, all $h \geq 0$, and all $n \geq 0$. Using (5.46), (5.47) and the fact that $q+1 \leq 2^{q+1}$ for all $q \geq 0$, we deduce the estimates (5.43).

In the first part of the proof of Lemma 5.3, we show the existence of a staircase distribution solution of the problem (5.39), (5.40), which satisfies the estimates (5.31). As a starting point, it is easy to check that the problem (5.39), (5.40) has a formal solution of the form

$$\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon) = \sum_{\beta \geq 0} \mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) \frac{z^\beta}{\beta!}, \tag{5.48}$$

where $r \mapsto \mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)$ are distributions on $\mathbb{R}_+$, for which the next recursion holds:

$$\begin{aligned}
\mathbb{V}_{h,\beta+S,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) = &\sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0} \widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} \epsilon^{k_0-s} \\
&\times \frac{e^{i(s-k_0)\arg(\lambda)}}{re^{i\arg(\lambda)} + 1 + \lambda h} \left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right) \\
&+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} c_q^{k_0^1} (h\lambda)^q \\
&\times \epsilon^{k_0-s} \frac{e^{i(s-k_0-q)\arg(\lambda)}}{re^{i\arg(\lambda)} + 1 + \lambda h} \left(\sum_{(m,p)\in\mathcal{O}^2_{s-k_0-q}} \alpha^{2,q}_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right)
\end{aligned} \tag{5.49}$$

for all $\beta \geq 0$, $h \geq 0$, with initial conditions

$$\mathbb{V}_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) = v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon), \quad 0 \leq j \leq S-1, h \geq 0. \tag{5.50}$$

Using Corollary 2.10, Propositions 2.11 and 2.12, the estimates (5.9), and Remark 2.4, we deduce that $\mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) \in \mathfrak{D}'_{\beta,\check{\sigma},\epsilon}$ for all $h,\beta \geq 0$ and that the following inequalities hold for the real numbers $\mathbb{V}_{h,\beta,\mathcal{M}_{d_0}}(\epsilon) := \|\mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\|_{\beta,\check{\sigma},\epsilon,d}$: there exist constants $C^1_{23.0}$, $C^2_{23.0}$ (depending on $\mathcal{S}$, $\check{\sigma}$, S, ρ, μ) with

$$\begin{aligned}
\mathbb{V}_{h,\beta+S,\mathcal{M}_{d_0}}(\epsilon) \leq &\sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C^1_{23.0} \beta! A B^{-\beta_1} (\beta+S+1)^{(s-k_0)b} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0}}(\epsilon)}{\beta_2!} \\
&+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} C^2_{23.0} \beta! A B^{-\beta_1} \left| c_q^{k_0^1} \right| |\lambda|^q h^q \\
&\times (\beta+S+1)^{(s-k_0-q)b} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0}}(\epsilon)}{\beta_2!}
\end{aligned} \tag{5.51}$$

for all $\beta, h \geq 0$, where $A, B > 0$ are defined in Lemma 5.4. We define the following Cauchy problem:

$$\partial_x^S \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) = \sum_{(s,k_0,k_1)\in\mathcal{S}} C_{23.0}^1 (x\partial_x + S + 1)^{b(s-k_0)} \left(\frac{A}{1-x/B} \partial_x^{k_1} \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \right)$$
$$+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} C_{23.0}^2 \left| c_q^{k_0^1} \right| |\lambda|^q (x\partial_x + S + 1)^{b(s-k_0-q)} \tag{5.52}$$
$$\times \left(\frac{A}{1-x/B} (u\partial_u)^q \partial_x^{k_1} \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \right)$$

for given initial data

$$\left(\partial_x^j \mathbb{W}_{\mathcal{M}_{d_0}}\right)(u,0) = \mathbb{W}_{\mathcal{M}_{d_0},j}(u) = \sum_{h\geq 0} \sup_{\epsilon\in\mathcal{E}_0} \left\| v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) \right\|_{j,\check{\sigma},\epsilon,d} \frac{u^h}{h!} \in \mathbb{C}\{u\}, \quad 0 \leq j \leq S-1. \tag{5.53}$$

From the assumption (4.42) and the fact that $b > 1$, we deduce that

$$S > b(s - k_0 - q) + q + k_1 \tag{5.54}$$

for all $(s, k_0, k_1) \in \mathcal{S}$ and all $0 \leq q \leq k_0$. Hence, the assumption (2.108) is satisfied in Proposition 2.22 for the Cauchy problem (5.52), (5.53). Since the initial data $\mathbb{W}_{\mathcal{M}_{d_0},j}(u)$ is an analytic function on a disc containing some closed disc $D(0, U_0)$, for $0 \leq j \leq S-1$ and since the coefficients of (5.52) are analytic on $\mathbb{C} \times D(0,B)$, we deduce that all the hypotheses of Proposition 2.22 are fulfilled for the problem (5.52), (5.53). We deduce the existence of a formal solution $\mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \in G(U_{\mathcal{M}_{d_0}}, X_{\mathcal{M}_{d_0}})$, where $0 < U_{\mathcal{M}_{d_0}} < U_0$ (depending on $\mathcal{S}$) and $0 < X_{\mathcal{M}_{d_0}} \leq B/2$ (depending on S, $\check{\sigma}$, $|\lambda|$, ρ, μ, U_0, $\mathcal{S}$, A, B).

Now, let $\mathbb{W}_{\mathcal{M}_{d_0}}(u,x) = \sum_{h,\beta\geq 0} w_{h,\beta,\mathcal{M}_{d_0}} (u^h/h!)(x^\beta/\beta!)$ be its Taylor expansion at the origin. Then, the sequence $w_{h,\beta,\mathcal{M}_{d_0}}$ satisfies the next equalities:

$$w_{h,\beta+S,\mathcal{M}_{d_0}} = \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C_{23.0}^1 \beta! A B^{-\beta_1} (\beta + S + 1)^{(s-k_0)b} \frac{w_{h,\beta_2+k_1,\mathcal{M}_{d_0}}}{\beta_2!}$$
$$+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} C_{23.0}^2 \beta! A B^{-\beta_1} \left| c_q^{k_0^1} \right| |\lambda|^q h^q \tag{5.55}$$
$$\times (\beta + S + 1)^{(s-k_0-q)b} \frac{w_{h,\beta_2+k_1,\mathcal{M}_{d_0}}}{\beta_2!}$$

for all $\beta, h \geq 0$, with

$$w_{h,j,\mathcal{M}_{d_0}} = \sup_{\epsilon\in\mathcal{E}_0} \left\| v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) \right\|_{j,\check{\sigma},\epsilon,d}, \quad h \geq 0, \ 0 \leq j \leq S-1. \tag{5.56}$$

Gathering the inequalities (5.51), the equalities (5.55) with the initial conditions (5.56), one gets

$$\sup_{\epsilon\in\mathcal{E}_0}\left|\mathbb{V}_{h,\beta,\mathcal{M}_{d_0}}(\epsilon)\right| \leq w_{h,\beta,\mathcal{M}_{d_0}} \tag{5.57}$$

for all $h, \beta \geq 0$. From (5.57) and the fact that $\mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \in G(U_{\mathcal{M}_{d_0}}, X_{\mathcal{M}_0})$, we get a constant $C_{\mathcal{M}_{d_0}} > 0$ such that

$$\begin{aligned}
&\sup_{\epsilon\in\mathcal{E}_0}\left\|\mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} \\
&\quad\leq C_{\mathcal{M}_{d_0}}(h+\beta)!\left(\frac{1}{U_{\mathcal{M}_{d_0}}}\right)^h\left(\frac{1}{X_{\mathcal{M}_{d_0}}}\right)^\beta \leq C_{\mathcal{M}_{d_0}} h!\beta!\left(\frac{2}{U_{\mathcal{M}_{d_0}}}\right)^h\left(\frac{2}{X_{\mathcal{M}_{d_0}}}\right)^\beta
\end{aligned} \tag{5.58}$$

for all $h, \beta \geq 0$. From this last estimates (5.58), we deduce that for all $h \geq 0$, $\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)$ belongs to $\mathfrak{D}'(\check{\sigma},\epsilon,\delta_{\mathcal{M}_{d_0}})$ for $0 < \delta_{\mathcal{M}_{d_0}} \leq X_{\mathcal{M}_{d_0}}/4$ and moreover that

$$\sum_{h\geq 0}\sup_{\epsilon\in\mathcal{E}_0}\left\|\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_{\mathcal{M}_{d_0}})}\frac{u^h}{h!} \in \mathbb{C}\{u\} \tag{5.59}$$

holds. This yields the property (5.31).

In the second part of the proof, we show (5.30). One defines the distribution

$$\mathbb{V}^{\Delta}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon) := \mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon) - \mathbb{V}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon) \tag{5.60}$$

for all $r \geq 0$, all $z \in D(0,\delta_{\mathcal{M}_{d_0}}) \cap D(0,\check{\delta})$, with $0 < \check{\delta} < x_1/2$, all $\epsilon \in \mathcal{E}_0$. If one writes the Taylor expansion

$$\mathbb{V}^{\Delta}_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon) = \sum_{\beta\geq 0}\mathbb{V}^{\Delta}_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)\frac{z^\beta}{\beta!} \tag{5.61}$$

for $z \in D(0, \delta_{\mathcal{M}_{d_0}}) \cap D(0, \check{\delta})$, then the coefficients $\mathbb{V}^{\Delta}_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)$ satisfy the following recursion:

$$
\begin{aligned}
\mathbb{V}^{\Delta}_{h,\beta+S,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon) = {} & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} \epsilon^{k_0-s} \left(\frac{e^{i(s-k_0)\zeta_n}}{re^{i\zeta_n}+1+\lambda h} \right) \\
& \times \left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}^{\Delta}_{h,\beta_2+k_1,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right) \\
& + \sum_{k_0^1+k_0^2=k_0,\, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} c_q^{k_0^1} (h\lambda)^q \\
& \times \epsilon^{k_0-s} \frac{e^{i(s-k_0-q)\zeta_n}}{re^{i\zeta_n}+1+\lambda h} \times \left(\sum_{(m,p)\in\mathcal{O}^2_{s-k_0-q}} \alpha^{2,q}_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}^{\Delta}_{h,\beta_2+k_1,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right) \\
& + \mathbb{B}_{h,\beta,n}(r,\epsilon),
\end{aligned}
\tag{5.62}
$$

where

$$
\begin{aligned}
\mathbb{B}_{h,\beta,n}(r,\epsilon) = {} & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} \epsilon^{k_0-s} \left(\frac{e^{i(s-k_0)\arg(\lambda)}}{re^{i\arg(\lambda)}+1+\lambda h} - \frac{e^{i(s-k_0)\zeta_n}}{re^{i\zeta_n}+1+\lambda h} \right) \\
& \times \left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right) \\
& + \sum_{k_0^1+k_0^2=k_0,\, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} \beta! \frac{\epsilon^{k_0}\widetilde{b}_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!} c_q^{k_0^1} (h\lambda)^q \\
& \times \epsilon^{k_0-s} \left(\frac{e^{i(s-k_0-q)\arg(\lambda)}}{re^{i\arg(\lambda)}+1+\lambda h} - \frac{e^{i(s-k_0-q)\zeta_n}}{re^{i\zeta_n}+1+\lambda h} \right) \\
& \times \left(\sum_{(m,p)\in\mathcal{O}^2_{s-k_0-q}} \alpha^{2,q}_{m,p} r^m \partial_r^{-p} \frac{\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)}{\beta_2!} \right)
\end{aligned}
\tag{5.63}
$$

for all $h \geq 0$ and all $\beta \geq 0$. Now, we put $\mathbb{V}^{\Delta}_{h,\beta,n}(\epsilon) = ||\mathbb{V}^{\Delta}_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)||_{\beta,\check{\sigma},\epsilon,d}$. Using Corollary 2.10, Propositions 2.11 and 2.12, and Lemma 5.4, we get that there exist constants $C^1_{23.1}, C^2_{23.1}$ (depending on $\mathcal{S}$, $\check{\sigma}$, S, ρ, μ) such that the following inequalities:

$$\begin{aligned}
\mathbb{V}^{\Delta}_{h,\beta+S,n}(\epsilon) \leq & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C^1_{23.1}\beta! AB^{-\beta_1}(\beta+S+1)^{(s-k_0)b}\frac{\mathbb{V}^{\Delta}_{h,\beta_2+k_1,n}(\epsilon)}{\beta_2!} \\
&+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\sum_{\beta_1+\beta_2=\beta} C^2_{23.1}\beta! AB^{-\beta_1} \\
&\times \left|c_q^{k_0^1}\right| |\lambda|^q h^q (\beta+S+1)^{(s-k_0-q)b}\frac{\mathbb{V}^{\Delta}_{h,\beta_2+k_1,n}(\epsilon)}{\beta_2!} + \mathbb{B}_{h,\beta,n}(\epsilon)
\end{aligned} \tag{5.64}$$

hold for all $h, \beta \geq 0$, where $A, B > 0$ are defined in Lemma 5.4 and $\mathbb{B}_{h,\beta,n}(\epsilon)$ is a sequence which satisfies the next estimates: there exist constants $C^3_{23.1}, C^4_{23.1} > 0$ (depending on $\mathcal{S}$,$\check{\sigma}$,S,ρ',μ') with

$$\begin{aligned}
\mathbb{B}_{h,\beta,n}(\epsilon) \leq & \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C^3_{23.1}\beta! D_n B^{-\beta_1} \\
&\times (\beta+S+1)^{(s-k_0)b}\frac{\left\|\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta_2+k_1,\check{\sigma},\epsilon,d}}{\beta_2!} \\
&+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1}\sum_{\beta_1+\beta_2=\beta} C^4_{23.1}\beta! D_n B^{-\beta_1} \\
&\times \left|c_q^{k_0^1}\right| |\lambda|^q h^q (\beta+S+1)^{(s-k_0-q)b}\frac{\left\|\mathbb{V}_{h,\beta_2+k_1,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta_2+k_1,\check{\sigma},\epsilon,d}}{\beta_2!}
\end{aligned} \tag{5.65}$$

for all $h, \beta, n \geq 0$, where D_n, $n \geq 0$ is the sequence defined in Lemma 5.4.

We consider the following sequence of Cauchy problem:

$$\begin{aligned}
\partial_x^S \mathbb{W}^{\Delta}_n(u,x) = & \sum_{(s,k_0,k_1)\in\mathcal{S}} C^1_{23.1}(x\partial_x+S+1)^{b(s-k_0)}\left(\frac{A}{1-x/B}\partial_x^{k_1}\mathbb{W}^{\Delta}_n(u,x)\right) \\
&+ \sum_{k_0^1+k_0^2=k_0, k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!}\sum_{q=1}^{k_0^1} C^2_{23.1}\left|c_q^{k_0^1}\right| |\lambda|^q (x\partial_x+S+1)^{b(s-k_0-q)} \\
&\times \left(\frac{A}{1-x/B}(u\partial_u)^q\partial_x^{k_1}\mathbb{W}^{\Delta}_n(u,x)\right) + \mathbb{D}_n(u,x),
\end{aligned} \tag{5.66}$$

where

$$\mathbb{D}_n(u,x) = \sum_{(s,k_0,k_1)\in\mathcal{S}} C_{23.1}^3 (x\partial_x + S + 1)^{b(s-k_0)} \left(\frac{D_n}{1-x/B} \partial_x^{k_1} \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \right)$$
$$+ \sum_{k_0^1+k_0^2=k_0,k_0^1\geq 1} \frac{k_0!}{k_0^1!k_0^2!} \sum_{q=1}^{k_0^1} C_{23.1}^4 \left| c_q^{k_0^1} \right| |\lambda|^q (x\partial_x + S + 1)^{b(s-k_0-q)} \tag{5.67}$$
$$\times \left(\frac{D_n}{1-x/B} (u\partial_u)^q \partial_x^{k_1} \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \right)$$

and $\mathbb{W}_{\mathcal{M}_{d_0}}(u,x)$ is already defined as the solution of the problem (5.52), (5.53), for given initial data

$$\left(\partial_x^j \mathbb{W}_n^{\Delta}\right)(u,0) = \mathbb{W}_{j,n}^{\Delta}(u)$$
$$= \sum_{h\geq 0} \sup_{\epsilon\in\mathcal{E}_0} \left\| v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) - v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon) \right\|_{j,\breve{\sigma},\epsilon,d} \frac{u^h}{h!} \in \mathbb{C}\{u\}, \tag{5.68}$$
$$0 \leq j \leq S-1,$$

which are convergent near the origin with respect to u due to the assumption (5.8) and Remark 2.4. Moreover, the initial data satisfy the estimates

$$\left| \mathbb{W}_{j,n}^{\Delta}(u) \right| \leq \frac{I_{n,j}}{1-|u|/u_j} \tag{5.69}$$

for all $|u| < u_j$, $0 \leq j \leq S-1$, all $n \geq 0$.

From the assumption (4.42) and the fact that $b > 1$, we deduce that

$$S > b(s-k_0-q) + q + k_1 \tag{5.70}$$

for all $(s,k_0,k_1) \in \mathcal{S}$ and all $0 \leq q \leq k_0$. Therefore, the assumption (2.108) is satisfied in Proposition 2.22 for the problem (5.66), (5.68).

On the other hand, from Lemmas 2.20 and 2.21, there exist a constant $D_{\mathcal{M}_{d_0}} > 0$ (depending on $\mathcal{S}, \breve{\sigma}, S, \rho', \mu', |\lambda|, B, U_{\mathcal{M}_{d_0}}, X_{\mathcal{M}_{d_0}}$), a constant $0 < U_{1,\mathcal{M}_{d_0}} < U_{\mathcal{M}_{d_0}}$, and a constant $0 < X_{1,\mathcal{M}_{d_0}} < X_{\mathcal{M}_{d_0}}$ such that

$$\|\mathbb{D}_n(u,x)\|_{(U_{1,\mathcal{M}_{d_0}},X_{1,\mathcal{M}_{d_0}})} \leq D_n D_{\mathcal{M}_{d_0}} \left\| \mathbb{W}_{\mathcal{M}_{d_0}}(u,x) \right\|_{(U_{\mathcal{M}_{d_0}},X_{\mathcal{M}_{d_0}})} \leq D_n D_{\mathcal{M}_{d_0}} C_{\mathcal{M}_{d_0}} \tag{5.71}$$

for all $n \geq 0$, where the constant $C_{\mathcal{M}_{d_0}}$ is introduced in (5.58).

Since the initial data $\mathbb{W}_{j,n}^{\Delta}(u)$ is an analytic function on some disc containing the closed disc $\overline{D}(0,u_j/2)$, for $0 \leq j \leq S-1$ and the coefficients of (5.66) are analytic on $\mathbb{C} \times D(0,B)$, we deduce that all the hypotheses of Proposition 2.22 for the problem (5.66), (5.68) are fulfilled.

We deduce the existence of a formal solution $\mathbb{W}_n^{\Delta}(u,x) \in G(U_1, X_1)$ of (5.66), (5.68), where $0 < U_1 < \min(U_{1,\mathcal{M}_{d_0}}, \min_{0\le j\le S-1} u_j/2)$ (depending on $\mathcal{S}$) and $0 < X_1 \le \min(B/2, X_{1,\mathcal{M}_{d_0}})$ (depending on S, $\check{\sigma}$, $|\lambda|$, u_j, for $0 \le j \le S-1$, $\mathcal{S}$, A, B, ρ, μ).

Moreover, from (2.111) and (5.71), there exist constants $M_1 > 0$ (depending on S, $\check{\sigma}$, λ, u_j, for $0 \le j \le S-1$, $\mathcal{S}$, A, B, ρ, μ) and $M_2 > 0$ (depending on S, u_j for $0 \le j \le S-1$, B, $\mathcal{S}$) such that

$$\left\| \mathbb{W}_n^{\Delta}(u,x) \right\|_{(U_1,X_1)} \le M_1 \max_{0\le j\le S-1} I_{n,j} + D_n M_2 D_{\mathcal{M}_{d_0}} C_{\mathcal{M}_{d_0}} \tag{5.72}$$

for all $n \ge 0$. Now, let $\mathbb{W}_n^{\Delta}(u,x) = \sum_{h,\beta\ge 0} w_{h,\beta,n}^{\Delta} (u^h/h!)(x^\beta/\beta!)$ be its Taylor expansion at the origin. Then, the sequence $w_{h,\beta,n}^{\Delta}$ satisfies the following equalities:

$$\begin{aligned} w_{h,\beta+S,n}^{\Delta} = &\sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C_{23.1}^{1} \beta! A B^{-\beta_1} (\beta+S+1)^{(s-k_0)b} \frac{w_{h,\beta_2+k_1,n}^{\Delta}}{\beta_2!} \\ &+ \sum_{k_0^1+k_0^2=k_0, k_0^1\ge 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} C_{23.1}^{2} \beta! A B^{-\beta_1} \left| c_q^{k_0^1} \right| |\lambda|^q h^q (\beta+S+1)^{(s-k_0-q)b} \frac{w_{h,\beta_2+k_1,n}^{\Delta}}{\beta_2!} \\ &+ \mathbb{D}_{h,\beta,n}, \end{aligned} \tag{5.73}$$

where

$$\begin{aligned} \mathbb{D}_{h,\beta,n} = &\sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{\beta_1+\beta_2=\beta} C_{23.1}^{3} \beta! D_n B^{-\beta_1} (\beta+S+1)^{(s-k_0)b} \frac{w_{h,\beta_2+k_1,\mathcal{M}_{d_0}}}{\beta_2!} \\ &+ \sum_{k_0^1+k_0^2=k_0, k_0^1\ge 1} \frac{k_0!}{k_0^1! k_0^2!} \sum_{q=1}^{k_0^1} \sum_{\beta_1+\beta_2=\beta} C_{23.1}^{4} \beta! D_n B^{-\beta_1} \left| c_q^{k_0^1} \right| |\lambda|^q h^q (\beta+S+1)^{(s-k_0-q)b} \frac{w_{h,\beta_2+k_1,\mathcal{M}_{d_0}}}{\beta_2!} \end{aligned} \tag{5.74}$$

for all $h, \beta, n \ge 0$, with

$$w_{h,j,n}^{\Delta} = \sup_{\epsilon\in\mathcal{E}_0} \left\| v_{h,j,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon) - v_{h,j,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},\epsilon) \right\|_{j,\check{\sigma},\epsilon,d}, \quad \forall h \ge 0,\ \forall 0 \le j \le S-1 . \tag{5.75}$$

Gathering the inequalities (5.64), (5.65) and the equalities (5.73), with the initial conditions (5.75), one gets that

$$\sup_{\epsilon\in\mathcal{E}_0} \left| \mathbb{V}_{h,\beta,n}^{\Delta}(\epsilon) \right| \le w_{h,\beta,n}^{\Delta} \tag{5.76}$$

for all $h, \beta \ge 0$ and all $n \ge 0$.

From (5.76) and the estimates (5.72), we deduce that

$$\sup_{\epsilon\in\mathcal{E}_0}\left\|V^{\Delta}_{h,\beta,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d}$$
$$\leq\left(M_1\max_{0\leq j\leq S-1}I_{n,j}+D_nM_2D_{\mathcal{M}_{d_0}}C_{\mathcal{M}_{d_0}}\right)(h+\beta)!\left(\frac{1}{U_1}\right)^h\left(\frac{1}{X_1}\right)^\beta \tag{5.77}$$
$$\leq\left(M_1\max_{0\leq j\leq S-1}I_{n,j}+D_nM_2D_{\mathcal{M}_{d_0}}C_{\mathcal{M}_{d_0}}\right)h!\beta!\left(\frac{2}{U_1}\right)^h\left(\frac{2}{X_1}\right)^\beta$$

for all $h,\beta\geq 0$, all $n\geq 0$. From (5.77), we get that

$$\sup_{\epsilon\in\mathcal{E}_0}\left\|V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(r,z,\epsilon)-\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_D)}$$
$$\leq\left(M_1\max_{0\leq j\leq S-1}I_{n,j}+D_nM_2D_{\mathcal{M}_{d_0}}C_{\mathcal{M}_{d_0}}\right)h!\left(\frac{2}{U_1}\right)^h \tag{5.78}$$

for all $h\geq 0$ and all $0<\delta_D\leq X_1/4$. This yields the estimates (5.30). □

In the next lemma, we express $Z_0(t,z,\epsilon)$ as a Laplace transform of a staircase distribution.

Lemma 5.5. *Let $\check{\sigma}>\widetilde{\sigma}>\sigma r_b(S-1)$. Then, one can write the solution $Z_0(t,z,\epsilon)$ of* (4.90), (4.91) *in the form of a Laplace transform in direction* $\arg(\lambda)$

$$Z_0(t,z,\epsilon)=\mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t) \tag{5.79}$$

for all $(t,z,\epsilon)\in(\mathcal{T}\cap D(0,\iota''))\times D(0,\delta_{D,Z_0})\times(\mathcal{E}_0\cap\mathcal{E}_1)$, where $\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\in\mathfrak{D}'(\check{\sigma},\epsilon,\delta_{D,Z_0})$ (with $\delta_{D,Z_0}=\min(\delta_D,\delta_{Z_0})$) solves the following Cauchy problem:

$$\left(re^{i\arg(\lambda)}+1\right)\partial_z^S\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)$$
$$=\sum_{(s,k_0,k_1)\in\mathcal{S}}\epsilon^{k_0-s}b_{s,k_0,k_1}(z,\epsilon)\left(e^{i(s-k_0)\arg(\lambda)}\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}}\alpha^1_{m,p}r^m\partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right), \tag{5.80}$$

where the sets $\mathcal{O}^1_{s-k_0}$ and the integers $\alpha^1_{m,p}$ are introduced in (4.20), *with initial data*

$$\left(\partial_z^j\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}\right)(r,0,\epsilon)=\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon),\quad 0\leq j\leq S-1. \tag{5.81}$$

Proof. From Proposition 4.17, we can write the solution $Z_0(t,z,\epsilon)$ of (4.90), (4.91) in the form

$$\begin{aligned} Z_0(t,z,\epsilon) &= \sum_{h\geq 0} \frac{\exp(-h\lambda/\epsilon t)}{h!} \frac{1}{\epsilon t} \int_{L_{\zeta_n}} V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(\tau,z,\epsilon) \exp\left(-\frac{\tau}{\epsilon t}\right) d\tau \\ &= \sum_{h\geq 0} \frac{\exp\left(-h|\lambda|e^{i\arg(\lambda)}/\epsilon t\right)}{h!} \frac{e^{i\zeta_n}}{\epsilon t} \int_0^{+\infty} V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right) \exp\left(-r\frac{e^{i\zeta_n}}{\epsilon t}\right) dr \end{aligned} \tag{5.82}$$

for all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{Z_0}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$ and all $n \geq 0$. Now, we write

$$\mathcal{L}_{\zeta_n}\left(V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right)\right)(\epsilon t) = \mathcal{L}_{\arg(\lambda)}\left(V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right) \tag{5.83}$$

for all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{Z_0}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$ and all $n \geq 0$. Now, we define $\delta_{D,Z_0} = \min(\delta_D,\delta_{Z_0})$. From the continuity estimates (3.5) for the Laplace transform, we deduce that for given $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, $t \in \mathcal{T} \cap D(0,\iota'')$, there exists a constant $C_{\epsilon,t}$ (depending on ϵ, t) such that

$$\begin{aligned} &\left|\mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t) - \mathcal{L}_{\arg(\lambda)}\left(V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right)\right| \\ &\quad \leq C_{\epsilon,t}\left\|\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon) - V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}(re^{i\zeta_n},z,\epsilon)\right\|_{(\breve{\sigma},\epsilon,d,\delta_{D,Z_0})} \\ &\qquad + \left|\mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)\left(\epsilon t e^{i(\arg(\lambda)-\zeta_n)}\right) - \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t)\right| \end{aligned} \tag{5.84}$$

for all $z \in D(0,\delta_{D,Z_0})$, all $n \geq 0$. By letting n tend to $+\infty$ in this latter inequality and using the estimates (5.30), we obtain

$$\mathcal{L}_{\zeta_n}\left(V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right)\right)(\epsilon t) = \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t) \tag{5.85}$$

for all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{D,Z_0}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$ and all $n \geq 0$.

On the other hand, from Corollary 2.10, we have that for all $h \geq 0$, the distribution $\partial_r^{-h}(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon))$ belongs to $\mathfrak{D}'(\breve{\sigma},\epsilon,\delta_{D,Z_0})$ and that there exists a universal constant $C_3 > 0$ such that

$$\left\|\partial_r^{-h}(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon))\right\|_{\breve{\sigma},\epsilon,d,\delta_{D,Z_0}} \leq C_3\left(\frac{|\epsilon|}{\breve{\sigma}}\right)^h \left\|\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{\breve{\sigma},\epsilon,d,\delta_{D,Z_0}} \tag{5.86}$$

for all $h \geq 0$.

From (5.85) and using Propositions 3.3 and 3.7, we can write

$$\begin{aligned}
&\frac{\exp(-h|\lambda|e^{i\arg(\lambda)}/\epsilon t)}{h!}\frac{e^{i\zeta_n}}{\epsilon t}\int_0^{+\infty} V_{h,S_{d_0,\vartheta_n},\mathcal{E}_0}\left(re^{i\zeta_n},z,\epsilon\right)\exp\left(-r\frac{e^{i\zeta_n}}{\epsilon t}\right)dr \\
&\quad=\left(\frac{e^{i\arg(\lambda)}}{\epsilon t}\right)^h\frac{\exp(-h|\lambda|e^{i\arg(\lambda)}/\epsilon t)}{h!}\mathcal{L}_{\arg(\lambda)}\left(\partial_r^{-h}\left(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)\right)(\epsilon t) \\
&\quad=\mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t),
\end{aligned} \tag{5.87}$$

where

$$\mathbb{V}_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)=\frac{\left(f_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r-|\lambda|h,z,\epsilon)1_{[|\lambda|h,+\infty)}(r)\right)^{(h)}}{h!}\in\mathfrak{D}'(\check{\sigma},\epsilon,\delta_{D,Z_0}) \tag{5.88}$$

with $f_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)=\partial_r^{-h}(\mathbb{V}_{h,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon))\in\mathfrak{D}'(\check{\sigma},\epsilon,\delta_{D,Z_0})$, for all $h\geq 0$, all $0\leq j\leq S-1$. From Proposition 3.6, we have a universal constant $A>0$ and a constant $B(\check{\sigma},b,\epsilon)$ (depending on $\check{\sigma}$, b, and ϵ, which tend to zero as $\epsilon\to 0$) such that

$$\left\|\mathbb{V}_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{\check{\sigma},\epsilon,d,\delta_{D,Z_0}}\leq A\frac{(B(\check{\sigma},b,\epsilon))^h}{h!}\left\|f_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{\check{\sigma},\epsilon,d,\delta_{D,Z_0}}. \tag{5.89}$$

From the convergence of the series (5.31) near the origin and using (5.86), (5.89), we deduce the distribution

$$\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)=\sum_{h\geq 0}\mathbb{V}_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon)\in\mathfrak{D}'(\check{\sigma},\epsilon,\delta_{D,Z_0}), \tag{5.90}$$

if $\epsilon_0>0$ is chosen small enough. Finally, by the continuity estimates (3.5) of the Laplace transform $\mathcal{L}_{\arg(\lambda)}$ and the formula (5.82), (5.87), we get the expression (5.79). Moreover, from the formulas in Proposition 3.3, as $Z_0(t,z,\epsilon)$ solves the problem (4.90), (4.91), we deduce that the distribution $\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)$ solves the Cauchy problem (5.80), (5.81). □

Step 2. In this step, we show that the function $X_{0,1}(t,z,\epsilon)$ can be expressed as a Laplace transform of some staircase distribution in direction $\arg(\lambda)$, satisfying the problem (5.80), (5.81).

From the assumption (5.23), we deduce from Proposition 4.12, that the function $V_{0,S_{d_1},\mathcal{E}_1}(\tau,z,\epsilon)$ constructed in (4.80) has an analytic continuation denoted by $V_{0,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,z,\epsilon)$ on $(S_{d_1,\delta_n}\cup D(0,\tau_0))\times D(0,\delta_{\mathcal{E}_1})\times(\mathcal{E}_0\cap\mathcal{E}_1)$ and satisfies estimates (4.62) for all $n\geq 0$, where $\delta_{\mathcal{E}_1}>0$ depends on S,$u_{\mathcal{E}_1}$ (which denotes a common radius of absolute convergence of the series (5.23), $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$. This constant $\delta_{\mathcal{E}_1}$ is, therefore, independent of n. Now, one defines the functions

$$\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) = V_{0,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},z,\epsilon\right) \tag{5.91}$$

for all $r \geq 0$, all $z \in D(0,\delta_{\mathcal{E}_1})$, and all $n \geq 0$.

Lemma 5.6. *Let $\check{\sigma} > \tilde{\sigma} > \sigma r_b(S-1)$ as in Lemma 5.3. Then, there exists $0 < \delta_{D_{0,1}} < \min(\delta_{\mathcal{E}_1},\delta_{D,Z_0})$ (depending on $\mathcal{S}, S, \check{\sigma}, |\lambda|, A, B, \rho, \mu$ and $\widetilde{A}, \widetilde{B}, \widetilde{\rho}, \widetilde{\mu}$ introduced in Lemma 5.7), there exist $\widetilde{M}_1, \widetilde{M}_{1'}$ (depending on $\mathcal{S}, S, \check{\sigma}, |\lambda|, A, B, \rho, \mu, \widetilde{A}, \widetilde{B}, \widetilde{\rho}, \widetilde{\mu}$ and $\widetilde{\rho}', \widetilde{\mu}'$ introduced in Lemma 5.7) such that*

$$\sup_{\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1} \left\| \mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) - \mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon) \right\|_{(\check{\sigma},\epsilon,d,\delta_{D_{0,1}})} \leq \left(\widetilde{M}_1 \max_{0 \leq j \leq S-1} J_{n,j} + \widetilde{M}_{1'} \widetilde{D}_n \right) \tag{5.92}$$

for all $n \geq 0$, where $\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)$ is defined in Lemma 5.5 and solves the problem (5.80), (5.81) *and $\widetilde{D}_n$ is the sequence (which tends to zero as $n \to +\infty$) defined in Lemma 5.7.*

Proof. From the estimates (4.54), we can write

$$V_{0,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,z,\epsilon) = \sum_{\beta \geq 0} V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon)\frac{z^\beta}{\beta!}, \tag{5.93}$$

where $V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon)$ are holomorphic functions such that there exist a constant u_1 with $0 < u_1 < u_{\mathcal{E}_1}$ (depending on $u_{\mathcal{E}_1}$, $\mathcal{S}$, and b, σ), a constant x_1 such that $0 < x_1 < \rho$ (depending on S, $u_{\mathcal{E}_1}$, $\mathcal{S}$, b, σ, $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, where $x_0 < \rho$), and a constant $C_{\Omega(d_1,\mathcal{E}_1),n} > 0$ (depending on $\max_{0\leq j\leq S-1} W_{j,S_{d_1,\delta_n},\mathcal{E}_1}(u_{\mathcal{E}_1})$ (where $W_{j,S_{d_1,\delta_n},\mathcal{E}_1}$ are defined in (5.23)), $|\lambda|$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|b|_{s,k_0,k_1}(x_0)$, $\max_{(s,k_0,k_1)\in\mathcal{S}}|\widetilde{b}|_{s,k_0,k_1}(x_0)$, S, $u_{\mathcal{E}_1}$, x_0, $\mathcal{S}$, b) with

$$\left|V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,\epsilon)\right| \leq C_{\Omega(d_1,\mathcal{E}_1),n}\beta!\left(\frac{2}{x_1}\right)^\beta\left(1+\frac{|\tau|^2}{|\epsilon|^2}\right)^{-1}\exp\left(\frac{\sigma}{2|\epsilon|}r_b(\beta)|\tau|\right) \tag{5.94}$$

for all $\tau \in S_{d_1,\delta_n} \cup D(0,\tau_0)$, $\epsilon \in \mathcal{E}_1$, all $\beta \geq 0$, and all $n \geq 0$. We deduce that

$$\left|V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},\epsilon\right)\right| \leq C_{\Omega(d_1,\mathcal{E}_1),n}\left(\frac{2}{x_1}\right)^\beta \beta! \exp\left(\frac{\sigma}{2|\epsilon|}r_b(\beta)r\right) \tag{5.95}$$

for all $r \geq 0$, all $\epsilon \in \mathcal{E}_1$, all $\beta \geq 0$, and all $n \geq 0$. In particular, $r \mapsto V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)$ belongs to $L_{\beta,\check{\sigma}/2,\epsilon}$. From Proposition 2.7, we deduce that $r \mapsto V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)$ belongs to $\mathfrak{D}'_{\beta,\check{\sigma},\epsilon}$. From Proposition 2.7 and (5.95), we get a universal constant $C_1 > 0$ such that

$$\begin{aligned} \left\|V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} &\leq C_1\left\|V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)\right\|_{\beta,\check{\sigma}/2,\epsilon} \\ &\leq C_1 C_{\Omega(d_1,\mathcal{E}_1),n}\frac{2|\epsilon|}{\check{\sigma}-\sigma}\left(\frac{2}{x_1}\right)^\beta \beta! \end{aligned} \tag{5.96}$$

for all $\beta \geq 0$ and all $n \geq 0$. From (5.96), we deduce that the distribution

$$\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) = \sum_{\beta\geq 0} V_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},\epsilon\right)\frac{z^\beta}{\beta!} \in \mathfrak{D}'\left(\check{\sigma},\epsilon,\check{\delta}\right) \tag{5.97}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, all $\check{\delta} < x_1/2$, and all $n \geq 0$.

From (4.20), (4.21), we have that the distribution $\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon)$ solves the following problem:

$$\begin{aligned}
&\left(re^{i\xi_n}+1\right)\partial_z^S \mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon)\\
&\quad= \sum_{(s,k_0,k_1)\in\mathcal{S}} \epsilon^{k_0-s} b_{s,k_0,k_1}(z,\epsilon)\left(e^{i(s-k_0)\xi_n}\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}} \alpha^1_{m,p} r^m \partial_r^{-p}\partial_z^{k_1}\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon)\right),
\end{aligned} \tag{5.98}$$

where $\mathcal{O}^1_{s-k_0}$ is the set and $\alpha^1_{m,p}$ are the integers from (5.80), with initial data

$$\left(\partial_z^j \mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}\right)(r,0,\epsilon) = v_{0,j,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},\epsilon\right), \quad 0 \leq j \leq S-1. \tag{5.99}$$

In the next lemma, we give estimates for the coefficients of (5.98) and (5.80). The proof is exactly the same as the one described for Lemma 5.4.

Lemma 5.7. *Let*

$$b_{s,k_0,k_1}(z,\epsilon) = \sum_{\beta\geq 0} b_{s,k_0,k_1,\beta}(\epsilon)\frac{z^\beta}{\beta!} \tag{5.100}$$

be the convergent Taylor expansion of b_{s,k_0,k_1} with respect to z near 0. Then, there exist positive constants $\widetilde{A}$, $\widetilde{B}$, $\widetilde{\rho}$, $\widetilde{\rho}'$, $\widetilde{\mu}$, $\widetilde{\mu}'$ and a sequence $\widetilde{D}_n > 0$ such that $\lim_{n\to+\infty}\widetilde{D}_n = 0$ with

$$\begin{aligned}
\left|\partial_r^q\left(\frac{b_{s,k_0,k_1,\beta}(\epsilon)e^{i(s-k_0)\arg(\lambda)}}{re^{i\arg(\lambda)}+1}\right)\right| &\leq \widetilde{A}\widetilde{B}^{-\beta}\frac{\beta!q!}{\left(\widetilde{\rho}(r+\widetilde{\mu})\right)^{q+1}},\\
\left|\partial_r^q\left(\frac{b_{s,k_0,k_1,\beta}(\epsilon)e^{i(s-k_0)\xi_n}}{re^{i\xi_n}+1}\right)\right| &\leq \widetilde{A}\widetilde{B}^{-\beta}\frac{\beta!q!}{\left(\widetilde{\rho}(r+\widetilde{\mu})\right)^{q+1}},\\
\left|\partial_r^q\left(\frac{b_{s,k_0,k_1,\beta}(\epsilon)e^{i(s-k_0)\arg(\lambda)}}{re^{i\arg(\lambda)}+1}\right)-\partial_r^q\left(\frac{b_{s,k_0,k_1,\beta}(\epsilon)e^{i(s-k_0)\xi_n}}{re^{i\xi_n}+1}\right)\right| &\leq \widetilde{D}_n\widetilde{B}^{-\beta}\frac{\beta!q!}{\left(\widetilde{\rho}'(r+\widetilde{\mu}')\right)^{q+1}}
\end{aligned} \tag{5.101}$$

for all $q \geq 0$, all $\beta \geq 0$, all $n \geq 0$, all $r \geq 0$ and all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$.

Now, we consider the distribution

$$\mathbb{V}^{\Delta}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) := \mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon) - \mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) \tag{5.102}$$

for all $r \geq 0$, all $z \in D(0,\delta_{D,Z_0}) \cap D(0,\check{\delta})$, with $0 < \check{\delta} < x_1/2$ and δ_{D,Z_0} defined in Lemma 5.5, for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. One writes the Taylor expansions as follows:

$$\begin{gathered}\mathbb{V}^{\Delta}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) = \sum_{\beta\geq 0}\mathbb{V}^{\Delta}_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon)\frac{z^\beta}{\beta!},\\ \mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon) = \sum_{\beta\geq 0}\mathbb{V}_{\beta,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)\frac{z^\beta}{\beta!},\end{gathered} \tag{5.103}$$

for $z \in D(0,\delta_{D,Z_0}) \cap D(0,\check{\delta})$; then the coefficients $\mathbb{V}^{\Delta}_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon)$ satisfy the next recursion:

$$\begin{aligned}\mathbb{V}^{\Delta}_{0,\beta+S,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon) = &\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta}\beta!\frac{b_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!}\epsilon^{k_0-s}\frac{e^{i(s-k_0)\xi_n}}{re^{i\xi_n}+1}\\ &\times\left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}}\alpha^1_{m,p}r^m\partial_r^{-p}\frac{\mathbb{V}^{\Delta}_{0,\beta_2+k_1,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon)}{\beta_2!}\right)\\ &+\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta}\beta!\frac{b_{s,k_0,k_1,\beta_1}(\epsilon)}{\beta_1!}\epsilon^{k_0-s}\left(\frac{e^{i(s-k_0)\arg(\lambda)}}{re^{i\arg(\lambda)}+1}-\frac{e^{i(s-k_0)\xi_n}}{re^{i\xi_n}+1}\right)\\ &\times\left(\sum_{(m,p)\in\mathcal{O}^1_{s-k_0}}\alpha^1_{m,p}r^m\partial_r^{-p}\frac{\mathbb{V}_{\beta_2+k_1,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)}{\beta_2!}\right)\end{aligned} \tag{5.104}$$

for all $h \geq 0$, all $\beta \geq 0$. We put $\mathbb{V}^{\Delta}_{0,\beta,n,\mathcal{E}_1}(\epsilon) = ||\mathbb{V}^{\Delta}_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon)||_{\beta,\check{\sigma},\epsilon,d}$. Using Corollary 2.10, Propositions 2.11 and 2.12 and Lemma 5.7, we get a constant $C^1_{23.2} > 0$ (depending on $\mathcal{S},\check{\sigma},S,\widetilde{\rho},\widetilde{\mu}$) and $C^2_{23.2} > 0$ (depending on $\mathcal{S},\check{\sigma},S,\widetilde{\rho}',\widetilde{\mu}'$) such that the next inequalities:

$$\begin{aligned}\mathbb{V}^{\Delta}_{0,\beta+S,n,\mathcal{E}_1}(\epsilon) \leq &\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta}C^1_{23.2}\beta!\widetilde{A}\widetilde{B}^{-\beta_1}(\beta+S+1)^{b(s-k_0)}\frac{\mathbb{V}^{\Delta}_{0,\beta_2+k_1,n,\mathcal{E}_1}(\epsilon)}{\beta_2!}\\ &+\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta}C^2_{23.2}\beta!\widetilde{D}_n\widetilde{B}^{-\beta_1}(\beta+S+1)^{b(s-k_0)}\\ &\times\frac{\left\|\mathbb{V}_{\beta_2+k_1,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta_2+k_1,\check{\sigma},\epsilon,d}}{\beta_2!}\end{aligned} \tag{5.105}$$

hold for all $\beta \geq 0$, where $\widetilde{A},\widetilde{B} > 0$ and the sequence $\widetilde{D}_n$, $n \geq 0$ are defined in Lemma 5.7.

We consider the following sequence of Cauchy problems:

$$\partial_x^S \mathbb{W}^{\Delta}_{n,\mathcal{E}_1}(x) = \sum_{(s,k_0,k_1)\in\mathcal{S}} C^1_{23.2}(x\partial_x + S + 1)^{b(s-k_0)}\left(\frac{\widetilde{A}}{1-x/\widetilde{B}}\partial_x^{k_1}\mathbb{W}^{\Delta}_{n,\mathcal{E}_1}(x)\right) + \widetilde{\mathbb{D}}_n(x), \tag{5.106}$$

where

$$\widetilde{\mathbb{D}}_n(x) = \sum_{(s,k_0,k_1)\in\mathcal{S}} C^2_{23.2}(x\partial_x + S + 1)^{b(s-k_0)}\left(\frac{\widetilde{D}_n}{1-x/\widetilde{B}}\partial_x^{k_1}\mathbb{W}_{\arg(\lambda),\mathcal{E}_0}(x)\right) \tag{5.107}$$

with

$$\mathbb{W}_{\arg(\lambda),\mathcal{E}_0}(x) = \sum_{\beta\geq 0}\sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left\|\mathbb{V}_{\beta,\arg(\lambda),S_{d_0},\mathcal{E}_0}(x)\right\|_{\beta,\check{\sigma},\epsilon,d}\frac{x^\beta}{\beta!} \tag{5.108}$$

for given initial data

$$\begin{aligned}&\left(\partial_x^j \mathbb{W}^{\Delta}_{n,\mathcal{E}_1}\right)(0)\\ &\quad = \mathbb{W}^{\Delta}_{j,n,\mathcal{E}_1} = \sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left\|\mathbb{V}_{j,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon) - v_{0,j,S_{d_1,\delta_n},\mathcal{E}_1}(re^{i\xi_n},\epsilon)\right\|_{j,\check{\sigma},\epsilon,d}, \quad 0\leq j\leq S-1,\end{aligned} \tag{5.109}$$

which are finite positive numbers due to the assumption (5.27) and Remark 2.4. Moreover, the initial data satisfy the estimates

$$\left|\mathbb{W}^{\Delta}_{j,n,\mathcal{E}_1}\right| \leq J_{n,j} \tag{5.110}$$

for all $0\leq j\leq S-1$ and all $n\geq 0$.

On the other hand, we have that $\mathbb{W}_{\arg(\lambda),\mathcal{E}_0}(x)$ is convergent for all $|x|\leq X_{\mathcal{M}_{d_0}}/4$ (where $X_{\mathcal{M}_{d_0}}$ is chosen in (5.58)). Indeed, we know, from (5.90), that

$$\mathbb{V}_{h,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,z,\epsilon) = \sum_{\beta\geq 0}\mathbb{V}_{h,\beta,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\frac{z^\beta}{\beta!} \tag{5.111}$$

is convergent for all $|z|<\delta_{D,Z_0}$, all $r>0$, and all $h\geq 0$. From (5.86) and (5.89), we know that

$$\left\|\mathbb{V}_{h,\beta,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} \leq C_3 A\frac{(|\epsilon|B(\check{\sigma},b,\epsilon)/\check{\sigma})^h}{h!}\left\|\mathbb{V}_{h,\beta,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} \tag{5.112}$$

for all $h \geq 0$ and all $\beta \geq 0$. From (5.58) and (5.112), we deduce that

$$\begin{aligned}\left\|\mathbb{V}_{\beta,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} &= \left\|\sum_{h\geq 0}\mathbb{V}_{h,\beta,\lambda,\mathcal{M}_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d} \\ &\leq C_3 A C_{\mathcal{M}_{d_0}}\beta!\left(\frac{2}{X_{\mathcal{M}_{d_0}}}\right)^{\beta}\sum_{h\geq 0}\left(\frac{2|\epsilon|B(\check{\sigma},b,\epsilon)}{\check{\sigma}U_{\mathcal{M}_{d_0}}}\right)^{h}\end{aligned} \tag{5.113}$$

and this last sum is convergent provided that ϵ_0 is small enough. We deduce that $\mathbb{W}_{\arg(\lambda),\mathcal{E}_0}(x)$ belongs to $G(U, X_{\mathcal{M}_{d_0}}/4)$, for any $U > 0$. Let $\widetilde{C}_{\mathcal{M}_{d_0}} := ||\mathbb{W}_{\arg(\lambda),\mathcal{E}_0}(x)||_{(U,X_{\mathcal{M}_{d_0}}/4)}$.

From Lemmas 2.20 and 2.21, we get constants $\widetilde{D}_{\mathcal{M}_{d_0}} > 0$ (depending on $\mathcal{S},\check{\sigma},S,\widetilde{\rho}',\widetilde{\mu}',\widetilde{B},U,X_{\mathcal{M}_{d_0}}$), $0 < \widetilde{U}_{1,\mathcal{M}_{d_0}} < U$, and $0 < \widetilde{X}_{1,\mathcal{M}_{d_0}} < X_{\mathcal{M}_{d_0}}/4$ such that

$$\left\|\widetilde{\mathbb{D}}_n(x)\right\|_{(\widetilde{U}_{1,\mathcal{M}_{d_0}},\widetilde{X}_{1,\mathcal{M}_{d_0}})} \leq \widetilde{D}_n\widetilde{D}_{\mathcal{M}_{d_0}}\widetilde{C}_{\mathcal{M}_{d_0}} \tag{5.114}$$

for all $n \geq 0$.

From the assumption (4.42) and the fact that $b > 1$, we deduce that

$$S > b(s-k_0)+k_1 \tag{5.115}$$

for all $(s,k_0,k_1) \in \mathcal{S}$. Hence, the assumption (2.108) is satisfied in Proposition 2.22 for the problem (5.106), (5.109). Moreover, the initial data $\mathbb{W}^{\Delta}_{j,n}$ can be seen as constant functions (therefore analytic) with respect to a variable u on the closed disc $\overline{D}(0,U)$ for any given $U > 0$ and the coefficients of (5.106) are analytic with respect to x on $\overline{D}(0,\widetilde{B}/2)$ and constant (therefore analytic) with respect to u on $\overline{D}(0,U)$. We deduce that all the hypotheses of Proposition 2.22 for the problem (5.106), (5.109) are fulfilled. A direct computation shows that the problem (5.106), (5.109) has a unique formal solution $\mathbb{W}^{\Delta}_{n,\mathcal{E}_1}(x) = \sum_{\beta\geq 0} w^{\Delta}_{\beta,n,\mathcal{E}_1}x^{\beta}/\beta!$, with $w^{\Delta}_{\beta,n,\mathcal{E}_1} \in \mathbb{C}$. From Proposition 2.22, we deduce that $\mathbb{W}^{\Delta}_{n,\mathcal{E}_1}(x) \in G(\widetilde{U}_1,\widetilde{X}_1)$ where $0 < \widetilde{U}_1 < \widetilde{U}_{1,\mathcal{M}_{d_0}}$ (depending on $\mathcal{S}$) and $0 < \widetilde{X}_1 < \min(\widetilde{B}/2,\widetilde{X}_{1,\mathcal{M}_{d_0}})$ (depending on $\mathcal{S},S,\check{\sigma},\widetilde{A},\widetilde{B},\widetilde{\rho},\widetilde{\mu}$). Moreover, from (2.111) and (5.114), there exist constants $\widetilde{M}_1 > 0$ (depending on $\mathcal{S},S,\check{\sigma},\widetilde{A},\widetilde{B},\widetilde{\rho},\widetilde{\mu}$) and $\widetilde{M}_2 > 0$ (depending on $\mathcal{S},\widetilde{B},S$) such that

$$\left\|\mathbb{W}^{\Delta}_{n,\mathcal{E}_1}(x)\right\|_{(\widetilde{U}_1,\widetilde{X}_1)} \leq \widetilde{M}_1\max_{0\leq j\leq S-1}J_{n,j} + \widetilde{D}_n\widetilde{M}_2\widetilde{D}_{\mathcal{M}_{d_0}}\widetilde{C}_{\mathcal{M}_{d_0}} \tag{5.116}$$

for all $n \geq 0$.

Now, the coefficients $w^{\Delta}_{\beta,n,\mathcal{E}_1}$ satisfy the following equalities:

$$\begin{aligned} w^{\Delta}_{\beta+S,n,\mathcal{E}_1} = &\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta} C^1_{23.2}\beta!\tilde{A}\tilde{B}^{-\beta_1}(\beta+S+1)^{b(s-k_0)}\frac{w^{\Delta}_{\beta_2+k_1,n,\mathcal{E}_1}}{\beta_2!} \\ &+\sum_{(s,k_0,k_1)\in\mathcal{S}}\sum_{\beta_1+\beta_2=\beta} C^2_{23.2}\beta!\tilde{D}_n\tilde{B}^{-\beta_1}(\beta+S+1)^{b(s-k_0)} \\ &\times\frac{\sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left\|\mathbb{V}_{\beta_2+k_1,\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,\epsilon)\right\|_{\beta_2+k_1,\check{\sigma},\epsilon,d}}{\beta_2!} \end{aligned} \tag{5.117}$$

for all $\beta\ge 0$ and all $n\ge 0$, with

$$w^{\Delta}_{j,n,\mathcal{E}_1}=\mathbb{W}^{\Delta}_{j,n,\mathcal{E}_1},\quad 0\le j\le S-1. \tag{5.118}$$

Gathering the inequalities (5.105) and the equalities (5.117), with the initial data (5.118), one gets that

$$\sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left|\mathbb{V}^{\Delta}_{0,\beta,n,\mathcal{E}_1}(\epsilon)\right|\le w^{\Delta}_{\beta,n,\mathcal{E}_1} \tag{5.119}$$

for all $\beta,n\ge 0$.

From (5.119) and the estimates (5.116), we deduce that

$$\sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left\|\mathbb{V}^{\Delta}_{0,\beta,S_{d_1,\delta_n},\mathcal{E}_1}(r,\epsilon)\right\|_{\beta,\check{\sigma},\epsilon,d}\le\left(\tilde{M}_1\max_{0\le j\le S-1}J_{n,j}+\tilde{D}_n\tilde{M}_2\tilde{D}_{\mathcal{M}_{d_0}}\tilde{C}_{\mathcal{M}_{d_0}}\right)\beta!\left(\frac{1}{\tilde{X}_1}\right)^{\beta} \tag{5.120}$$

for all $\beta,n\ge 0$. From (5.120), we get that

$$\begin{aligned} &\sup_{\epsilon\in\mathcal{E}_0\cap\mathcal{E}_1}\left\|\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon)-\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_{D_{0,1}})} \\ &\quad\le 2\left(\tilde{M}_1\max_{0\le j\le S-1}J_{n,j}+\tilde{D}_n\tilde{M}_2\tilde{D}_{\mathcal{M}_{d_0}}\tilde{C}_{\mathcal{M}_{d_0}}\right) \end{aligned} \tag{5.121}$$

for all $n\ge 0$ and for all $0<\delta_{D_{0,1}}<\tilde{X}_1/2$. This implies the estimates (5.92). □

In the following lemma, we express the function $X_{0,1}(t,z,\epsilon)$ as Laplace transform of a staircase distribution.

Lemma 5.7. *Let $\check{\sigma}>\tilde{\sigma}>\sigma r_b(S-1)$ as in Lemma 5.3. Then, one can write the function $X_{0,1}(t,z,\epsilon)$, which by construction of Proposition 4.12, solves the singularly perturbed Cauchy problem*

$$\epsilon t^2\partial_t\partial_z^S X_{0,1}(t,z,\epsilon)+(\epsilon t+1)\partial_z^S X_{0,1}(t,z,\epsilon)=\sum_{(s,k_0,k_1)\in\mathcal{S}}b_{s,k_0,k_1}(z,\epsilon)t^s\left(\partial_t^{k_0}\partial_z^{k_1}X_{0,1}\right)(t,z,\epsilon) \tag{5.122}$$

for given initial data

$$\left(\partial_z^j X_{0,1}\right)(t,0,\epsilon) = \varphi_{0,1,j}(\epsilon t,\epsilon), \quad 0 \leq j \leq S-1 \tag{5.123}$$

in the form of a Laplace transform in direction $\arg(\lambda)$

$$X_{0,1}(t,z,\epsilon) = \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t) \tag{5.124}$$

for all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{D_{0,1}}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$, *where* $\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon) \in \mathfrak{D}'(\check{\sigma},\epsilon,\delta_{D_{0,1}})$ *solves the Cauchy problem* (5.80), (5.81).

Proof. From Proposition 4.12 and the assumption (5.23), we get that the function $X_{0,1}(t,z,\epsilon)$ can be expressed as a Laplace transform in the direction ξ_n,

$$\begin{aligned} X_{0,1}(t,z,\epsilon) &= \frac{1}{\epsilon t}\int_{L_{\xi_n}} V_{0,S_{d_1,\delta_n},\mathcal{E}_1}(\tau,z,\epsilon)\exp\left(-\frac{\tau}{\epsilon t}\right)d\tau \\ &= \frac{e^{i\xi_n}}{\epsilon t}\int_0^{+\infty} V_{0,S_{d_1,\delta_n},\mathcal{E}_1}\left(re^{i\xi_n},z,\epsilon\right)\exp\left(-\frac{re^{i\xi_n}}{\epsilon t}\right)dr \end{aligned} \tag{5.125}$$

for all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{\mathcal{E}_1}) \times (\mathcal{E}_0 \cap \mathcal{E}_1)$, all $n \geq 0$. Now, let $t \in \mathcal{T} \cap D(0,\iota'')$, $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. For all $n \geq 0$, we can rewite $X_{0,1}(t,z,\epsilon)$ as a Laplace transform in the direction $\arg(\lambda)$ as follows:

$$X_{0,1}(t,z,\epsilon) = \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon)\right)\left(\epsilon t e^{i(\arg(\lambda)-\xi_n)}\right) \tag{5.126}$$

for all $z \in D(0,\delta_{\mathcal{E}_1})$. Using the expression (5.126), we deduce that from the estimates (3.5), there exists a constant $C_{(t,\epsilon)} > 0$ such that

$$\begin{aligned} &\left|X_{0,1}(t,z,\epsilon) - \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t)\right| \\ &\quad \leq C_{(t,\epsilon)}\left\|\mathbb{V}_{0,S_{d_1,\delta_n},\mathcal{E}_1}(r,z,\epsilon) - \mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right\|_{(\check{\sigma},\epsilon,d,\delta_{D_{0,1}})} \\ &\qquad + \left|\mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)\left(\epsilon t e^{i(\arg(\lambda)-\xi_n)}\right) - \mathcal{L}_{\arg(\lambda)}\left(\mathbb{V}_{\arg(\lambda),S_{d_0},\mathcal{E}_0}(r,z,\epsilon)\right)(\epsilon t)\right| \end{aligned} \tag{5.127}$$

for all $n \geq 0$ and all $z \in D(0,\delta_{D_{0,1}})$. By letting n tend to $+\infty$ and using the estimates (5.92), we get the formula (5.124). □

Now, we are in the position to state the main result of our work.

Theorem 5.8. *Let the assumptions* (4.42), (4.44), (4.67), (4.69), (4.70), (5.1), (5.4), (5.8), (5.20), (5.23), *and* (5.27) *hold. Then, if one denote by op¿*$(\mathcal{G}_{\kappa_0})$ *(resp. op*$(\mathcal{G}_{\kappa_1})$*) the opening of the sector* $\mathcal{G}_{\kappa_0}$ *(resp.* $\mathcal{G}_{\kappa_1}$*), one has that for all* $t \in \mathcal{T} \cap D(0,\iota'')$, $z \in D(0,\delta_{D_{0,1}})$, *the function* $s \mapsto g_0(s,t,z)$

(constructed in Proposition 4.15) can be analytically continued along any path Γ in the punctured sector

$$\dot{S}_{\kappa_0,\kappa_1,t,\lambda} = \left\{ s \in \frac{\mathbb{C}^*}{\kappa_0} - \frac{op(\mathcal{G}_{\kappa_0})}{2} < \arg(s) < \kappa_1 + \frac{op(\mathcal{G}_{\kappa_1})}{2} \right\} \setminus \bigcup_{k=1}^{\infty} \left\{ \frac{\lambda k}{t} \right\}, \tag{5.128}$$

as a function denoted by $g_0^{\Gamma,t,z}(s)$. Moreover, for all $k \geq 1$, and any path $\Gamma_{0,k} \subset \dot{S}_{\kappa_0,\kappa_1,t,\lambda}$ from 0 to a neighborhood of $\lambda k/t$, there exists a constant $C_k > 0$ such that

$$\left| g_0^{\Gamma_{0,k},t,z}(s) \right| \leq C_k \left| \log\left(s - \frac{\lambda k}{t} \right) \right| \tag{5.129}$$

as s tends to $\lambda k/t$ in a sector centered at $\lambda k/t$.

Proof. The proof is based on the following version of a result on analytic continuation of Borel transforms obtained by Fruchard and Schäfke in [3]. This result extends a former statement obtained by the same authors in [28].

Theorem (FS). *Let $r > 0$ and let $g : D(0,r) \to \mathbb{C}$ be a holomorphic function that can be analytically continued as a function g^+ (resp., g^-) with exponential growth of order 1 on an unbounded sector S_{κ^+,δ^+} (resp. S_{κ^-,δ^-}) centered at 0, with bisecting direction κ^+ (resp. κ^-) and opening δ^+ (resp. δ^-). Let $C > r$ be a real number and let $m \geq 1$ be an integer. Let $\{a_k \in \mathbb{C}^*, 1 \leq k \leq m\} \subset D(0,C)$ be a set of aligned points and let $\alpha > 0$ with $\arg(a_k) = \alpha \in (\kappa^-, \kappa^+)$, for all $1 \leq k \leq m$. For all integers $1 \leq k \leq m$, let S_k be an unbounded open sector centered at a_k, with bisecting direction which is parallel to κ^-, and opening $\mu > 0$ such that the $S_k \cap D(0,C)$ do not intersect for all $1 \leq k \leq m$.*

Now, for all $1 \leq k \leq m$, let g_k be a holomorphic and bounded function on a small neighborhood of 0 and with exponential growth of order 1 on the sector $S_k - a_k = \{s \in \mathbb{C}/s + a_k \in S_k\}$ with bisecting direction κ^-. We consider the Laplace transforms

$$f^+(\epsilon) = \int_{L_{\kappa^+}} g^+(s) e^{-s/\epsilon} ds, \qquad f^-(\epsilon) = \int_{L_{\kappa^-}} g^-(s) e^{-s/\epsilon} ds, \qquad f_k^-(\epsilon) = \int_{L_{\kappa^-}} g_k(s) e^{-s/\epsilon} ds \tag{5.130}$$

for all $k \geq 1$, where L_{κ^+} is the half-line starting from 0 in the direction κ^+ and L_{κ^-} is the half-line starting from 0 in the direction κ^-. The function f^+ (resp. f^-) defines a holomorphic and bounded function on an open sector $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) with finite radius, with bisecting direction κ^+ (resp. κ^-) and opening $\pi + \delta^+$ (resp. $\pi + \delta^-$). The sectors $\mathcal{E}^+, \mathcal{E}^-$ are chosen in such a way that $\mathcal{E}^+ \cap \mathcal{E}^-$ is contained in a sector with direction α and with opening less than π. Assume that the following Stokes relation

$$f^+(\epsilon) = f^-(\epsilon) + \sum_{k=1}^{m} \frac{e^{-a_k/\epsilon}}{k!} f_k^-(\epsilon) + \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon} \right) \tag{5.131}$$

holds for all $\epsilon \in \mathcal{E}^+ \cap \mathcal{E}^-$*, where* $\mathcal{O}(e^{-Ce^{i\alpha}/\epsilon})$ *is a holomorphic function* $R(\epsilon)$ *on* $\mathcal{E}^+ \cap \mathcal{E}^-$ *such that there exists a constant* $H > 0$ *with*

$$|R(\epsilon)| \leq H\left|e^{-Ce^{i\alpha}/\epsilon}\right| = He^{-(C/|\epsilon|)\cos(\alpha-\arg(\epsilon))} \tag{5.132}$$

for all $\epsilon \in \mathcal{E}^+ \cap \mathcal{E}^-$.

Then, the function $g : D(0,r) \to \mathbb{C}$ *can be analytically continued along any path* Γ *in the punctured sector*

$$\dot{S}_{\kappa^-,\kappa^+,C} = \left\{ s \in \frac{\mathbb{C}^*}{|s|} < C, \kappa^- - \frac{\delta^-}{2} < \arg(s) < \kappa^+ + \frac{\delta^+}{2} \right\} \setminus \bigcup_{k=1}^{m} \{a_k\}. \tag{5.133}$$

Moreover, for all $1 \leq k \leq m$*, and any path* $\Gamma_{0,k} \subset \dot{S}_{\kappa^-,\kappa^+,C}$ *from 0 to a neighborhood of* a_k*, if we denote by* $g^{\Gamma_{0,k}}(s)$ *the analytic continuation of* g *along* $\Gamma_{0,k}$*, then there exists a constant* $C_k > 0$ *such that*

$$\left|g^{\Gamma_{0,k}}(s)\right| \leq C_k \left|\log(s - a_k)\right| \tag{5.134}$$

as s *tends to* a_k *in a sector centered at* a_k.

Proof. For the sake of completeness, we give a sketch of proof of this theorem. In the first step, let us consider the following sums of Cauchy integrals

$$h(t) = \frac{1}{2i\pi}\sum_{k=1}^{m}\frac{1}{k!}\int_{L_{a_k,\kappa^-,C}} \frac{g_k(\tau - a_k)}{\tau - t}d\tau, \tag{5.135}$$

where $L_{a_k,\kappa^-,C}$ is the segment starting from a_k in the direction κ^- with length C. The multivalued function $h(t)$ can be analytically continued along any path Γ in $\mathbb{C} \setminus \{a_1,\ldots,a_m\}$ by deforming the path of integration $L_{a_k,\kappa^-,C}$ in the sector S_k and keeping the endpoints of the segment $L_{a_k,\kappa^-,C}$ fixed for all $1 \leq k \leq m$. Moreover, let $1 \leq k \leq m$ and $t \in L_{a_k,\kappa^-} \setminus \{a_k\}$, where L_{a_k,κ^-} denotes the half-line starting from a_k in the direction κ^-. We denote by $h^{\Gamma_{a_k,t,\rho}}(t)$ the analytic continuation of $h(t)$ along a loop $\Gamma_{a_k,t,\rho}$ around a_k constructed as follows: the loop follows a segment starting from t in the direction a_k then turns around a_k along a circle $\Gamma_{a_k,\rho}$ of small radius $\rho > 0$ positively oriented and then goes back to t following the same segment. We have that

$$h(t) - h^{\Gamma_{a_k,t,\rho}}(t) = \frac{g_k(t - a_k)}{k!}. \tag{5.136}$$

Indeed, by the Cauchy theorem, one can write $h(t) - h^{\Gamma_{a_k,t,\rho}}(t)$ as a Cauchy integral

$$I_k = \frac{1}{2i\pi k!}\int_{\mathcal{C}_{a_k,C}} \frac{g_k(\tau - a_k)}{\tau - t}d\tau, \tag{5.137}$$

where $\mathcal{C}_{a_k,C}$ is a positively oriented closed curve enclosing t starting from a_k and containing the point $a_k + Ce^{i\kappa^-}$. By the residue theorem, one gets that $I_k = g_k(t-a_k)/k!$. From the relation (5.136), we also deduce the existence of a holomorphic function $b(t)$ near a_k such that

$$h(t) = -\frac{g_k(t-a_k)}{2i\pi k!}\log(t-a_k) + b(t) \tag{5.138}$$

for all t near a_k, for a well-chosen determination of the logarithm $\log(x)$.

In the second step, let us define the truncated Laplace transforms and Laplace transforms

$$\begin{aligned} H^+_{C'}(\epsilon) &= \int_{L_{\kappa^+,C'}} h(s)e^{-s/\epsilon}ds, \qquad H^-_{C'}(\epsilon) = \int_{L_{\kappa^-,C'}} h(s)e^{-s/\epsilon}ds, \\ H^+(\epsilon) &= \int_{L_{\kappa^+}} h(s)e^{-s/\epsilon}ds, \qquad H^-(\epsilon) = \int_{L_{\kappa^-}} h(s)e^{-s/\epsilon}ds, \end{aligned} \tag{5.139}$$

where $L_{\kappa^+,C'}$ is the segment starting from 0 to $C'e^{i\kappa^+}$ and $L_{\kappa^-,C'}$ is the segment starting from 0 to $C'e^{i\kappa^-}$, for any fixed $C' > C$. By the Cauchy formula, one can write the difference $H^+_{C'}(\epsilon) - H^-_{C'}(\epsilon)$ as the sum

$$\begin{aligned} &H^+_{C'}(\epsilon) - H^-_{C'}(\epsilon) \\ &\quad = -\sum_{k=1}^{m}\int_{\Gamma_{a_k,\rho}} h(s)e^{-s/\epsilon}ds + \int_{L_{a_k,\rho,C',\kappa^-}}\left(h(s) - h^{\Gamma_{a_k,s,\rho}}(s)\right)e^{-s/\epsilon}ds + \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon}\right), \end{aligned} \tag{5.140}$$

where L_{a_k,ρ,C',κ^-} is the segment starting from $a_k + \rho e^{i\kappa^-}$ to $a_k + C'e^{i\kappa^-}$ for any $\rho > 0$ small enough. Due to the decomposition (5.138), $h(s)$ is integrable at a_k. By letting ρ tending to 0 and C' tending to infinity, using the relation (5.136) in (5.140), ones gets that

$$\begin{aligned} H^+(\epsilon) - H^-(\epsilon) &= \sum_{k=1}^{m}\frac{1}{k!}\int_{L_{a_k,\kappa^-}} g_k(s-a_k)e^{-s/\epsilon}ds + \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon}\right) \\ &= \sum_{k=1}^{m}\frac{e^{-a_k/\epsilon}}{k!}\int_{L_{\kappa^-}} g_k(s)e^{-s/\epsilon}ds + \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon}\right), \end{aligned} \tag{5.141}$$

where L_{a_k,κ^-} is the half-line starting from a_k in the direction κ^-.

Now, one considers the differences $D^+(\epsilon) = f^+(\epsilon) - H^+(\epsilon)$ and $D^-(\epsilon) = f^-(\epsilon) - H^-(\epsilon)$. From the Stokes relations (5.131) and (5.141), one deduces that

$$D^+(\epsilon) - D^-(\epsilon) = \mathcal{O}\left(e^{-Ce^{i\alpha}/\epsilon}\right) \tag{5.142}$$

for all $\epsilon \in \mathcal{E}^+ \cap \mathcal{E}^-$. Using a similar Borel transform integral representation as in the proof of Theorem 1 in [28], one can show that the difference $g(s) - h(s)$, which is by construction analytic near the origin in $\mathbb{C}$, can be analytically continued to a function $G(s)$, which is holomorphic on the sector $S_{\kappa^-,\kappa^+,C} = \{s \in \mathbb{C}^*/|s| < C, \kappa^- < \arg(s) < \kappa^+\}$. Since h can be analytically continued along any path in $\mathbb{C} \setminus \{a_1,\ldots,a_n\}$, one gets that the function g can be analytically continued along any path in $\dot{S}_{\kappa^-,\kappa^+,C}$ and from the decomposition (5.138) one deduces the estimates (5.134).

Now, we return to the proof of Theorem 5.8. From the formula (4.76) and Proposition 5.2, the following equality

$$\begin{aligned}\int_{L_{\kappa_1}} g_{0,1}(s,t,z)e^{-s/\epsilon}ds = &\int_{L_{\kappa_0}} g_{0,0}(s,t,z)e^{-s/\epsilon}ds \\ &+ \sum_{h\geq 1}\frac{\exp(-h\lambda/\epsilon t)}{h!}\int_{L_{\kappa_0}} g_{h,0}(s,t,z)e^{-s/\epsilon}ds\end{aligned} \tag{5.143}$$

holds for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$, and all $t \in \mathcal{T} \cap D(0,\iota'')$, all $z \in D(0,\delta_{D_{0,1}})$. Let $t \in \mathcal{T} \cap D(0,\iota'')$ and $z \in D(0,\delta_{D_{0,1}})$ fixed. Let $m \geq 1$ be an integer. From the estimates (4.78), we get that

$$\begin{aligned}&\sum_{h\geq m+1}\left|\frac{\exp(-h\lambda/\epsilon t)}{h!}\int_{L_{\kappa_0}} g_{h,0}(s,t,z)e^{-s/\epsilon}ds\right| \\ &\quad\leq 2\tilde{C}_0\sum_{h\geq m+1}\left|\exp\left(-h\frac{\lambda}{\epsilon t}\right)\right|\left(\frac{2}{u_1}\right)^h \\ &\quad\leq 2\tilde{C}_0\left(\frac{2}{u_1}\right)^{m+1}\left|\exp\left(-(m+1)\frac{\lambda}{\epsilon t}\right)\right|\frac{1}{1-2\left|\exp(-\lambda/\epsilon t)\right|/u_1}\end{aligned} \tag{5.144}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. From (5.143) and (5.144), we deduce that the following Stokes relation

$$\begin{aligned}\int_{L_{\kappa_1}} g_{0,1}(s,t,z)e^{-s/\epsilon}ds = &\int_{L_{\kappa_0}} g_{0,0}(s,t,z)e^{-s/\epsilon}ds \\ &+ \sum_{h=1}^{m}\frac{\exp(-h\lambda/\epsilon t)}{h!}\int_{L_{\kappa_0}} g_{h,0}(s,t,z)e^{-s/\epsilon} + \mathcal{O}\left(e^{-(m+1)\lambda/(\epsilon t)}\right).\end{aligned} \tag{5.145}$$

Holds, where $\mathcal{O}(e^{-(m+1)\lambda/(\epsilon t)})$ is a holomorphic function $R(\epsilon)$ on $\mathcal{E}_0 \cap \mathcal{E}_1$ such that there exists a constant $H > 0$ with

$$|R(\epsilon)| \leq H\left|e^{-(m+1)\lambda/(\epsilon t)}\right| \tag{5.146}$$

for all $\epsilon \in \mathcal{E}_0 \cap \mathcal{E}_1$. We can apply Theorem (FS) with $a_k = k\lambda/t$, for $1 \le k \le m$, $C = |\lambda|(m+1)/|t|$ to get that the function $s \mapsto g_0(s,t,z)$ (constructed in Proposition 4.15) can be analytically continued along any path in the punctured sector

$$\dot{S}_{\kappa_0,\kappa_1,t,\lambda,m} = \left\{ s \in \frac{\mathbb{C}^*}{|s|} < \frac{|\lambda|(m+1)}{|t|}, \kappa_0 - \frac{\mathrm{op}(\mathcal{G}_{\kappa_0})}{2} < \arg(s) < \kappa_1 + \frac{\mathrm{op}(\mathcal{G}_{\kappa_1})}{2} \right\} \setminus \bigcup_{k=1}^{m} \left\{ \frac{\lambda k}{t} \right\} \tag{5.147}$$

as a function denoted by $g_0^{\Gamma,t,z}(s)$. Moreover, for all $1 \le k \le m$, and any path $\Gamma_{0,k} \subset \dot{S}_{\kappa_0,\kappa_1,t,\lambda,m}$ from 0 to a neighborhood of $\lambda k/t$, there exists a constant $C_k > 0$ such that $|g_0^{\Gamma_{0,k},t,z}(s)| \le C_k |\log(s - \lambda k/t)|$ as s tends to $\lambda k/t$ in a sector centered at $\lambda k/t$. Since this result is true for all $m \ge 1$, Theorem 5.8 follows. □

In the next result, Onee show that under the additional hypothesis that the coefficients of (4.90) are polynomials in the parameter ϵ, the function $g_0(s,t,z)$ solves a singular linear partial differential equation in $\mathbb{C}^3$.

Corollary 5.9. *Let the assumptions of Theorem 5.8 hold. We assume moreover that, for all tuple (s,k_0,k_1) chosen in the set $\mathcal{S}$, the coefficients $b_{s,k_0,k_1}(z,\epsilon)$ belong to $\mathbb{C}\{z\}[\epsilon]$ with the following expansion in ϵ:*

$$b_{s,k_0,k_1}(z,\epsilon) = \sum_{m=k_0}^{d_{s,k_0,k_1}} b^m_{s,k_0,k_1}(z)\epsilon^m \tag{5.148}$$

for some $d_{s,k_0,k_1} \ge k_0$. Then, for all $K \in \mathbb{N}$ with $K \ge 1$ and $K \ge \max\{d_{s,k_0,k_1} \in \mathbb{N}/(s,k_0,k_1) \in \mathcal{S}\}$, the function $g_0(u,t,z)$ (constructed in Proposition 4.15) satisfies the following singular linear partial differential equation

$$\begin{aligned} &t^2\partial_t\partial_u^{K-1}\partial_z^S g_0(u,t,z) + \partial_u^K\partial_z^S g_0(u,t,z) \\ &\quad = -t\partial_u^{K-1}\partial_z^S g_0(u,t,z) + \sum_{(s,k_0,k_1)\in\mathcal{S}} \sum_{m=k_0}^{d_{s,k_0,k_1}} b^m_{s,k_0,k_1}(z) t^s \left(\partial_u^{K-m}\partial_t^{k_0}\partial_z^{k_1} g_0\right)(u,t,z) \end{aligned} \tag{5.149}$$

for all $(u,t,z) \in D(0,s_0) \times (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{D_{0,1}})$. From Theorem 5.8, for all $(t,z) \in (\mathcal{T} \cap D(0,\iota'')) \times D(0,\delta_{D_{0,1}})$, this solution $g_0(u,t,z)$ can be analytically continued with respect to u along any path in the punctured sector $\dot{S}_{\kappa_0,\kappa_1,t,\lambda}$ with logarithmic estimates (5.129) *near the singular points $\lambda k/t$ for all $k \ge 1$.*

Proof. From Proposition 4.15, we have that the function

$$X_{0,0}(t,z,\epsilon) = \epsilon^{-1}\int_{L_{\kappa_0}} g_{0,0}(s,t,z)e^{-s/\epsilon}ds \tag{5.150}$$

solves (5.122) on $(\mathcal{T} \cap D(0, \iota'')) \times D(0, \delta_{D_{0,1}}) \times \mathcal{E}_0$. From the formulas in Proposition 4.2, we deduce that the function $g_{0,0}(u, t, z)$ solves the singular integrodifferential equation

$$t^2 \partial_t \partial_u^{-1} \partial_z^S g_{0,0}(u, t, z) + \partial_z^S g_{0,0}(u, t, z)$$
$$= -t \partial_u^{-1} \partial_z^S g_{0,0}(u, t, z) + \sum_{(s,k_0,k_1) \in \mathcal{S}} \sum_{m=k_0}^{d_{s,k_0,k_1}} b_{s,k_0,k_1}^m(z) t^s \left(\partial_u^{-m} \partial_t^{k_0} \partial_z^{k_1} g_{0,0} \right)(u, t, z) \tag{5.151}$$

for all $(u, t, z) \in (\mathcal{G}_{\kappa_0} \cup D(0, s_0)) \times (\mathcal{T} \cap D(0, \iota'')) \times D(0, \delta_{D_{0,1}})$. Since $g_0(u, t, z)$ is holomorphic on $D(0, s_0) \times (\mathcal{T} \cap D(0, \iota'')) \times D(0, \delta_{D_{0,1}})$ and has $g_{0,0}(u, t, z)$ as analytic continuation on $(\mathcal{G}_{\kappa_0} \cup D(0, s_0)) \times (\mathcal{T} \cap D(0, \iota'')) \times D(0, \delta_{D_{0,1}})$, we get that $g_0(u, t, z)$ also solves (5.151). By differentiating K times of each hand side of the equation with respect to u, one gets that $g_0(u, t, z)$ solves the partial differential equation (5.149). □

References

[1] T. Mandai, “Existence and nonexistence of null-solutions for some non-Fuchsian partial differential operators with T-dependent coefficients,” *Nagoya Mathematical Journal*, vol. 122, pp. 115–137, 1991.

[2] S. Malek, “On the summability of formal solutions for doubly singular nonlinear partial differential equations,” *Journal of Dynamical and Control Systems*, vol. 18, no. 1, pp. 45–82, 2012.

[3] A. Fruchard and R. Schäfke, “On the parametric resurgence for a certain singularly perturbed linear differential equation of second order,” in *Asymptotics in Dynamics, Geometry and PDEs; Generalized Borel Summation*, vol. 2 of *Crm series*, pp. 213–243, Scuola Normale Superiore, 2011.

[4] T. Aoki, T. Kawai, and Y. Takei, “The Bender-Wu analysis and the Voros theory. II,” in *Algebraic Analysis and Around*, vol. 54 of *Advanced Studies in Pure Mathematics*, pp. 19–94, Mathematical Society of Japan, Tokyo, Japan, 2009.

[5] É. Delabaere, H. Dillinger, and F. Pham, “Résurgence de Voros et périodes des courbes hyperelliptiques,” *Université de Grenoble. Annales de l'Institut Fourier*, vol. 43, no. 1, pp. 163–199, 1993.

[6] J. Écalle, *Les Fonctions Résurgentes*, Publications Mathématiques d'Orsay, 1981.

[7] B. Braaksma and R. Kuik, “Resurgence relations for classes of differential and difference equations,” *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, vol. 13, no. 4, pp. 479–492, 2004.

[8] O. Costin, “On Borel summation and Stokes phenomena for rank-q nonlinear systems of ordinary differential equations,” *Duke Mathematical Journal*, vol. 93, no. 2, pp. 289–344, 1998.

[9] J. Écalle, *Les Fonctions Résurgentes. Tome II*, vol. 6, Département de Mathématique, Université de Paris-Sud, Orsay, France, 1981.

[10] B. Malgrange, “Introduction aux travaux de J. Écalle,” *L'Enseignement Mathématique*, vol. 31, no. 3-4, pp. 261–282, 1985.

[11] J. Écalle, “Classification analytique des champs hamiltoniens locaux. Invariants holomorphes et hamiltoniens étrangers,” *Astérisque*, no. 150-151, pp. 67–86, 1987.

[12] D. Sauzin, “Mould expansions for the saddle-node and resurgence monomials,” in *Renormalization and Galois Theories*, vol. 15 of *IRMA Lectures in Mathematics and Theoretical Physics*, pp. 83–163, European Mathematical Society, Zurich, Switzerland, 2009.

[13] C. Olivé, D. Sauzin, and T. M. Seara, “Resurgence in a Hamilton-Jacobi equation,” *Annales de l'Institut Fourier*, vol. 53, no. 4, pp. 1185–1235, 2003.

[14] S. Malek, “On the Stokes phenomenon for holomorphic solutions of integro-differential equations with irregular singularity,” *Journal of Dynamical and Control Systems*, vol. 14, no. 3, pp. 371–408, 2008.

[15] J. Leray, “Problème de Cauchy. I. Uniformisation de la solution du problème linéaire analytique de Cauchy près de la variété qui porte les données de Cauchy,” *Bulletin de la Société Mathématique de France*, vol. 85, pp. 389–429, 1957.

[16] Y. Hamada, “The singularities of the solutions of the Cauchy problem,” *Publications of the Research Institute for Mathematical Sciences*, vol. 5, pp. 21–40, 1969.

[17] S. Ōuchi, "An integral representation of singular solutions of linear partial differential equations in the complex domain," *Journal of the Faculty of Science. University of Tokyo. Section IA*, vol. 27, no. 1, pp. 37–85, 1980.

[18] S. Ōuchi, "The behaviour near the characteristic surface of singular solutions of linear partial differential equations in the complex domain," *Proceedings of the Japan Academy, Series A*, vol. 65, no. 4, pp. 102–105, 1989.

[19] Y. Hamada, J. Leray, and C. Wagschal, "Systèmes d'équations aux dérivées partielles à caractéristiques multiples: problème de Cauchy ramifié; hyperbolicité partielle," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 55, no. 3, pp. 297–352, 1976.

[20] K. Igari, "On the branching of singularities in complex domains," *Japan Academy. Proceedings. Series A*, vol. 70, no. 5, pp. 128–130, 1994.

[21] B. Sternin and V. Shatalov, *Differential Equations on Complex Manifolds*, vol. 276, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1994.

[22] C. Wagschal, "Sur le problème de Cauchy ramifié," *Journal de Mathématiques Pures et Appliquées*, vol. 53, pp. 147–163, 1974.

[23] H. Yamane, "Ramified Cauchy problem for a class of Fuchsian operators with tangent characteristics," *Journal de Mathématiques Pures et Appliquées*, vol. 79, no. 3, pp. 271–294, 2000.

[24] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.

[25] R. Kuik, *Transseries in difference and differential equations [thesis]*, Rijksuniversiteit, Groningen, The Netherlands, 2003.

[26] S. Malek, "On singularly perturbed partial integro-differential equations with irregular singularity," *Journal of Dynamical and Control Systems*, vol. 13, no. 3, pp. 419–449, 2007.

[27] W. Balser, *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Springer, New York, NY, USA, 2000.

[28] A. Fruchard and R. Schäfke, "On the Borel transform," *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, vol. 323, no. 9, pp. 999–1004, 1996.