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Research Article

Optimal Bounds for Seiffert Mean in terms of One-Parameter Means

Hua-Nan Hu,[1] Guo-Yan Tu,[2] and Yu-Ming Chu[3]

[1] *Acquisitions & Cataloging Department of Library, Huzhou Teachers College, Huzhou 313000, China*
[2] *Department of Basic Course Teaching, Tongji Zhejiang College, Jiaxing 314000, China*
[3] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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The authors present the greatest value r_1 and the least value r_2 such that the double inequality $J_{r1}(a,b) < T(a,b) < J_{r2}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $T(a,b)$ and $J_p(a,b)$ denote the Seiffert and pth one-parameter means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the pth one-parameter mean $J_p(a,b)$ and the Seiffert mean $T(a,b)$ of two positive real numbers a and b are defined by

$$J_p(a,b) = \begin{cases} \dfrac{p\left(a^{p+1} - b^{p+1}\right)}{(p+1)\left(a^p - b^p\right)}, & a \neq b,\ p \neq 0, -1, \\ \dfrac{a-b}{\log a - \log b}, & a \neq b,\ p = 0, \\ \dfrac{ab(\log a - \log b)}{a-b}, & a \neq b,\ p = -1, \\ a, & a = b, \end{cases} \tag{1.1}$$

$$T(a,b) = \begin{cases} \dfrac{a-b}{2\arctan((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for J_p and T can be found in the literature [1–14].

It is well known that the one-parameter mean $J_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many mean values are the special case of the one-parameter mean, for example:

$$\begin{aligned} J_1(a,b) &= \frac{(a+b)}{2}, && \text{the arithmetic mean,} \\ J_{1/2}(a,b) &= \frac{\left(a+\sqrt{ab}+b\right)}{3}, && \text{the Heronian mean,} \\ J_{-1/2}(a,b) &= \sqrt{ab}, && \text{the geometric mean,} \\ J_{-2}(a,b) &= \frac{2ab}{(a+b)}, && \text{the harmonic mean.} \end{aligned} \tag{1.3}$$

Seiffert [4] proved that the double inequality

$$M_1(a,b) < T(a,b) < M_2(a,b) \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$, where $M_r(a,b) = [(a^r + b^r)/2]^{1/r} (r \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the rth power mean of a and b.

In [15–17], the authors presented the best possible bounds for the Seiffert mean in terms of the Lehmer, power-type Heron, and one-parameter Gini means as follows:

$$\begin{gathered} L_0(a,b) < T(a,b) < L_{1/3}(a,b), \\ H_{\log 3/\log(\pi/2)}(a,b) < T(a,b) < H_{5/2}(a,b), \\ S_1(a,b) < T(a,b) < S_{5/3}(a,b). \end{gathered} \tag{1.5}$$

for all $a,b > 0$ with $a \neq b$, where $L_r(a,b) = (a^{r+1} + b^{r+1})/(a^r + b^r)$, $H_k(a,b) = [(a^k + (ab)^{k/2} + b^k)/3]^{1/k} (k \neq 0)$ and $H_0(a,b) = \sqrt{ab}$, and $S_\alpha(a,b) = [(a^{\alpha-1} + b^{\alpha-1})/(a+b)]^{1/(\alpha-2)} (\alpha \neq 2)$ and $S_2(a,b) = (a^a b^b)^{1/(a+b)}$ denote the Lehmer, power-type Heron, and one-parameter Gini means of a and b, respectively.

The purpose of this paper is to answer the question: what are the greatest value r_1 and the least value r_2 such that the double inequality

$$J_{r_1}(a,b) < T(a,b) < J_{r_2}(a,b) \tag{1.6}$$

holds for all $a,b > 0$ with $a \neq b$?

2. Lemma

In order to establish our main result we need the following lemma.

Lemma 2.1. *If* $p = 2/(\pi - 2) = 1.75\cdots$, $t \geq 1$ *and* $g(t) = -(p-1)t^{2p+2} + (p+1)t^{2p} + p(p+1)t^{p+3} - 2(p+1)^2 t^{p+2} + 2p(p+3)t^{p+1} - 2(p+1)^2 t^p + p(p+1)t^{p-1} + (p+1)t^2 - (p-1)$, *then there exists* $\lambda \in (1, \infty)$ *such that* $g(t) > 0$ *for* $t \in (1, \lambda)$ *and* $g(t) < 0$ *for* $t \in (\lambda, \infty)$.

Proof. Let $g_1(t) = g'(t)/t$, $g_2(t) = t^{4-p} g_1'(t)$ and $g_3(t) = t^{4-p} g_2^{(5)}(t)/[4p^2(p-1)^2(p+1)^2]$. Then simple computations lead to

$$g(1) = 0, \tag{2.1}$$

$$\lim_{t \to +\infty} g(t) = -\infty, \tag{2.2}$$

$$\begin{aligned} g_1(t) = {} & -2(p-1)(p+1)t^{2p} + 2p(p+1)t^{2p-2} + p(p+1)(p+3) \\ & \times t^{p+1} - 2(p+1)^2(p+2)t^p + 2p(p+1)(p+3)t^{p-1} \\ & - 2p(p+1)^2 t^{p-2} + p(p+1)(p-1)t^{p-3} + 2(p+1), \end{aligned} \tag{2.3}$$

$$g_1(1) = 0, \tag{2.4}$$

$$\lim_{t \to +\infty} g_1(t) = -\infty, \tag{2.5}$$

$$\begin{aligned} g_2(t) = {} & -4p(p-1)(p+1)t^{p+3} + 4p(p+1)(p-1)t^{p+1} + p(p+1)^2 \\ & \times (p+3)t^4 - 2p(p+1)^2(p+2)t^3 + 2p(p-1)(p+1)(p+3) \\ & \times t^2 - 2p(p+1)^2(p-2)t + p(p+1)(p-1)(p-3), \end{aligned} \tag{2.6}$$

$$g_2(1) = 0, \tag{2.7}$$

$$\lim_{t \to +\infty} g_2(t) = -\infty, \tag{2.8}$$

$$\begin{aligned} g_2'(t) = {} & -4p(p-1)(p+1)(p+3)t^{p+2} + 4p(p+1)^2(p-1)t^p \\ & + 4p(p+1)^2(p+3)t^3 - 6p(p+1)^2(p+2)t^2 \\ & + 4p(p-1)(p+1)(p+3)t - 2p(p+1)^2(p-2), \end{aligned} \tag{2.9}$$

$$g_2'(1) = 0, \tag{2.10}$$

$$\lim_{t \to +\infty} g_2'(t) = -\infty, \tag{2.11}$$

$$\begin{aligned} g_2''(t) = {} & -4p(p-1)(p+1)(p+2)(p+3)t^{p+1} + 4p^2(p+1)^2 \\ & \times (p-1)t^{p-1} + 12p(p+1)^2(p+3)t^2 - 12p(p+1)^2 \\ & \times (p+2)t + 4p(p-1)(p+1)(p+3), \end{aligned} \tag{2.12}$$

$$g_2''(1) = 12p(2-p)(p+1)^2 > 0, \tag{2.13}$$

$$\lim_{t \to +\infty} g_2''(t) = -\infty, \tag{2.14}$$

$$g_2'''(t) = -4p(p-1)(p+1)^2(p+2)(p+3)t^p + 4p^2(p+1)^2 \times (p-1)^2 t^{p-2} + 24p(p+1)^2(p+3)t - 12p(p+1)^2(p+2), \tag{2.15}$$

$$g_2'''(1) = 12p(2-p)(2p+3)(p+1)^2 > 0, \tag{2.16}$$

$$\lim_{t\to+\infty} g_2'''(t) = -\infty, \tag{2.17}$$

$$g_2^{(4)}(t) = -4p^2(p-1)(p+1)^2(p+2)(p+3)t^{p-1} + 4p^2(p+1)^2 \times (p-1)^2(p-2)t^{p-3} + 24p(p+1)^2(p+3), \tag{2.18}$$

$$\begin{aligned} g_2^{(4)}(1) &= 8p(p+1)^2\left(-4p^3 + 2p^2 + 5p + 9\right) \\ &> 8p(p+1)^2\left[-4\times 1.8^3 + 2\times 1.75^2 + 5\times 1.75 + 9\right] \\ &= 4.376p(p+1)^2 > 0, \end{aligned} \tag{2.19}$$

$$\lim_{t\to+\infty} g_2^{(4)}(t) = -\infty, \tag{2.20}$$

$$\begin{aligned} g_3(t) &= -(p+2)(p+3)t^2 + (p-2)(p-3) \\ &\le -(p+2)(p+3) + (p-2)(p-3) \\ &= -10p < 0 \end{aligned} \tag{2.21}$$

for $t \in [1,\infty)$.

From the inequality (2.21) we clearly see that $g_2^{(4)}(t)$ is strictly decreasing in $[1,\infty)$. Then (2.19) and (2.20) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g_2^{(4)}(t) > 0$ for $t \in [1,\lambda_1)$ and $g_2^{(4)}(t) < 0$ for $t \in (\lambda_1,\infty)$. Hence, $g_2'''(t)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,\infty)$.

It follows from (2.16) and (2.17) together with the monotonicity of $g_2'''(t)$ that there exists $\lambda_2 > 1$ such that $g_2'''(t) > 0$ for $t \in [1,\lambda_2)$ and $g_2'''(t) < 0$ for $t \in (\lambda_2,\infty)$. Therefore, $g_2''(t)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,\infty)$.

From (2.13) and (2.14) together with the monotonicity of $g_2''(t)$ we know that there exists $\lambda_3 > 1$ such that $g_2''(t) > 0$ for $t \in [1,\lambda_3)$ and $g_2''(t) < 0$ for $t \in (\lambda_3,\infty)$. So, $g_2'(t)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,\infty)$.

Equations (2.10) and (2.11) together with the monotonicity of $g_2'(t)$ imply that there exists $\lambda_4 > 1$ such that $g_2'(t) > 0$ for $t \in (1,\lambda_4)$ and $g_2'(t) < 0$ for $t \in (\lambda_4,\infty)$. Hence, $g_2(t)$ is strictly increasing in $[1,\lambda_4]$ and strictly decreasing in $[\lambda_4,\infty)$.

It follows from (2.7) and (2.8) together with the monotonicity of $g_2(t)$ that there exists $\lambda_5 > 1$ such that $g_2(t) > 0$ for $t \in (1,\lambda_5)$ and $g_2(t) < 0$ for $t \in (\lambda_5,\infty)$. Therefore, $g_1(t)$ is strictly increasing in $[1,\lambda_5]$ and strictly decreasing in $[\lambda_5,\infty)$.

From (2.4) and (2.5) together with the monotonicity of $g_1(t)$ we clearly see that there exists $\lambda_6 > 1$ such that $g_1(t) > 0$ for $t \in (1,\lambda_6)$ and $g_1(t) < 0$ for $t \in (\lambda_6,\infty)$. So, $g(t)$ is strictly increasing in $[1,\lambda_6]$ and strictly decreasing in $[\lambda_6,\infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of $g(t)$. □

3. Main Result

Theorem 3.1. *The double inequality*

$$J_{2/(\pi-2)}(a,b) < T(a,b) < J_2(a,b) \tag{3.1}$$

holds for all $a,b>0$ with $a\neq b$, and $J_{2/(\pi-2)}(a,b)$ and $J_2(a,b)$ are the best possible lower and upper one-parameter mean bounds for the Seiffert mean $T(a,b)$, respectively.

Proof. Without loss of generality, we assume that $a>b$. Let $t=a/b>1$. Then from (1.1) and (1.2) we have

$$J_2(a,b)-T(a,b)=\frac{b(t^2+t+1)}{6(t+1)\arctan((t-1)/(t+1))}\left[4\arctan\frac{t-1}{t+1}-\frac{3(t^2-1)}{t^2+t+1}\right]. \tag{3.2}$$

Let

$$f(t)=4\arctan\frac{t-1}{t+1}-\frac{3(t^2-1)}{t^2+t+1}. \tag{3.3}$$

Then simple computations lead to

$$\begin{gathered} f(1)=0, \\ f'(t)=\frac{(t-1)^4}{(t^2+1)(t^2+t+1)^2}>0, \end{gathered} \tag{3.4}$$

for $t>1$.

Therefore, $T(a,b)<J_2(a,b)$ for all $a,b>0$ with $a\neq b$ follows from (3.2)–(3.4).

Next, we prove that

$$T(a,b)>J_{2/(\pi-2)}(a,b) \tag{3.5}$$

for all $a,b>0$ with $a\neq b$.

Let $p=2/(\pi-2)=1.75\cdots$. Then (1.1) and (1.2) lead to

$$\begin{aligned} &T(a,b)-J_p(a,b) \\ &\quad=\frac{bp(t^{p+1}-1)}{2(p+1)(t^p-1)\arctan((t-1)/(t+1))}\left[\frac{(p+1)(t-1)(t^p-1)}{p(t^{p+1}-1)}-2\arctan\frac{t-1}{t+1}\right]. \end{aligned} \tag{3.6}$$

Let

$$G(t) = \frac{(p+1)(t-1)(t^p-1)}{p(t^{p+1}-1)} - 2\arctan\frac{t-1}{t+1}. \tag{3.7}$$

Then simple computations lead to

$$\lim_{t\to 1} G(t) = \lim_{t\to+\infty} G(t) = 0, \tag{3.8}$$

$$G'(t) = \frac{g(t)}{p(t^{p+1}-1)^2(t^2+1)}, \tag{3.9}$$

where $g(t)$ is defined as in Lemma 2.1.

From Lemma 2.1 and (3.9) we know that there exists $\lambda > 1$ such that $G(t)$ is strictly increasing in $[1,\lambda]$ and strictly decreasing in $[\lambda,\infty)$. Then (3.8) leads to that

$$G(t) > 0, \tag{3.10}$$

for $t > 1$.

Therefore, the inequality (3.5) follows from (3.6), (3.7), and (3.10).

Finally, we prove that $J_{2/(\pi-2)}(a,b)$ and $J_2(a,b)$ are the best possible lower and upper one-parameter mean bounds for the Seiffert mean $T(a,b)$, respectively.

Let $p = 2/(\pi-2)$, $0 < \varepsilon < 2$ and $x > 0$. Then from (1.1) and (1.2) one has

$$\lim_{x\to+\infty}\frac{J_{p+\varepsilon}(x,1)}{T(x,1)} = \frac{p+\varepsilon}{p+\varepsilon+1}\times\frac{\pi}{2} > \frac{p}{p+1}\times\frac{\pi}{2} = 1, \tag{3.11}$$

$$T(1+x,1) - J_{2-\varepsilon}(1+x,1) = \frac{h(x)}{2(3-\varepsilon)\left[(1+x)^{2-\varepsilon}-1\right]\arctan(x/(2+x))}, \tag{3.12}$$

where

$$h(x) = (3-\varepsilon)x\left[(1+x)^{2-\varepsilon}-1\right] - 2(2-\varepsilon)\left[(1+x)^{3-\varepsilon}-1\right]\arctan\frac{x}{2+x}. \tag{3.13}$$

Letting $x \to 0$ and making use of Taylor expansion we get

$$\begin{aligned} h(x) = {} & (3-\varepsilon)x\left[(2-\varepsilon)x + \frac{(2-\varepsilon)(1-\varepsilon)}{2}x^2 - \frac{\varepsilon(1-\varepsilon)(2-\varepsilon)}{6}x^3 + o\left(x^3\right)\right] \\ & - 2(2-\varepsilon)\left[(3-\varepsilon)x + \frac{(3-\varepsilon)(2-\varepsilon)}{2}x^2 + \frac{(1-\varepsilon)(2-\varepsilon)(3-\varepsilon)}{6}x^3 + o\left(x^3\right)\right] \\ & \times\left[\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} + o\left(x^3\right)\right] = \frac{1}{12}\varepsilon(2-\varepsilon)(3-\varepsilon)x^4 + o\left(x^4\right). \end{aligned} \tag{3.14}$$

The inequality (3.11) implies that for any $0 < \varepsilon < 2$, there exists $X = X(\varepsilon) > 1$, such that $T(x,1) < J_{2/(\pi-2)+\varepsilon}(x,1)$ for $x \in (X, +\infty)$.

Equations (3.12)–(3.14) imply that for any $0 < \varepsilon < 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $T(1+x,1) > J_{2-\varepsilon}(1+x,1)$ for $x \in (0,\delta)$. □

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