

Research Article

Strong Convergence Theorems for Modifying Halpern Iterations for Quasi- ϕ -Asymptotically Nonexpansive Multivalued Mapping in Banach Spaces with Applications

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An iterative sequence for quasi- ϕ -asymptotically nonexpansive multivalued mapping for modifying Halpern's iterations is introduced. Under suitable conditions, some strong convergence theorems are proved. The results presented in the paper improve and extend the corresponding results in the work by Chang et al. 2011.

1. Introduction

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space X . A mapping $T : D \rightarrow D$ is said to be *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D , respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max \left\{ \sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1) \right\} \quad (1.1)$$

for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x - y\|, y \in A_1\}$. The multivalued mapping $T : D \rightarrow CB(D)$ is called *nonexpansive*, if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \rightarrow N(D)$, if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

Let X be a real Banach space with dual X^* . We denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad x \in X, \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space X is said to be *strictly convex*, if $\|(x+y)/2\| \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space is said to be *uniformly convex*, if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \|(x_n + y_n)/2\| = 0$.

The norm of Banach space X is said to be *Gâteaux differentiable*, if for each $x, y \in S(x)$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (1.3)$$

exists, where $S(x) = \{x : \|x\| = 1, x \in X\}$. In this case, X is said to be *smooth*. The norm of Banach space X is said to be *Fréchet differentiable*, if for each $x \in S(x)$, the limit (1.3) is attained *uniformly*, for $y \in S(x)$, and the norm is uniformly Fréchet differentiable if the limit (1.3) is attained uniformly for $x, y \in S(x)$. In this case, X is said to be *uniformly smooth*.

Remark 1.1. The following basic properties for Banach space X and for the normalized duality mapping J can be found in Cioranescu [1].

- (1) X (X^* , resp.) is uniformly convex if and only if X^* (X , resp.) is uniformly smooth.
- (2) If X is smooth, then J is single-valued and norm-to-weak* continuous.
- (3) If X is reflexive, then J is onto.
- (4) If X is strictly convex, then $Jx \cap Jy \neq \emptyset$, for all $x, y \in X$.
- (5) If X has a Fréchet differentiable norm, then J is norm-to-norm continuous.
- (6) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X .
- (7) Each uniformly convex Banach space X has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset X$, if $x_n \rightarrow x \in X$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x \in X$.
- (8) If X is a reflexive and strictly convex Banach space with a strictly convex dual X^* and $J^* : X^* \rightarrow X$ is the normalized duality mapping in X^* , then $J^{-1} = J^*$, $JJ^* = I_{X^*}$ and $J^*J = I_X$.

Next, we assume that X is a smooth, strictly convex, and reflexive Banach space and D is a nonempty, closed and convex subset of X . In the sequel, we always use $\phi : X \times X \rightarrow R^+$ to denote the *Lyapunov functional* defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X. \quad (1.4)$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad (1.5)$$

$$\phi(y, x) = \phi(y, z) + \phi(z, x) + 2\langle z - y, Jx - Jz \rangle, \quad x, y, z \in X,$$

$$\phi\left(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)\right) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \quad (1.6)$$

for all $\lambda \in [0, 1]$ and $x, y, z \in X$.

Following Alber [2], the generalized projection $\Pi_D : X \rightarrow D$ is defined by

$$\Pi_D(x) = \arg \inf_{y \in D} \phi(y, x), \quad \forall x \in X. \quad (1.7)$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

Example 1.2 (see [3]). Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty closed convex subset D of X , then Π_D is a closed and quasi- ϕ -nonexpansive from X onto D .

In 1953, Mann [4] introduced the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad (1.8)$$

where the initial guess $x_1 \in D$ is arbitrary, and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T . However, even in a Hilbert space, Mann iteration may fail to converge strongly [5]. Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [6] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad (1.9)$$

where $u, x_1 \in D$ are arbitrary given and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Another approach was proposed by Nakajo and Takahashi [7]. They generated a sequence as follows:

$$\begin{aligned} x_1 &\in X \text{ is arbitrary,} \\ y_n &= \alpha_n u + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in D : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in D : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_1 \quad (n = 1, 2, \dots), \end{aligned} \quad (1.10)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ and P_K denotes the metric projection from a Hilbert space H onto a closed and convex subset K of H . It should be noted here that the iteration

previous works only in Hilbert space setting. To extend this iteration to a Banach space, the concept of relatively nonexpansive mappings are introduced (see [8–12]).

Inspired by Matsushita and Takahashi, in this paper, we introduce modifying Halpern-Mann iterations sequence for finding a fixed point of multivalued mapping $T : D \rightarrow \text{CB}(D)$.

2. Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.1 (see [2]). *Let X be a smooth, strictly convex, and reflexive Banach space, and let D be a nonempty closed and convex subset of X . Then the following conclusions hold*

- (a) $\phi(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;
- (b) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \leq \phi(x, y)$, for all $x \in D$, for all $y \in X$;
- (c) if $x \in X$ and $z \in D$, then $z = \Pi_D x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0$, for all $y \in D$.

Remark 2.2. If H is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and Π_D is the metric projection P_D of H onto D .

Definition 2.3. A point $p \in D$ is said to be an asymptotic fixed point of $T : D \rightarrow \text{CB}(D)$, if there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightarrow x \in X$ and $d(x_n, T(x_n)) \rightarrow 0$. Denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$.

Definition 2.4. (1) A multivalued mapping $T : D \rightarrow \text{CB}(D)$ is said to be relatively nonexpansive, if $F(T) \neq \emptyset$, $\widehat{F}(T) = F(T)$, and $\phi(p, z) \leq \phi(p, x)$, for all $x \in D$, $p \in F(T)$, $z \in T(x)$.

(2) A multivalued mapping $T : D \rightarrow \text{CB}(D)$ is said to be closed, if for any sequence $\{x_n\} \subset D$ with $x_n \rightarrow x \in D$ and $d(y, T(x_n)) \rightarrow 0$, then $d(y, T(x)) = 0$.

Next, we present an example of relatively nonexpansive multivalued mapping.

Example 2.5 (see [13]). Let X be a smooth, strictly convex, and reflexive Banach space, let D be a nonempty closed and convex subset of X , and let $f : D \times D \rightarrow R$ be a bifunction satisfying the conditions: (A1) $f(x, x) = 0$, for all $x \in D$; (A2) $f(x, y) + f(y, x) \leq 0$, for all $x, y \in D$; (A3) $\lim_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$, for each $x, y, z \in D$; (A4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous, for each $x \in D$. The “so-called” equilibrium problem for f is to find a $x^* \in D$ such that $f(x^*, y) \geq 0$, for all $y \in D$. The set of its solutions is denoted by $\text{EP}(f)$.

Let $r > 0$, $x \in X$ and define mapping $T_r : X \rightarrow D$ as follows:

$$T_r(x) = \left\{ x \in D, f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in X, \quad (2.1)$$

then (1) T_r is single-valued, and so $\{z\} = T_r(x)$; (2) T_r is a relatively nonexpansive mapping, therefore it is a closed and quasi- ϕ -nonexpansive mapping; (3) $F(T_r) = \text{EP}(f)$.

Definition 2.6. (1) A multivalued mapping $T : D \rightarrow \text{CB}(D)$ is said to be quasi- ϕ -nonexpansive, if $F(T) \neq \emptyset$, and $\phi(p, z) \leq \phi(p, x)$, for all $x \in D$, $p \in F(T)$, $z \in T(x)$.

(2) A multivalued mapping $T : D \rightarrow \text{CB}(D)$ is said to be quasi- ϕ -asymptotically nonexpansive, if $F(T) \neq \emptyset$, and there exists a real sequence $k_n \subset [1, +\infty)$, $k_n \rightarrow 1$ such that

$$\phi(p, z_n) \leq k_n \phi(p, x), \quad \forall x \in D, p \in F(T), z_n \in T^n(x). \quad (2.2)$$

(3) A multivalued mapping $T : D \rightarrow \text{CB}(D)$ is said to be totally quasi- ϕ -asymptotically nonexpansive, if $F(T) \neq \emptyset$, and there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}$ with $v_n, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\zeta : R^+ \rightarrow R^+$ with $\zeta(0) = 0$ such that

$$\phi(p, z_n) \leq \phi(p, x) + v_n \zeta[\phi(p, x)] + \mu_n, \quad \forall x \in D, \forall n \geq 1, p \in F(T), z_n \in T^n(x). \quad (2.3)$$

Remark 2.7. From the definitions, it is obvious that a relatively nonexpansive multivalued mapping is a quasi- ϕ -nonexpansive multivalued mapping, and a quasi- ϕ -nonexpansive multivalued mapping is a quasi- ϕ -asymptotically nonexpansive multivalued mapping, but the converse is not true.

Lemma 2.8. *Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and let D be a nonempty closed and convex subset of X . Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D such that $x_n \rightarrow p$ and $\phi(x_n, y_n) \rightarrow 0$, where ϕ is the function defined by (1.4), then $y_n \rightarrow p$.*

Proof. For $\phi(x_n, y_n) \rightarrow 0$, we have $(\|x_n\| - \|y_n\|)^2 \rightarrow 0$. This implies that $\|y_n\| \rightarrow \|p\|$ and so $\|Jy_n\| \rightarrow \|Jp\|$. Since D is uniformly smooth, X^* is reflexive and $JX = X^*$, therefore, there exist a subsequence $\{Jy_{n_i}\} \subset \{Jy_n\}$ and a point $x \in X$ such that $Jy_{n_i} \rightarrow Jx$. Because the norm $\|\cdot\|$ is weakly lower semi continuous, we have

$$\begin{aligned} 0 &= \lim_{n_i \rightarrow \infty} \phi(x_{n_i}, y_{n_i}) = \lim_{n_i \rightarrow \infty} \left\{ \|x_{n_i}\|^2 - 2\langle x_{n_i}, Jy_{n_i} \rangle + \|Jy_{n_i}\|^2 \right\} \\ &\geq \|p\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 = \phi(p, x). \end{aligned} \quad (2.4)$$

By Lemma 2.1(a), we have $p = x$. Hence we have $Jy_{n_i} \rightarrow Jp$. Since $\|Jy_{n_i}\| \rightarrow \|Jp\|$ and X^* has the Kadec-Klee property, we have $Jy_{n_i} \rightarrow Jp$. By Remark 1.1, it follows that $y_{n_i} \rightarrow p$. Since $\|Jy_{n_i}\| \rightarrow \|Jp\|$, by using the Kadec-Klee property of X , we get $y_{n_i} \rightarrow p$. If there exists another subsequence $\{Jy_{n_j}\} \subset \{Jy_n\}$ such that $y_{n_j} \rightarrow q$, then we have

$$\begin{aligned} 0 &= \lim_{n_j \rightarrow \infty} \phi(x_{n_j}, y_{n_j}) = \lim_{n_j \rightarrow \infty} \left\{ \|x_{n_j}\|^2 - 2\langle x_{n_j}, Jy_{n_j} \rangle + \|Jy_{n_j}\|^2 \right\} \\ &= \|p\|^2 - 2\langle p, Jq \rangle + \|q\|^2 = \phi(p, q). \end{aligned} \quad (2.5)$$

This implies that $p = q$. So $y_n \rightarrow p$. The conclusion of Lemma 2.8 is proved. \square

Lemma 2.9. *Let X and D be as in Lemma 2.8. Let $T : D \rightarrow \text{CB}(D)$ be a closed and quasi- ϕ -asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{k_n\} \subset [1, +\infty)$, if $k_n \rightarrow 1$, then the fixed point set $F(T)$ of T is a closed and convex subset of D .*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$, such that $x_n \rightarrow x^*$. Since T is quasi- ϕ -asymptotically nonexpansive multivalued mapping, we have

$$\phi(x_n, z) \leq k_1 \phi(x_n, x^*) \quad (2.6)$$

for all $z \in Tx^*$ and for all $n \in N$. Therefore,

$$\phi(x^*, z) = \lim_{n \rightarrow \infty} \phi(x_n, z) \leq \lim_{n \rightarrow \infty} k_1 \phi(x_n, x^*) = k_1 \phi(x^*, x^*) = 0. \quad (2.7)$$

By Lemma 2.1, we obtain $z = x^*$, Hence, $Tx^* = \{x^*\}$. So, we have $x^* \in F(T)$. This implies that $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0, 1)$, and put $w = tp + (1-t)q$. we prove that $w \in F(T)$. Indeed, in view of the definition of ϕ , let $z_n \in T^n w$, we have

$$\begin{aligned} \phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n \rangle + \|z_n\|^2 \\ &= \|w\|^2 - 2\langle tp + (1-t)q, Jz_n \rangle + \|z_n\|^2 \\ &= \|w\|^2 + t\phi(p, z_n) + (1-t)\phi(q, z_n) - t\|p\|^2 - (1-t)\|q\|^2. \end{aligned} \quad (2.8)$$

Since

$$\begin{aligned} &t\phi(p, z_n) + (1-t)\phi(q, z_n) \\ &\leq tk_n\phi(p, w) + (1-t)k_n\phi(q, w) \\ &= t\left\{\|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 + (k_n - 1)\phi(p, w)\right\} \\ &\quad + (1-t)\left\{\|q\|^2 - 2\langle q, Jw \rangle + \|w\|^2 + (k_n - 1)\phi(q, w)\right\} \\ &= t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w). \end{aligned} \quad (2.9)$$

Substituting (2.8) into (2.9) and simplifying it, we have

$$\phi(w, z_n) \leq t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w) \rightarrow 0, \quad (\text{as } n \rightarrow \infty). \quad (2.10)$$

Hence, we have $z_n \rightarrow w$. This implies that $z_{n+1} \in TT^n w \rightarrow w$. Since T is closed, we have $Tw = \{w\}$, that is, $w \in F(T)$. This completes the proof of Lemma 2.9. \square

Definition 2.10. A mapping $T : D \rightarrow CB(D)$ is said to be uniformly L -Lipschitz continuous, if there exists a constant $L > 0$ such that $\|x_n - y_n\| \leq L\|x - y\|$, where $x, y \in D$, $x_n \in T^n x$, $y_n \in T^n y$.

3. Main Results

Theorem 3.1. Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, let D be a nonempty, closed and convex subset of X , and let $T : D \rightarrow CB(D)$ be a closed and

uniformly L -Lipschitz continuous quasi- ϕ -asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{k_n\} \subset [1, +\infty)$ and $k_n \rightarrow 1$ satisfying condition (2.2). Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. If $\{x_n\}$ is the sequence generated by

$$\begin{aligned} x_1 \in X \quad \text{is arbitrary}; \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)Jz_n], \quad z_n \in T^n x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \dots), \end{aligned} \quad (3.1)$$

where $\xi_n = (k_n - 1)\sup_{p \in F(T)} \phi(p, x_n)$, $F(T)$ is the fixed point set of T , and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1} . If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Proof. (I) First, we prove that D_n are closed and convex subsets in D . By the assumption that $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \geq 1$. In view of the definition of ϕ , we have

$$\begin{aligned} D_{n+1} &= \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\} \\ &= \{z \in D : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\} \cap D_n \\ &= \left\{ z \in D : 2\alpha_n \langle z, Jx_1 \rangle + 2(1 - \alpha_n) \langle z, Jx_n \rangle - 2 \langle z, Jz_n \rangle \right. \\ &\quad \left. \leq \alpha_n \|x_1\|^2 + (1 - \alpha_n) \|x_n\|^2 - \|z_n\|^2 \right\} \cap D_n. \end{aligned} \quad (3.2)$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $F(T) \subset D_n$, for all $n \geq 1$. In fact, it is obvious that $F(T) \subset D_1$. Suppose $F(T) \subset D_n$, for some $n \geq 1$. Hence, for any $u \in F(T) \subset D_n$, by (1.6), we have

$$\begin{aligned} \phi(u, y_n) &= \phi\left(u, J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jz_n)\right) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, z_n) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n)k_n \phi(u, x_n) \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n)\{\phi(u, x_n) + (k_n - 1)\phi(u, x_n)\} \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \left\{ \phi(u, x_n) + (k_n - 1) \sup_{u \in F(T)} \phi(u, x_n) \right\} \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, x_n) + \xi_n. \end{aligned} \quad (3.3)$$

This shows that $u \in F(T) \subset D_{n+1}$ and so $F(T) \subset D_n$.

(III) Now, we prove that $\{x_n\}$ converges strongly to some point p^* . In fact, since $x_n = \Pi_{D_n} x_1$, from Lemma 2.1(c), we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in D_n. \quad (3.4)$$

Again since $F(T) \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in F(T). \quad (3.5)$$

It follows from Lemma 2.1(b) that for each $u \in F(T)$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{D_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \quad (3.6)$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n} x_1$ and $x_{n+1} = \Pi_{D_{n+1}} x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists. Since X is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in $D = D_1$). Since D_n is closed and convex and $D_{n+1} \subset D_n$. This implies that D_n is weakly closed and $p^* \in D_n$ for each $n \geq 1$. In view of $x_{n_i} = \Pi_{D_{n_i}} x_1$, we have

$$\phi(x_{n_i}, x_1) \leq \phi(p^*, x_1), \quad \forall n_i \geq 1. \quad (3.7)$$

Since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\begin{aligned} \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_1) &= \liminf_{n_i \rightarrow \infty} \left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1 \rangle + \|x_1\|^2 \right) \\ &\geq \|p^*\|^2 - 2\langle p^*, Jx_1 \rangle + \|x_1\|^2 \\ &= \phi(p^*, x_1), \end{aligned} \quad (3.8)$$

and so

$$\phi(p^*, x_1) \leq \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_1) \leq \limsup_{n_i \rightarrow \infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1). \quad (3.9)$$

This shows that $\lim_{n_i \rightarrow \infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$, and we have $\|x_{n_i}\| \rightarrow \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, by the virtue of Kadec-Klee property of X , we obtain that $x_{n_i} \rightarrow p^*$. Since $\{\phi(x_n, x_1)\}$ is convergent, this together with $\lim_{n_i \rightarrow \infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$ shows that $\lim_{n \rightarrow \infty} \phi(x_n, x_1) = \phi(p^*, x_1)$. If there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightarrow q$, then from Lemma 2.1, we have

$$\begin{aligned} \phi(p^*, q) &= \lim_{n_i, n_j \rightarrow \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{n_i, n_j \rightarrow \infty} \phi(x_{n_i}, \Pi_{D_{n_j}} x_1) \\ &\leq \lim_{n_i, n_j \rightarrow \infty} \left[\phi(x_{n_i}, x_1) - \phi(\Pi_{D_{n_j}} x_1, x_1) \right] = \lim_{n_i, n_j \rightarrow \infty} \left[\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1) \right] \\ &= \phi(p^*, x_1) - \phi(p^*, x_1) = 0, \end{aligned} \quad (3.10)$$

that is, $p^* = q$ and hence

$$x_n \rightarrow p^*. \quad (3.11)$$

By the way, from (3.11), it is easy to see that

$$\xi_n = (k_n - 1) \sup_{p \in F(T)} \phi(p, x_n) \longrightarrow 0. \quad (3.12)$$

(IV) Now, we prove that $p^* \in F(T)$. In fact, since $x_{n+1} \in D_{n+1}$, from (3.1), (3.11), and (3.12), we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \xi_n \longrightarrow 0. \quad (3.13)$$

Since $x_n \rightarrow p^*$, it follows from (3.13) and Lemma 2.8 that

$$y_n \longrightarrow p^*. \quad (3.14)$$

Since $\{x_n\}$ is bounded and T is quasi- ϕ -asymptotically nonexpansive multivalued mapping, $T^n x_n$ is bounded. In view of $\alpha_n \rightarrow 0$. Hence from (3.1), we have that

$$\lim_{n \rightarrow \infty} \|Jy_n - Jz_n\| = \lim_{n \rightarrow \infty} \|Jx_1 - Jz_n\| = 0. \quad (3.15)$$

Since $Jy_n \rightarrow Jp^*$, this implies $Jz_n \rightarrow Jp^*$. From Remark 1.1, it yields that

$$z_n \rightarrow p^*. \quad (3.16)$$

Again since

$$\|z_n\| - \|p^*\| = \|Jz_n\| - \|Jp^*\| \leq \|Jz_n - Jp^*\| \longrightarrow 0, \quad (3.17)$$

this together with (3.16) and the Kadec-Klee-property of X shows that

$$z_n \longrightarrow p^*. \quad (3.18)$$

On the other hand, by the assumptions that T is L -Lipschitz continuous, thus we have

$$\begin{aligned} d(Tz_n, z_n) &\leq d(Tz_n, z_{n+1}) + \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\leq (L + 1)\|x_{n+1} - x_n\| + \|z_{n+1} - x_{n+1}\| + \|x_n - z_n\|. \end{aligned} \quad (3.19)$$

From (3.18) and $x_n \rightarrow p^*$, we have that $d(Tz_n, z_n) \rightarrow 0$. In view of the closeness of T , it yields that $T(p^*) = \{p^*\}$, this implies that $p^* \in F(T)$.

(V) Finally, we prove that $p^* = \Pi_{F(T)} x_1$ and so $x_n \rightarrow \Pi_{F(T)} x_1$. Let $w = \Pi_{F(T)} x_1$. Since $w \in F(T) \subset D_n$, we have $\phi(p^*, x_1) \leq \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(w, x_1), \quad (3.20)$$

which yields that $p^* = w = \Pi_{F(T)}x_1$. Therefore, $x_n \rightarrow \Pi_{F(T)}x_1$. This completes the proof of Theorem 3.1. \square

By Remark 2.7, the following corollaries are obtained.

Corollary 3.2. *Let X and D be as in Theorem 3.1, and let $T : D \rightarrow CB(D)$ be a closed and uniformly L -Lipschitz continuous a relatively nonexpansive multivalued mapping. Let $\{\alpha_n\}$ in $(0, 1)$ with $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_1 \in X \quad \text{is arbitrary}; \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)Jz_n], \quad z_n \in Tx_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \dots), \end{aligned} \quad (3.21)$$

where $F(T)$ is the set of fixed points of T , and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1} , then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Corollary 3.3. *Let X and D be as in Theorem 3.1, and let $T : D \rightarrow CB(D)$ be a closed and uniformly L -Lipschitz continuous quasi- ϕ -nonexpansive multivalued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$, and satisfying: $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be the sequence generated by (3.21). Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.*

4. Application

We utilize Corollary 3.3 to study a modified Halpern iterative algorithm for a system of equilibrium problems.

Theorem 4.1. *Let D , X , and $\{\alpha_n\}$ be the same as in Theorem 3.1. Let $f : D \times D \rightarrow \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4) as given in Example 2.5. Let $T_r : X \rightarrow D$ be a mapping defined by (2.1), that is,*

$$T_r(x) = \left\{ x \in D, f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in X. \quad (4.1)$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_1 \in X \quad \text{is arbitrary}; \quad D_1 = D, \\ f(u_n, y) + \frac{1}{r} \langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \quad \forall y \in D, \quad r > 0, \quad u_n \in T_r x_n, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)Ju_n], \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \dots). \end{aligned} \quad (4.2)$$

If $F(T_r) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$ which is a common solution of the system of equilibrium problems for f .

Proof. In Example 2.5, we have pointed out that $u_n = T_r(x_n)$, $F(T_r) = \text{EP}(f)$, and T_r is a closed and quasi- ϕ -nonexpansive mapping. Hence (4.2) can be rewritten as follows:

$$\begin{aligned} x_1 \in X \quad \text{is arbitrary}; \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)Ju_n], \quad u_n \in T_r x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \dots). \end{aligned} \tag{4.3}$$

Therefore the conclusion of Theorem 4.1 can be obtained from Corollary 3.3. \square

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