

Research Article

Further Research on the M/G/1 Retrial Queueing Model with Server Breakdowns

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We study spectral properties of the operator which corresponds to the M/G/1 retrial queueing model with server breakdowns and obtain that all points on the imaginary axis except zero belong to the resolvent set of the operator and 0 is not an eigenvalue of the operator. Our results show that the time-dependent solution of the model is probably strongly asymptotically stable.

1. Introduction

There has been considerable interest in retrial queueing systems, see Atencia et al. [1, 2], Choi et al. [3], Djellab [4], Gupur [5, 6], Kasim and Gupur [7], Li et al. [8], Li and Wang [9], Wang et al. [10, 11], and Yang and Templeton [12]. Many researchers studied the M/G/1 retrial queueing systems with server breakdowns in the steady-state case, see Atencia et al. [1, 2], Choi et al. [3], Djellab [4], Li and Wang [9], and Wang et al. [10]. Only few researchers studied transient solutions of M/G/1 retrial queueing systems, see Wang et al. [11], Gupur [5, 6], Kasim and Gupur [7]. And Wang et al. [11] studied the transient solution of the M/G/1 retrial queueing system with server failure by using Laplace transform and obtained the expression of the probability-generating function. In other words, they studied the existence of the time-dependent solution of the model. Gupur [5, 6, 13, 14], Gupur et al. [15], and Kasim and Gupur [7] did dynamic analysis for several queueing models including retrial queueing models by using functional analysis and obtained the existence and uniqueness of the time-dependent solution of several queueing models and asymptotic behavior of their time-dependent solutions. In this paper, by using the idea in Gupur [6] and Gupur et al. [15], we study the asymptotic behavior of the time-dependent solution of the M/G/1 retrial queueing system with server breakdowns in which the failure states of the server are absorbing states. This queueing system was studied by Wang et al. [10] in 2001. By using

the supplementary variable technique they established the corresponding queueing model and obtained explicit expressions of some reliability indices such as the availability, failure frequency for steady-state cases under the following hypothesis: “the time-dependent solution of the model converges to a steady-state solution.” By reading their paper we find that the above hypothesis, in fact, implies the following two hypotheses.

Hypothesis 1. The model has a nonnegative time-dependent solution.

Hypothesis 2. The time-dependent solution of the model converges to a steady-state solution.

In 2010, Gupur [6] studied the above two hypotheses. Firstly, he converted the model into an abstract Cauchy problem by selecting a suitable Banach space and defining the underlying operator which corresponds to the model and its domain. Next, by using the Hille-Yosida theorem and the Phillips theorem he proved that the model has a unique non-negative time-dependent solution and therefore obtained that Hypothesis 1 holds. Then, when the service completion rate is a constant, he studied the asymptotic behavior of its time-dependent solution. By studying resolvent set of of the adjoint operator of the operator which corresponds to the model (in this case, the M/G/1 retrial queueing model with server breakdowns is called the M/M/1 retrial queueing model with server breakdowns) he obtained the resolvent set of the operator on the imaginary axis: all points on the imaginary axis except zero belong to its resolvent set. And he proved that 0 is not an eigenvalue of the operator. Thus, he suggested that the time-dependent solution of the model probably strongly converges to zero. Until now, any other results about this model have not been found in the literature. In this paper, we try to study the asymptotic behavior of the time-dependent solution of the above model when the service completion rate is a function. Firstly, we convert the model into an abstract Cauchy problem by Gupur [6] and we determine the resolvent set of the adjoint operator of the operator corresponding to the model when the service completion rate satisfies a certain condition. Also, we prove that 0 is not an eigenvalue of the operator. If we can verify 0 is not in the residue spectrum of the operator, then from the above results we conclude that the time-dependent solution of the model strongly converges to zero. Naturally, the results obtained in [6] are now special cases of our results.

According to Wang et al. [10], the M/G/1 retrial queueing system with server breakdowns can be described by the following system of partial differential equations with integral boundary conditions:

$$\begin{aligned}
 \frac{dp_{I,0,0}(t)}{dt} &= -\lambda p_{I,0,0}(t) + \int_0^\infty \mu(x)p_{w,0,1}(x,t)dx, \\
 \frac{dp_{I,i,0}(t)}{dt} &= -(\lambda + i\theta)p_{I,i,0}(t) + \int_0^\infty \mu(x)p_{w,i,1}(x,t)dx, \quad i \geq 1, \\
 \frac{\partial p_{w,0,1}(x,t)}{\partial t} + \frac{\partial p_{w,0,1}(x,t)}{\partial x} &= -(\lambda + \alpha + \mu(x))p_{w,0,1}(x,t), \\
 \frac{\partial p_{w,i,1}(x,t)}{\partial t} + \frac{\partial p_{w,i,1}(x,t)}{\partial x} &= -(\lambda + \alpha + \mu(x))p_{w,i,1}(x,t) \\
 &\quad + \lambda p_{w,i-1,1}(x,t), \quad i \geq 1, \\
 p_{w,i,1}(0,t) &= \lambda p_{I,i,0}(t) + (i+1)\theta p_{I,i+1,0}(t), \quad i \geq 0, \\
 p_{I,0,0}(0) &= 1, \quad p_{I,i,0}(0) = 0, \quad i \geq 1; \quad p_{w,j,1}(x,0) = 0, \quad j \geq 0.
 \end{aligned}
 \tag{1.1}$$

Here $(x, t) \in [0, \infty) \times [0, \infty)$; $p_{I,i,0}(t)$ ($i \geq 0$) represents the probability that the server is idle and there are i customers in the retrial group at time t ; $p_{w,i,1}(x, t)$ represents the joint probability that at time t there are i customers in the retrial group and the server is up and customer is being served with elapsed service time x ; λ is the arrival rate of customers; α is the server failing rate; θ is the successive interretrial times of customers; $\mu(x)$ is the service completion rate at time x satisfying

$$\mu(x) \geq 0, \quad \int_0^\infty \mu(x) dx = \infty. \tag{1.2}$$

In this paper, we use the notations in Gupur [6]:

$$\Gamma = \begin{pmatrix} \lambda & \theta & 0 & 0 & 0 & \dots \\ 0 & \lambda & 2\theta & 0 & 0 & \dots \\ 0 & 0 & \lambda & 3\theta & 0 & \dots \\ 0 & 0 & 0 & \lambda & 4\theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.3}$$

Take a state space as follows:

$$X = \left\{ (p_I, p_w) \left| \begin{array}{l} p_I = (p_{I,0,0}, p_{I,1,0}, p_{I,2,0}, \dots) \in l^1 \\ p_w = (p_{w,0,1}, p_{w,1,1}, p_{w,2,1}, \dots) \\ \in L^1[0, \infty) \times L^1[0, \infty) \times L^1[0, \infty) \times \dots \\ \|(p_I, p_w)\| = \sum_{i=0}^\infty |p_{I,i,0}| + \sum_{i=0}^\infty \|p_{w,i,1}\|_{L^1[0, \infty)} < \infty \end{array} \right. \right\}. \tag{1.4}$$

It is obvious that X is a Banach space and also a Banach lattice. In the following we define operators and their domains:

$$A(p_I, p_w) = \begin{pmatrix} \begin{pmatrix} -\lambda & 0 & 0 & 0 & \dots \\ 0 & -(\lambda + \theta) & 0 & 0 & \dots \\ 0 & 0 & -(\lambda + 2\theta) & 0 & \dots \\ 0 & 0 & 0 & -(\lambda + 3\theta) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{I,0,0} \\ p_{I,1,0} \\ p_{I,2,0} \\ p_{I,3,0} \\ \vdots \end{pmatrix}, \\ \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & 0 & \dots \\ 0 & -\frac{d}{dx} & 0 & 0 & \dots \\ 0 & 0 & -\frac{d}{dx} & 0 & \dots \\ 0 & 0 & 0 & -\frac{d}{dx} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{w,0,1}(x) \\ p_{w,1,1}(x) \\ p_{w,2,1}(x) \\ p_{w,3,1}(x) \\ \vdots \end{pmatrix} \end{pmatrix},$$

$$D(A) = \left\{ (p_I, p_w) \in X \left| \begin{array}{l} p_{w,i,1}(x) (i \geq 0) \text{ are absolutely continuous} \\ \text{functions and satisfy } p_w(0) = \Gamma p_I, \\ \sum_{n=0}^{\infty} \left\| \frac{dp_{w,i,1}}{dx} \right\|_{L^1[0,\infty)} < \infty \end{array} \right. \right\},$$

$$U(p_I, p_w) = \left(\begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{I,0,0} \\ p_{I,1,0} \\ p_{I,2,0} \\ \vdots \end{pmatrix} \\ \begin{pmatrix} u & 0 & 0 & \cdots \\ \lambda & u & 0 & \cdots \\ 0 & \lambda & u & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{w,0,1}(x) \\ p_{w,1,1}(x) \\ p_{w,2,1}(x) \\ \vdots \end{pmatrix} \end{array} \right), \quad D(U) = X,$$
(1.5)

where $u = -(\lambda + \alpha + \mu(x))$.

$$E(p_I, p_w) = \left(\begin{array}{c} \begin{pmatrix} \int_0^{\infty} \mu(x) p_{w,0,1}(x) dx \\ \int_0^{\infty} \mu(x) p_{w,1,1}(x) dx \\ \int_0^{\infty} \mu(x) p_{w,2,1}(x) dx \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \end{array} \right), \quad D(E) = X.$$
(1.6)

Then the above system of (1.1) can be expressed as an abstract Cauchy problem:

$$\frac{d(p_I, p_w)(t)}{dt} = (A + U + E)(p_I, p_w)(t), \quad \forall t \in (0, \infty),$$

$$(p_I, p_w)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix} \right).$$
(1.7)

Gupur [6] have obtained the following results.

Theorem 1.1. *If $\bar{\mu} = \sup_{x \in [0, \infty)} \mu(x) < \infty$, then $A + U + E$ generates a positive contraction C_0 -semi group $T(t)$. And the system (1.7) has a unique nonnegative time-dependent solution $(p_I, p_w)(x, t) = T(t)(p_I, p_w)(0)$ satisfying $\|(p_I, p_w)(\cdot, t)\| \leq 1, \forall t \in [0, \infty)$.*

2. Asymptotic Behavior of the Time-Dependent Solution of the System (1.7)

In this section, firstly we determine the expression of $(A + U + E)^*$, the adjoint operator of $A + U + E$, next we study the resolvent set of $(A + U + E)^*$, through which we deduce the resolvent set of $A + U + E$ on the imaginary axis. Thirdly, we prove that 0 is not an eigenvalue of $A + U + E$. Thus, we state our main results in this paper.

It is easy to see that X^* , the dual space of X , is as follows (see Gupur [6]):

$$X^* = \left\{ (q_I^*, q_w^*) \left| \begin{array}{l} q_I^* = (q_{I,0,0}^*, q_{I,1,0}^*, q_{I,2,0}^*, \dots) \in l^\infty, \\ q_w^* = (q_{w,0,1}^*, q_{w,1,1}^*, q_{w,2,1}^*, \dots) \\ \in L^\infty[0, \infty) \times L^\infty[0, \infty) \times L^\infty[0, \infty) \times \dots \\ \|\|(q_I^*, Q_w^*)\|\| = \sup \left\{ \sup_{i \geq 0} |q_{I,i,0}^*|, \sup_{i \geq 0} \|q_{w,i,1}^*\|_{L^\infty[0, \infty)} \right\} < \infty \end{array} \right. \right\}. \quad (2.1)$$

It is not difficult to verify that X^* is a Banach space.

Lemma 2.1. $(A + U + E)^*$, the adjoint operator of $A + U + E$, is as follows:

$$(A + U + E)^*(q_I^*, q_w^*) = (\mathcal{L} + \mathcal{G} + \mathcal{H})(q_I^*, q_w^*), \quad \forall (q_I^*, q_w^*) \in D(\mathcal{L}), \quad (2.2)$$

where

$$\begin{aligned} \mathcal{L}(q_I^*, q_w^*) &= \left(\begin{array}{cccc} -\lambda & 0 & 0 & \dots \\ 0 & -(\lambda + \theta) & 0 & \dots \\ 0 & 0 & -(\lambda + 2\theta) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array} \begin{array}{c} q_{I,0,0}^* \\ q_{I,1,0}^* \\ q_{I,2,0}^* \\ \vdots \end{array} \right), \\ &\quad \left(\begin{array}{ccc} \frac{d}{dx} - (\lambda + \alpha + \mu(x)) & 0 & \dots \\ 0 & \frac{d}{dx} - (\lambda + \alpha + \mu(x)) & \dots \\ \vdots & \vdots & \ddots \end{array} \begin{array}{c} q_{w,0,1}^*(x) \\ q_{w,1,1}^*(x) \\ \vdots \end{array} \right), \\ \mathcal{G}(q_I^*, q_w^*) &= \left(\begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \mu(x) & 0 & 0 & \dots \\ 0 & \mu(x) & 0 & \dots \\ 0 & 0 & \mu(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{I,0,0}^* \\ q_{I,1,0}^* \\ q_{I,2,0}^* \\ \vdots \end{pmatrix} \right), \\ \mathcal{H}(q_I^*, q_w^*) &= \left(\begin{array}{cccc} \begin{pmatrix} \lambda & 0 & 0 & 0 & \dots \\ \theta & \lambda & 0 & 0 & \dots \\ 0 & 2\theta & \lambda & 0 & \dots \\ 0 & 0 & 3\theta & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{w,0,1}^*(0) \\ q_{w,1,1}^*(0) \\ q_{w,2,1}^*(0) \\ q_{w,3,1}^*(0) \\ \vdots \end{pmatrix}, \\ \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & \lambda & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda & 0 & \dots \\ 0 & 0 & 0 & 0 & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{w,0,1}^*(x) \\ q_{w,1,1}^*(x) \\ q_{w,2,1}^*(x) \\ q_{w,3,1}^*(x) \\ \vdots \end{pmatrix} \right), \end{array}$$

$$D(\mathcal{L}) = \left\{ (q_I^*, q_w^*) \in X^* \left| \begin{array}{l} q_{w,i,1}^*(x) \text{ are absolutely continuous} \\ \text{and satisfy } q_{w,i,1}^*(\infty) = \eta, \quad i \geq 0 \end{array} \right. \right\}$$

$$D(\mathcal{G}) = D(\mathcal{A}) = X^*. \tag{2.3}$$

Here, η in $D(\mathcal{L})$ is a nonzero constant which is irrelevant to i .

Proof. By using integration by parts and the boundary conditions on $(p_I, p_w) \in D(A)$, we have, for any $(q_I^*, q_w^*) \in D(\mathcal{L})$,

$$\begin{aligned} \langle (A + U + E)(p_I, p_w), (q_I^*, q_w^*) \rangle &= \sum_{i=0}^{\infty} \left[-(\lambda + i\theta)p_{I,i,0} + \int_0^{\infty} \mu(x)p_{w,i,1}(x)dx \right] q_{I,i,0}^* \\ &\quad + \int_0^{\infty} \left[-\frac{dp_{w,0,1}(x)}{dx} - (\lambda + \alpha + \mu(x))p_{w,0,1}(x) \right] q_{w,0,1}^*(x)dx \\ &\quad + \sum_{i=1}^{\infty} \int_0^{\infty} \left[-\frac{dp_{w,i,1}(x)}{dx} - (\lambda + \alpha + \mu(x))p_{w,i,1}(x) + \lambda p_{w,i-1,1}(x) \right] \\ &\quad \times q_{w,i,1}^*(x)dx \\ &= \sum_{i=0}^{\infty} -(\lambda + i\theta)p_{I,i,0}q_{I,i,0}^* + \sum_{i=0}^{\infty} \int_0^{\infty} \mu(x)p_{w,i,1}(x)q_{I,i,0}^*dx \\ &\quad + \sum_{i=0}^{\infty} \int_0^{\infty} -\frac{dp_{w,i,1}(x)}{dx} q_{w,i,1}^*(x)dx \\ &\quad - \sum_{i=0}^{\infty} \int_0^{\infty} (\lambda + \alpha + \mu(x))p_{w,i,1}(x)q_{w,i,1}^*(x)dx \\ &\quad + \sum_{i=1}^{\infty} \int_0^{\infty} \lambda p_{w,i-1,1}(x)q_{w,i,1}^*(x)dx \\ &= \sum_{i=0}^{\infty} -(\lambda + i\theta)p_{I,i,0}q_{I,i,0}^* + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x)\mu(x)q_{I,i,0}^*dx \\ &\quad + \sum_{i=0}^{\infty} \left[-p_{w,i,1}(x)q_{w,i,1}^*(x) \Big|_{x=0}^{x=\infty} + \int_0^{\infty} p_{w,i,1}(x) \frac{dq_{w,i,1}^*(x)}{dx} dx \right] \\ &\quad + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) [-(\lambda + \alpha + \mu(x))q_{w,i,1}^*(x)] dx \\ &\quad + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x)\lambda q_{w,i+1,1}^*(x)dx \\ &= \sum_{i=0}^{\infty} -(\lambda + i\theta)p_{I,i,0}q_{I,i,0}^* + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x)\mu(x)q_{I,i,0}^*dx \\ &\quad + \sum_{i=0}^{\infty} p_{w,i,1}(0)q_{w,i,1}^*(0) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \left[\frac{dq_{w,i,1}^*(x)}{dx} - (\lambda + \alpha + \mu(x))q_{w,i,1}^*(x) \right] dx \\
 & + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \lambda q_{w,i+1,1}^*(x) dx \\
 = & \sum_{i=0}^{\infty} -(\lambda + i\theta)p_{I,i,0}q_{I,i,0}^* + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \mu(x) q_{I,i,0}^* dx \\
 & + \sum_{i=0}^{\infty} [\lambda p_{I,i,0} + (i+1)\theta p_{I,i+1,0}] q_{w,i,1}^*(0) \\
 & + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \left[\frac{dq_{w,i,1}^*(x)}{dx} - (\lambda + \alpha + \mu(x))q_{w,i,1}^*(x) \right] dx \\
 & + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \lambda q_{w,i+1,1}^*(x) dx \\
 = & \sum_{i=0}^{\infty} -(\lambda + i\theta)p_{I,i,0}q_{I,i,0}^* \\
 & + \sum_{i=0}^{\infty} \int_0^{\infty} p_{w,i,1}(x) \left[\frac{dq_{w,i,1}^*(x)}{dx} - (\lambda + \alpha + \mu(x))q_{w,i,1}^*(x) \right. \\
 & \qquad \qquad \qquad \left. + \lambda q_{w,i+1,1}^*(x) + \mu(x)q_{I,i,0}^* \right] dx \\
 & + \sum_{i=0}^{\infty} p_{I,i,0} \lambda q_{w,i,1}^*(0) + \sum_{i=0}^{\infty} p_{I,i+1,0} (i+1) \theta q_{w,i,1}^*(0) \\
 = & \langle (p_I, p_w), (\mathcal{L} + \mathcal{G} + \mathcal{L})(q_I^*, q_w^*) \rangle.
 \end{aligned}
 \tag{2.4}$$

From the definition of the adjoint operator and (2.4) we know that the assertion of this lemma is true. □

Lemma 2.2. Assume that there exist two positive constant $\bar{\mu}, \underline{\mu}$ such that

$$0 < \underline{\mu} = \inf_{x \in [0, \infty)} \mu(x) \leq \bar{\mu} = \sup_{x \in [0, \infty)} \mu(x) < \infty. \tag{2.5}$$

If $\bar{\mu} < \alpha + \underline{\mu}$, then

$$\left\{ \gamma \in \mathbb{C} \left| \begin{aligned} & \sup \left\{ \frac{\lambda}{|\gamma + \lambda|}, \sup_{i \geq 1} \left\{ \frac{\lambda + i\theta}{|\gamma + \lambda + i\theta|} \right\}, \frac{\lambda \bar{\mu}}{|\gamma + \lambda| (\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\} \right. \\ & \left. \sup_{i \geq 1} \left\{ \frac{(\lambda + i\theta) \bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}) |\gamma + \lambda + i\theta|} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\} \right\} < 1, \\ & \operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu} > 0 \end{aligned} \right. \tag{2.6}$$

belongs to the resolvent set of $(A + U + E)^*$. In particular, all points on the imaginary axis except zero belong to the resolvent set of $(A + U + E)^*$, which implies that all points on the imaginary axis except zero belong to the resolvent set of $A + U + E$.

Proof. For any given $(y_I^*, y_w^*) \in X^*$, consider the equation $(\gamma I - \mathcal{L} - \mathcal{G})(q_I^*, q_w^*) = \mathcal{H}(y_I^*, y_w^*)$, that is,

$$(\gamma + \lambda)q_{I,0,0}^* = \lambda y_{w,0,1}^*(0), \quad (2.7)$$

$$(\gamma + \lambda + \theta)q_{I,1,0}^* = \theta y_{w,0,1}^*(0) + \lambda y_{w,1,1}^*(0), \quad (2.8)$$

$$(\gamma + \lambda + i\theta)q_{I,i,0}^* = i\theta y_{w,i-1,1}^*(0) + \lambda y_{w,i,1}^*(0), \quad i \geq 1, \quad (2.9)$$

$$\frac{dq_{w,i,1}^*(x)}{dx} = (\gamma + \lambda + \alpha + \mu(x))q_{w,i,1}^*(x) - \mu(x)q_{I,i,0}^* - \lambda y_{w,i+1,1}^*(x), \quad i \geq 0, \quad (2.10)$$

$$q_{w,i,1}^*(\infty) = \eta, \quad i \geq 0. \quad (2.11)$$

By solving (2.7)–(2.10), we have

$$q_{I,0,0}^* = \frac{\lambda}{\gamma + \lambda} y_{w,0,1}^*(0), \quad (2.12)$$

$$q_{I,i,0}^* = \frac{i\theta}{\gamma + \lambda + i\theta} y_{w,i-1,1}^*(0) + \frac{\lambda}{\gamma + \lambda + i\theta} y_{w,i,1}^*(0), \quad i \geq 1, \quad (2.13)$$

$$\begin{aligned} q_{w,i,1}^*(x) &= a_i e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \\ &\quad + e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \int_0^x \left[-\mu(\tau) q_{I,i,0}^* - \lambda y_{w,i+1,1}^*(\tau) \right] e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau \\ &= a_i e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \\ &\quad - e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \int_0^x \left[\mu(\tau) q_{I,i,0}^* + \lambda y_{w,i+1,1}^*(\tau) \right] e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau, \quad i \geq 0. \end{aligned} \quad (2.14)$$

Through multiplying $e^{-\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau}$ to two sides of (2.14), we obtain

$$e^{-\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} q_{w,i,1}^*(x) = a_i - \int_0^x \left[\mu(\tau) q_{I,i,0}^* + \lambda y_{w,i+1,1}^*(\tau) \right] e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau, \quad i \geq 0. \quad (2.15)$$

By combining (2.11) with (2.15), we deduce

$$a_i = \int_0^\infty \left[\mu(\tau) q_{I,i,0}^* + \lambda y_{w,i+1,1}^*(\tau) \right] e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau, \quad i \geq 0. \quad (2.16)$$

By inserting (2.16) into (2.14), we get

$$q_{w,i,1}^*(x) = e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \int_x^\infty \left[\mu(\tau) q_{I,i,0}^* + \lambda y_{w,i+1,1}^*(\tau) \right] e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau$$

$$\begin{aligned}
&= e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \int_x^\infty \mu(\tau) q_{I,i,0}^* e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau \\
&\quad + \lambda e^{\int_0^x (\gamma + \lambda + \alpha + \mu(\tau)) d\tau} \int_x^\infty y_{w,i+1,1}^*(\tau) e^{-\int_0^\tau (\gamma + \lambda + \alpha + \mu(s)) ds} d\tau
\end{aligned} \tag{2.17}$$

\Rightarrow

$$\begin{aligned}
\|q_{w,i,1}^*\|_{L^\infty[0,\infty)} &\leq \int_0^\infty e^{\int_0^x (\operatorname{Re} \gamma + \lambda + \alpha) d\tau} \int_x^\infty \mu(\tau) |q_{I,i,0}^*| e^{-\int_0^\tau (\operatorname{Re} \gamma + \lambda + \alpha) ds} e^{-\int_x^\tau \mu(s) ds} d\tau dx \\
&\quad + \lambda \int_0^\infty e^{\int_0^x (\operatorname{Re} \gamma + \lambda + \alpha) d\tau} \int_x^\infty \mu(\tau) |y_{w,i+1,1}^*(\tau)| e^{-\int_0^\tau (\operatorname{Re} \gamma + \lambda + \alpha) ds} e^{-\int_x^\tau \mu(s) ds} d\tau dx \\
&\leq \frac{\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} |q_{I,i,0}^*| \\
&\quad + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \|y_{w,i+1,1}^*\|_{L^\infty[0,\infty)}, \quad i \geq 0.
\end{aligned} \tag{2.18}$$

Equation (2.12) gives

$$|q_{I,0,0}^*| \leq \frac{\lambda}{|\gamma + \lambda|} |y_{w,0,1}^*(0)| \leq \frac{\lambda}{|\gamma + \lambda|} \|y_{w,0,1}^*\|_{L^\infty[0,\infty)}. \tag{2.19}$$

From (2.13) we estimate

$$\begin{aligned}
|q_{I,i,0}^*| &\leq \frac{i\theta}{|\gamma + \lambda + i\theta|} |y_{w,i-1,1}^*(0)| + \frac{\lambda}{|\gamma + \lambda + i\theta|} |y_{w,i,1}^*(0)| \\
&\leq \frac{i\theta}{|\gamma + \lambda + i\theta|} \|y_{w,i-1,1}^*\|_{L^\infty[0,\infty)} + \frac{\lambda}{|\gamma + \lambda + i\theta|} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)} \\
&\leq \frac{\lambda + i\theta}{|\gamma + \lambda + i\theta|} \sup_{i \geq 0} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)}, \quad i \geq 1.
\end{aligned} \tag{2.20}$$

By inserting (2.19) and (2.20) into (2.18), we have

$$\begin{aligned}
\|q_{w,0,1}^*\|_{L^\infty[0,\infty)} &\leq \frac{\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} \frac{\lambda}{|\gamma + \lambda|} \|y_{w,0,1}^*\|_{L^\infty[0,\infty)} \\
&\quad + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \|y_{w,1,1}^*\|_{L^\infty[0,\infty)} \\
&\leq \left\{ \frac{\lambda \bar{\mu}}{|\gamma + \lambda| (\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \sup_{i \geq 0} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)}, \\
\|q_{w,i,1}^*\|_{L^\infty[0,\infty)} & \leq \frac{\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} \frac{\lambda + i\theta}{|\gamma + \lambda + i\theta|} \sup_{i \geq 0} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)} \\
& + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \sup_{i \geq 0} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)} \\
& = \left\{ \frac{(\lambda + i\theta)\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})|\gamma + \lambda + i\theta|} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\} \\
& \times \sup_{i \geq 0} \|y_{w,i,1}^*\|_{L^\infty[0,\infty)}, \quad i \geq 0.
\end{aligned} \tag{2.21}$$

By combining (2.19) and (2.20) with (2.21), we derive

$$\begin{aligned}
\| (q_I^*, q_w^*) \| & \leq \sup \left\{ \frac{\lambda}{|\gamma + \lambda|}, \sup_{i \geq 1} \left\{ \frac{\lambda + i\theta}{|\gamma + \lambda + i\theta|} \right\}, \right. \\
& \left. \frac{\lambda \bar{\mu}}{|\gamma + \lambda|(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}}, \right. \\
& \left. \sup_{i \geq 1} \left\{ \frac{(\lambda + i\theta)\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})|\gamma + \lambda + i\theta|} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\} \right\}.
\end{aligned} \tag{2.22}$$

This shows that $[I - (\gamma I - \mathcal{L} - \mathcal{G})^{-1} \mathcal{A}]^{-1}$ exists and is bounded when γ belongs to the set

$$\left\{ \gamma \in \mathbb{C} \left| \begin{array}{l} \sup \left\{ \frac{\lambda}{|\gamma + \lambda|}, \sup_{i \geq 1} \left\{ \frac{\lambda + i\theta}{|\gamma + \lambda + i\theta|} \right\}, \frac{\lambda \bar{\mu}}{|\gamma + \lambda|(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}}, \right. \right. \\ \left. \left. \sup_{i \geq 1} \left\{ \frac{(\lambda + i\theta)\bar{\mu}}{(\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu})|\gamma + \lambda + i\theta|} + \frac{\lambda}{\operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu}} \right\} \right\} < 1, \right. \\ \left. \operatorname{Re} \gamma + \lambda + \alpha + \underline{\mu} > 0 \right. \end{array} \right\}. \tag{2.23}$$

Through discussing the solution of the equation $(\gamma I - \mathcal{L} - \mathcal{G})(q_I^*, q_w^*) = (y_I^*, y_w^*)$ for any given $(y_I^*, y_w^*) \in X^*$, it is not difficult to verify $(\gamma I - \mathcal{L} - \mathcal{G})^{-1}$ exists and is bounded when γ satisfies

(2.23). Therefore, by the resolvent equation

$$\begin{aligned} [\gamma I - (\mathcal{L} + \mathcal{G} + \mathcal{H})]^{-1} &= \left\{ (\gamma I - \mathcal{L} - \mathcal{G}) \left[I - (\gamma I - \mathcal{L} - \mathcal{G})^{-1} \mathcal{H} \right] \right\}^{-1} \\ &= \left[I - (\gamma I - \mathcal{L} - \mathcal{G})^{-1} \mathcal{H} \right]^{-1} (\gamma I - \mathcal{L} - \mathcal{G})^{-1}, \end{aligned} \quad (2.24)$$

we know that $(\gamma I - \mathcal{L} - \mathcal{G} - \mathcal{H})^{-1}$ exists and is bounded when γ belongs to the set (2.23), which means that (2.23) belongs to $\rho(\mathcal{L} + \mathcal{G} + \mathcal{H})$.

In particular, if $\gamma = ia, a \in \mathbb{R} \setminus \{0\}, i^2 = -1$, then all γ 's belong to (2.23). In fact, by using the condition on this lemma, we have

$$\begin{aligned} \frac{\lambda}{\sqrt{a^2 + \lambda^2}} &< 1, \\ \sup_{k \geq 1} \frac{\lambda + k\theta}{\sqrt{a^2 + (\lambda + k\theta)^2}} &< 1, \\ \underline{\mu} < \alpha + \underline{\mu} &\implies \lambda \underline{\mu} < (\alpha + \underline{\mu}) \sqrt{a^2 + \lambda^2} \\ &\implies \lambda \underline{\mu} + \lambda \sqrt{a^2 + \lambda^2} < (\alpha + \underline{\mu}) \sqrt{a^2 + \lambda^2} + \lambda \sqrt{a^2 + \lambda^2} \\ &\implies \lambda \underline{\mu} + \lambda \sqrt{a^2 + \lambda^2} < (\lambda + \alpha + \underline{\mu}) \sqrt{a^2 + \lambda^2} \\ &\implies \frac{\lambda \underline{\mu}}{\sqrt{a^2 + \lambda^2} (\lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\lambda + \alpha + \underline{\mu}} < 1, \\ \underline{\mu} < \alpha + \underline{\mu} &\implies (\lambda + k\theta) \underline{\mu} < (\alpha + \underline{\mu}) \sqrt{a^2 + (\lambda + k\theta)^2} \\ &\implies (\lambda + k\theta) \underline{\mu} + \lambda \sqrt{a^2 + (\lambda + k\theta)^2} < (\alpha + \underline{\mu}) \sqrt{a^2 + (\lambda + k\theta)^2} + \lambda \sqrt{a^2 + (\lambda + k\theta)^2} \\ &\implies (\lambda + k\theta) \underline{\mu} + \lambda \sqrt{a^2 + (\lambda + k\theta)^2} < (\lambda + \alpha + \underline{\mu}) \sqrt{a^2 + (\lambda + k\theta)^2}, \\ \sup_{k \geq 1} \left\{ \frac{(\lambda + k\theta) \underline{\mu}}{\sqrt{a^2 + (\lambda + k\theta)^2} (\lambda + \alpha + \underline{\mu})} + \frac{\lambda}{\lambda + \alpha + \underline{\mu}} \right\} &< 1. \end{aligned} \quad (2.25)$$

The above inequalities show that all points on the imaginary axis except zero belong to the resolvent set of $(A + U + E)^*$. From the relation between the spectrum of $A + U + E$ and spectrum of $(A + U + E)^*$ we know that all points on the imaginary axis except zero belong to the resolvent set of $A + U + E$. \square

Lemma 2.3. *If*

$$0 < \underline{\mu} = \inf_{x \in [0, \infty)} \mu(x) \leq \bar{\mu} = \sup_{x \in [0, \infty)} \mu(x) < \infty, \quad (2.26)$$

then 0 is not an eigenvalue of $A + U + E$.

Proof. We consider the equation $(A + U + E)(p_I, p_w) = 0$, which is equivalent to

$$\lambda p_{I,0,0} = \int_0^\infty \mu(x) p_{w,0,1}(x) dx, \quad (2.27)$$

$$(\lambda + i\theta) p_{I,i,0} = \int_0^\infty \mu(x) p_{w,i,1}(x) dx, \quad i \geq 1, \quad (2.28)$$

$$\frac{dp_{w,0,1}(x)}{dx} = -(\lambda + \alpha + \mu(x)) p_{w,0,1}(x), \quad (2.29)$$

$$\frac{dp_{w,i,1}(x)}{dx} = -(\lambda + \alpha + \mu(x)) p_{w,i,1}(x) + \lambda p_{w,i-1,1}(x), \quad i \geq 1, \quad (2.30)$$

$$p_{w,i,1}(0) = \lambda p_{I,i,0} + (i+1)\theta p_{I,i+1,0}, \quad i \geq 0. \quad (2.31)$$

It is hard to determine the concrete expressions of all $p_{I,i,0}, p_{w,i,1}(x)$ and to prove $(p_I, p_w) \in D(A)$. In the following we use another method. We introduce the probability-generating functions $Q_I(z) = \sum_{i=0}^\infty p_{I,i,0} z^i$, $Q_w(x, z) = \sum_{i=0}^\infty p_{w,i,1}(x) z^i$ for $|z| < 1$. Theorem 1.1 ensures that $Q_I(z), Q_w(x, z)$ are well-defined. By applying the basic knowledge of power series, the Fubini theorem and (2.27)-(2.28), we have

$$\begin{aligned} \lambda p_{I,0,0} + \sum_{i=1}^\infty (\lambda + i\theta) p_{I,i,0} z^i &= \int_0^\infty \mu(x) p_{w,0,1}(x) dx + \sum_{i=1}^\infty \int_0^\infty \mu(x) p_{w,i,1}(x) z^i dx \\ \implies \lambda Q_I(z) + \theta z \left(\sum_{i=0}^\infty p_{I,i,0} z^i \right)' &= \int_0^\infty \mu(x) \sum_{i=0}^\infty p_{w,i,1}(x) z^i dx \\ \implies \lambda Q_I(z) + \theta z \frac{d}{dz} Q_I(z) &= \int_0^\infty \mu(x) Q_w(x, z) dx. \end{aligned} \quad (2.32)$$

By (2.29) and (2.30), we deduce

$$\begin{aligned} \frac{dp_{w,0,1}(x)}{dx} + \sum_{i=1}^\infty \frac{dp_{w,i,1}(x)}{dx} z^i &= -(\lambda + \alpha + \mu(x)) p_{w,0,1}(x) \\ &\quad - (\lambda + \alpha + \mu(x)) \sum_{i=1}^\infty p_{w,i,1}(x) z^i + \sum_{i=1}^\infty \lambda p_{w,i-1,1}(x) z^i \\ \implies \\ \frac{\partial \sum_{i=0}^\infty p_{w,i,1}(x) z^i}{\partial x} &= -(\lambda + \alpha + \mu(x)) \sum_{i=0}^\infty p_{w,i,1}(x) z^i + \lambda \sum_{i=1}^\infty p_{w,i-1,1}(x) z^i \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \\
&\frac{\partial Q_w(x, z)}{\partial x} = -(\lambda + \alpha + \mu(x))Q_w(x, z) + \lambda z \sum_{i=0}^{\infty} p_{w,i,1}(x)z^i \\
&\Rightarrow \\
&\frac{\partial Q_w(x, z)}{\partial x} = (\lambda z - \lambda - \alpha - \mu(x))Q_w(x, z) \\
&\Rightarrow \\
&Q_w(x, z) = Q_w(0, z)e^{\int_0^x (\lambda z - \lambda - \alpha - \mu(\tau))d\tau}.
\end{aligned} \tag{2.33}$$

Equation (2.31) gives

$$\begin{aligned}
Q_w(0, z) &= \sum_{i=0}^{\infty} p_{w,i,1}(0)z^i = \lambda \sum_{i=0}^{\infty} p_{I,i,0}z^i + \sum_{i=0}^{\infty} (i+1)\theta p_{I,i+1,0}z^i \\
&= \lambda Q_I(z) + \theta \frac{d}{dz} Q_I(z).
\end{aligned} \tag{2.34}$$

By combining (2.34) with (2.33) and using (2.32), we calculate

$$\begin{aligned}
\lambda Q_I(z) + \theta z \frac{d}{dz} Q_I(z) &= \int_0^{\infty} \mu(x)Q_w(x, z)dx \\
&= \int_0^{\infty} \mu(x)Q_w(0, z)e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)]d\tau} dx \\
&= \left[\lambda Q_I(z) + \theta \frac{d}{dz} Q_I(z) \right] \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)]d\tau} dx \\
&\Rightarrow \\
\frac{dQ_I(z)}{Q_I(z)} &= \frac{\lambda \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)]d\tau} dx - 1}{\theta \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)]d\tau} dx} dz \\
&\Rightarrow \\
Q_I(z) &= Q_I(0) \exp \left\{ \frac{\lambda \int_0^z \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)]d\tau} dx - 1}{\theta \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)]d\tau} dx} ds \right\} \\
&= p_{I,0,0} \exp \left\{ \frac{\lambda \int_0^z \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)]d\tau} dx - 1}{\theta \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)]d\tau} dx} ds \right\}.
\end{aligned} \tag{2.35}$$

This together with (2.34) gives

$$Q_w(0, z) = \frac{\lambda(z-1)}{\left(z - \int_0^{\infty} \mu(x)e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)]d\tau} dx \right)} Q_I(z). \tag{2.36}$$

From (2.35), we derive

$$\begin{aligned}
\sum_{i=0}^{\infty} p_{I,i,0} &= \lim_{z \rightarrow 1} Q_I(z) \\
&= \lim_{z \rightarrow 1} p_{I,0,0} \exp \left\{ \frac{\lambda}{\theta} \int_0^z \frac{\int_0^{\infty} \mu(x) e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)] d\tau} dx - 1}{s - \int_0^{\infty} \mu(x) e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)] d\tau} dx} ds \right\} \\
&= p_{I,0,0} \exp \left\{ \frac{\lambda}{\theta} \int_0^1 \frac{\int_0^{\infty} \mu(x) e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)] d\tau} dx - 1}{s - \int_0^{\infty} \mu(x) e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)] d\tau} dx} ds \right\}.
\end{aligned} \tag{2.37}$$

In the following, by the Rouché theorem we know that $s - \int_0^{\infty} \mu(x) e^{\int_0^x [\lambda s - \lambda - \alpha - \mu(\tau)] d\tau} dx$ has a unique zero point inside the unit circle $|z| = 1$. And by the residue theorem we determine the above integral.

Let $f(z) = z$ and $g(z) = \int_0^{\infty} \mu(x) e^{-(\lambda + \alpha - \lambda z)x - \int_0^x \mu(\tau) d\tau} dx$. It is well known that $f(z)$ and $g(z)$ are differentiable inside and continuous on the contour $|z| = 1$. And $|f(z)| = 1$ on $|z| = 1$. Since, for $(\lambda + \alpha - \lambda \operatorname{Re} z) > 0$,

$$\begin{aligned}
|g(z)| &= \left| \int_0^{\infty} \mu(x) e^{-(\lambda + \alpha - \lambda z)x - \int_0^x \mu(\tau) d\tau} dx \right| \\
&\leq - \int_0^{\infty} e^{-(\lambda + \alpha - \lambda \operatorname{Re} z)x} dx e^{-\int_0^x \mu(\tau) d\tau} \\
&= - \left[e^{-(\lambda + \alpha - \lambda \operatorname{Re} z)x} e^{-\int_0^x \mu(\tau) d\tau} \right]_{x=0}^{x=\infty} \\
&\quad + (\lambda + \alpha - \lambda \operatorname{Re} z) \int_0^{\infty} e^{-(\lambda + \alpha - \lambda \operatorname{Re} z)x - \int_0^x \mu(\tau) d\tau} dx \\
&= 1 - (\lambda + \alpha - \lambda \operatorname{Re} z) \int_0^{\infty} e^{-(\lambda + \alpha - \lambda \operatorname{Re} z)x - \int_0^x \mu(\tau) d\tau} dx \\
&< 1 = |f(z)|,
\end{aligned} \tag{2.38}$$

we have $|f(z)| < |g(z)|$ on $|z| = 1$. Consequently, all conditions of Rouché's theorem are satisfied. Obviously $f(z) - g(z)$ has only one zero inside the unit circle $|z| = 1$, since $f(z)$ has one. If we denote this zero by z_0 , then it is a simple pole of

$$\frac{\int_0^{\infty} \mu(x) e^{-(\lambda + \alpha - \lambda z)x - \int_0^x \mu(\tau) d\tau} dx - 1}{z - \int_0^{\infty} \mu(x) e^{-(\lambda + \alpha - \lambda z)x - \int_0^x \mu(\tau) d\tau} dx}. \tag{2.39}$$

By the residue theorem, we calculate

$$\begin{aligned}
 & \int_0^1 \frac{\int_0^\infty \mu(x) e^{-(\lambda+\alpha-\lambda s)x - \int_0^x \mu(\tau) d\tau} dx - 1}{s - \int_0^\infty \mu(x) e^{-(\lambda+\alpha-\lambda s)x - \int_0^x \mu(\tau) d\tau} dx} ds \\
 &= \lim_{z \rightarrow z_0} \frac{\int_0^\infty \mu(x) e^{-(\lambda+\alpha-\lambda z)x - \int_0^x \mu(\tau) d\tau} dx - 1}{1 - \int_0^\infty \lambda x \mu(x) e^{-(\lambda+\alpha-\lambda z)x - \int_0^x \mu(\tau) d\tau} dx} \\
 &= \frac{\int_0^\infty \mu(x) e^{-(\lambda+\alpha-\lambda z_0)x - \int_0^x \mu(\tau) d\tau} dx - 1}{1 - \int_0^\infty \lambda x \mu(x) e^{-(\lambda+\alpha-\lambda z_0)x - \int_0^x \mu(\tau) d\tau} dx}.
 \end{aligned} \tag{2.40}$$

Thus, we obtain

$$\sum_{i=0}^{\infty} p_{I,i,0} = p_{I,0,0} \exp \left\{ \frac{\lambda}{\theta} \frac{\int_0^\infty \mu(x) e^{-(\lambda+\alpha-\lambda z_0)x - \int_0^x \mu(\tau) d\tau} dx - 1}{1 - \int_0^\infty \lambda x \mu(x) e^{-(\lambda+\alpha-\lambda z_0)x - \int_0^x \mu(\tau) d\tau} dx} \right\} < \infty. \tag{2.41}$$

From (2.36), we derive

$$\begin{aligned}
 \lim_{z \rightarrow 1} Q_w(0, z) &= \lim_{z \rightarrow 1} \frac{\lambda(z-1)}{z - \int_0^\infty \mu(x) e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)] d\tau} dx} Q_I(z) \\
 &= \lim_{z \rightarrow 1} \frac{\lambda(z-1)}{z - \int_0^\infty \mu(x) e^{\int_0^x [\lambda z - \lambda - \alpha - \mu(\tau)] d\tau} dx} \lim_{z \rightarrow 1} Q_I(z) \\
 &= \frac{\lambda(1-1)}{1 - \int_0^\infty \mu(x) e^{-\int_0^x [\alpha + \mu(\tau)] d\tau} dx} Q_I(1) \\
 &= 0.
 \end{aligned} \tag{2.42}$$

By combining (2.33) with (2.42), we estimate

$$\begin{aligned}
 \sum_{i=0}^{\infty} p_{w,i,1}(x) &= \lim_{z \rightarrow 1} Q_w(x, z) = \lim_{z \rightarrow 1} Q_w(0, z) e^{\int_0^x (\lambda z - \lambda - \alpha - \mu(\tau)) d\tau} \\
 &= \lim_{z \rightarrow 1} Q_w(0, z) e^{-\int_0^x (\alpha + \mu(\tau)) d\tau} \\
 &= 0.
 \end{aligned} \tag{2.43}$$

Since Theorem 1.1 shows that each $p_{w,i,1}(x)$ is nonnegative, (2.43) implies $p_{w,i,1}(x) = 0$ for all $i \geq 0$. This together with (2.27) and (2.28) gives $p_{I,i,0} = 0$ for all $i \geq 0$. In other words, $(A + U + E)(p_I, p_w) = 0$ has only zero solution; that is, 0 is not an eigenvalue of $A + U + E$. \square

Lemma 2.3 shows that the system (1.7) does not have nonzero steady-state solution.

If we can prove that 0 is not in the residue spectrum of $A + U + E$, then by Theorem 1.1, Lemmas 2.2 and 2.3, and ABLV Theorem (see [16] or [17]), we deduce that the time-dependent solution of system (1.7) is strongly asymptotically stable. This result is quite different from other queueing models, see Gupur [13–15] and Kasim and Gupur [7].

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