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*Research Article*

# Generalized Carleson Measure Spaces and Their Applications

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We introduce the generalized Carleson measure spaces $\mathrm{CMO}_r^{\alpha,q}$ that extend BMO. Using Frazier and Jawerth's $\varphi$-transform and sequence spaces, we show that, for $\alpha \in \mathbb{R}$ and $0 < p \le 1$, the duals of homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ for $1 < q < \infty$ and $0 < q \le 1$ are $\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$ and $\mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$ (for any $r \in \mathbb{R}$), respectively. As applications, we give the necessary and sufficient conditions for the boundedness of wavelet multipliers and paraproduct operators acting on homogeneous Triebel-Lizorkin spaces.

## 1. Introduction

In 1972, Fefferman and Stein [1] proved that the dual of $H^1$ is the BMO space. In 1990, Frazier and Jawerth [2, Theorem 5.13] generalized the above duality to homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$. More precisely, they showed that the dual of $\dot{F}_1^{\alpha,q}$ is $\dot{F}_\infty^{-\alpha,q'}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, where $q'$ is the conjugate index of $q$. Throughout the paper, $q'$ is interpreted as $q' = \infty$ whenever $0 < q \le 1$, and $q' = q/(q-1)$ for $1 < q \le \infty$. Note that $\dot{F}_1^{0,2} = H^1$ and $\mathrm{BMO} = \dot{F}_\infty^{0,2}$. For $\alpha \in \mathbb{R}$, $0 < p < 1$, and $0 < q < \infty$, it is known (cf. [2–4]) that the dual of $\dot{F}_p^{\alpha,q}$ is $\dot{F}_\infty^{-\alpha+(n/p)-n,\infty}$. Here, we will give another characterization for the duals of $\dot{F}_p^{\alpha,q}$ in terms of the generalized Carleson measure spaces for $\alpha \in \mathbb{R}$, $0 < p \le 1$, and $0 < q < \infty$.

We say that a cube $Q \subseteq \mathbb{R}^n$ is *dyadic* if $Q = Q_{j\mathbf{k}} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \le x_i < 2^{-j}(k_i+1), i = 1, 2, \dots, n\}$ for some $j \in \mathbb{Z}$ and $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q$ and by $x_Q = 2^{-j}\mathbf{k}$ the "left lower corner" of $Q$ when $Q = Q_{j\mathbf{k}}$. We use $\sup_P$ and $\sum_P$ to express the supremum and summation taken over all dyadic cubes $P$, respectively. Also, denote the summation taken over all dyadic cubes $Q$ contained in $P$ by $\sum_{Q \subseteq P}$. For any dyadic cubes $P$ and $Q$, either $P$ and $Q$ are nonoverlapping or one contains the other. For any function $f$ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and dyadic cube $Q = Q_{j\mathbf{k}}$, set

$$\begin{aligned} f_Q(x) &= |Q|^{-1/2} f\left(\frac{x - x_Q}{\ell(Q)}\right) = 2^{jn/2} f\left(2^j x - \mathbf{k}\right), \\ f_j(x) &= 2^{jn} f\left(2^j x\right), \\ \tilde{f}(x) &= \overline{f(-x)}. \end{aligned} \tag{1.1}$$

It is clear that $\tilde{g}_j * f(x_Q) = |Q|^{-1/2} \langle f, g_Q \rangle$, where $\langle f, g \rangle$ denotes the paring in the usual sense for $g$ in a Fréchet space $X$ and $f$ in the dual of $X$.

Choose a fixed function $\varphi$ in Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, the collection of rapidly decreasing $C^\infty$ functions on $\mathbb{R}^n$, satisfying

$$\begin{aligned} &\operatorname{supp}(\widehat{\varphi}) \subseteq \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}, \\ &\left|\widehat{\varphi}(\xi)\right| \ge c > 0 \quad \text{if } \frac{3}{5} \le |\xi| \le \frac{5}{3}. \end{aligned} \tag{1.2}$$

For $\alpha \in \mathbb{R}$ and $0 < p, q \le +\infty$, we say that $f$ belongs to the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ if $f \in \mathcal{S}'/\mathcal{P}$, the tempered distributions modulo polynomials, satisfies

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \begin{cases} \left\| \left\{ \sum_{k \in \mathbb{Z}} \left(2^{k\alpha} |\varphi_k * f|\right)^q \right\}^{1/q} \right\|_{L^p} < \infty & \text{for } 0 < p < \infty, \\ \sup_P \left\{ |P|^{-1} \int_P \sum_{k=-\log_2 \ell(P)}^{\infty} \left(2^{k\alpha} |\varphi_k * f(x)|\right)^q dx \right\}^{1/q} < \infty & \text{for } p = \infty. \end{cases} \tag{1.3}$$

When $0 < p < \infty$ and $q = \infty$, the above $\ell^q$-norm is modified to be the supremum norm as usual, and $\dot{F}_\infty^{\alpha,\infty}$ is defined to be $\dot{B}_\infty^{\alpha,\infty}$, which is

$$\|f\|_{\dot{F}_\infty^{\alpha,\infty}} := \sup_{k \in \mathbb{Z}} \sup_{\substack{x \in Q \\ \ell(Q) = 2^{-k}}} 2^{k\alpha} |\varphi_k * f(x)| \approx \sup_Q |Q|^{-(\alpha/n)-(1/2)} \left|\langle f, \varphi_Q \rangle\right| < \infty. \tag{1.4}$$

We now introduce a new space $\mathrm{CMO}_r^{\alpha,q}$ as follows.

*Definition 1.1.* Let $\varphi \in \mathcal{S}$ satisfy (1.2). For $\alpha, r \in \mathbb{R}$ and $0 < q \le \infty$, the *generalized Carleson measure spaces* $\mathrm{CMO}_r^{\alpha,q}$ is the collection of all $f \in \mathcal{S}'/\mathcal{P}$ satisfying $\|f\|_{\mathrm{CMO}_r^{\alpha,q}} < \infty$, where

$$\|f\|_{\mathrm{CMO}_r^{\alpha,q}} := \begin{cases} \sup_P \left\{ |P|^{-r} \int_P \sum_{Q \subseteq P} \left( |Q|^{-(\alpha/n)-(1/2)} \left|\langle f, \varphi_Q \rangle\right| \chi_Q(x) \right)^q dx \right\}^{1/q}, & 0 < q < \infty, \\ \sup_P \sup_{Q \subseteq P} |Q|^{-(\alpha/n)-(1/2)} \left|\langle f, \varphi_Q \rangle\right| = \sup_Q |Q|^{-(\alpha/n)-(1/2)} \left|\langle f, \varphi_Q \rangle\right|, & q = \infty, \end{cases} \tag{1.5}$$

and $\chi_Q$ denotes the characteristic function of $Q$.

*Remark 1.2.* By definition, we immediately have $\mathrm{CMO}_r^{\alpha,\infty} = \dot{F}_\infty^{\alpha,\infty}$ for $\alpha, r \in \mathbb{R}$, and it is easy to check $\mathrm{CMO}_r^{\alpha,q} = \{0\}$ for $r < 0$ and $0 < q < \infty$. Note that the zero element in $\mathrm{CMO}_r^{\alpha,q}$ means the class of polynomials. Also note that $\mathrm{CMO}_0^{\alpha,q} = \dot{F}_q^{\alpha,q}$ with equivalent norms for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. It follows from Proposition 3.3 that $\mathrm{CMO}_1^{\alpha,q} = \dot{F}_\infty^{\alpha,q}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. In particular, $\mathrm{CMO}_1^{0,2} = \mathrm{BMO}$, and hence the spaces $\mathrm{CMO}_r^{\alpha,q}$ generalize BMO.

*Remark 1.3.* For a dyadic cube $P$, denote by $k_P = -\log_2 \ell(P)$; that is, $k_P$ is the integer so that $\ell(P) = 2^{-k_P}$. In [5, 6], Yang and Yuan introduced the so-called "unified and generalized" Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{\alpha,\tau}$ with four parameters by

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\tau}} := \sup_P |P|^{-\tau} \left\{ \int_P \left[ \sum_{k \geq k_P} \left( 2^{k\alpha} \left| \varphi_k * f(x) \right| \right)^q \right]^{p/q} dx \right\}^{1/p} < \infty, \tag{1.6}$$

for $\alpha, \tau \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $f \in \mathcal{S}'/\mathcal{P}$. Note that in [5] the space $\dot{F}_{p,q}^{\alpha,\tau}$ was defined for $\tau \in [0,\infty)$, $p \in (1,\infty)$, and $q \in (1,\infty]$. It follows from [6, Theorem 3.1] that

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\tau}} \approx \sup_P |P|^{-\tau} \left\{ \int_P \left[ \sum_{Q \subset P} \left( |Q|^{(-\alpha/n)-(1/2)} \left| \langle f, \varphi_Q \rangle \right| \chi_Q(x) \right)^q \right]^{p/q} dx \right\}^{1/p}. \tag{1.7}$$

It is clear that $\mathrm{CMO}_r^{\alpha,q} = \dot{F}_{q,q}^{\alpha,r/q}$ for $0 < q < \infty$, and hence $\mathrm{CMO}_r^{\alpha,q}$ "looks like" a special case of $\dot{F}_{p,q}^{\alpha,\tau}$. In fact, it was proved in [7, 8] that the space $\dot{F}_{p,q}^{\alpha,\tau}$ is the "same" as the space $\mathrm{CMO}_{\tau q+1-q/p}^{\alpha,q}$.

The definition of $\mathrm{CMO}_r^{\alpha,q}$ is independent of the choice of $\varphi \in \mathcal{S}$ satisfying (1.2). To show that, we need the following Plancherel-Pôlya inequalities.

**Theorem 1.4** (Plancherel-Pôlya inequality for $0 < q < \infty$). *Let $\varphi, \phi \in \mathcal{S}$ satisfy* (1.2). *For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, if $f \in \mathcal{S}'/\mathcal{P}$ satisfies*

$$\sup_P \left\{ |P|^{-r} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{\substack{Q \subseteq P \\ \ell(Q)=2^{-k}}} \left( 2^{k\alpha} \sup_{u \in Q} \left| \tilde{\varphi}_k * f(u) \right| \right)^q |Q| \right\}^{1/q} < \infty, \tag{1.8}$$

*then*

$$\begin{aligned} &\sup_P \left\{ |P|^{-r} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{\substack{Q \subseteq P \\ \ell(Q)=2^{-k}}} \left( 2^{k\alpha} \sup_{u \in Q} \left| \tilde{\phi}_k * f(u) \right| \right)^q |Q| \right\}^{1/q} \\ &\quad \approx \sup_P \left\{ |P|^{-r} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{\substack{Q \subseteq P \\ \ell(Q)=2^{-k}}} \left( 2^{k\alpha} \inf_{u \in Q} \left| \tilde{\varphi}_k * f(u) \right| \right)^q |Q| \right\}^{1/q}. \end{aligned} \tag{1.9}$$

**Theorem 1.5** (Plancherel-Pôlya inequality for $q=\infty$). *Let $\varphi,\phi\in\mathcal{S}$ satisfy* (1.2). *For $\alpha, r\in\mathbb{R}$, if $f\in\mathcal{S}'/\mathcal{P}$ satisfies*

$$\sup_Q\left(|Q|^{-(\alpha/n)-r}\sup_{u\in Q}\left|\widetilde{\varphi}_{k_Q}*f(u)\right|\right)<\infty, \tag{1.10}$$

*then*

$$\sup_Q\left(|Q|^{-(\alpha/n)-r}\sup_{u\in Q}\left|\widetilde{\phi}_{k_Q}*f(u)\right|\right)\approx\sup_Q\left(|Q|^{-(\alpha/n)-r}\inf_{u\in Q}\left|\widetilde{\varphi}_{k_Q}*f(u)\right|\right). \tag{1.11}$$

*Remark 1.6.* Let $\varphi,\phi\in\mathcal{S}$ satisfy (1.2). Denote by $\mathrm{CMO}_r^{\alpha,q}(\varphi)$ the collection of all $f\in\mathcal{S}'/\mathcal{P}$ satisfying $\|f\|_{\mathrm{CMO}_r^{\alpha,q}(\varphi)}<\infty$ defined in Definition 1.1 with respect to $\varphi$. Then, by Theorem 1.4,

$$\begin{aligned}\|f\|_{\mathrm{CMO}_r^{\alpha,q}(\phi)}&\le\sup_P\left\{|P|^{-r}\sum_{k=-\log_2\ell(P)}^{\infty}\sum_{\substack{Q\subseteq P\\ \ell(Q)=2^{-k}}}\left(2^{k\alpha}\sup_{u\in Q}\left|\widetilde{\phi}_k*f(u)\right|\right)^q|Q|\right\}^{1/q}\\&\le C\sup_P\left\{|P|^{-r}\sum_{k=-\log_2\ell(P)}^{\infty}\sum_{\substack{Q\subseteq P\\ \ell(Q)=2^{-k}}}\left(2^{k\alpha}\inf_{u\in Q}\left|\widetilde{\varphi}_k*f(u)\right|\right)^q|Q|\right\}^{1/q}\\&\le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}(\varphi)}\quad\text{for } 0<q<\infty.\end{aligned} \tag{1.12}$$

Similarly, $\|f\|_{\mathrm{CMO}_r^{\alpha,q}(\varphi)}\le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}(\phi)}$ by interchanging the roles of $\varphi$ and $\phi$. Hence, the definition of $\mathrm{CMO}_r^{\alpha,q}(\varphi)$ is independent of the choice of $\varphi$ and, for short, denoted by $\mathrm{CMO}_r^{\alpha,q}$. Also, Theorem 1.5 shows that $\mathrm{CMO}_r^{\alpha,\infty}$ is independent of the choice of $\varphi$ satisfying (1.2) in the same argument.

*Remark 1.7.* The classical Plancherel-Pôlya inequality [9] concludes that if $\{x_k\}$ is an appropriate set of points in $\mathbb{R}^n$, for example, lattice points, where the length of the mesh is sufficiently small, then

$$\left(\sum_{k=1}^{\infty}\left|f(x_k)\right|^p\right)^{1/p}\approx\|f\|_p \tag{1.13}$$

for all $0<p\le\infty$ with a modification if $p=\infty$.

Using the Calderón reproducing formula (either continuous or discrete version), several authors obtain the variant Plancherel-Pôlya inequalities [10–13]. These inequalities give characterizations of the Besov spaces and the Triebel-Lizorkin spaces. Moreover, using these inequalities, one can show that the Littlewood-Paley $g$-function and Lusin area $S$-function are equivalent in $L^p$-norm.

Define a linear map $S_\varphi$ from $\mathcal{S}'/\mathcal{P}$ into the family of complex sequences by

$$S_\varphi(f) = \{\langle f, \varphi_Q\rangle\}_Q. \tag{1.14}$$

Let $\mathcal{S}_0$ denote the family of $f \in \mathcal{S}$ satisfying $\int x^{\mathbf{k}} f(x)dx = 0$ for all $\mathbf{k} \in (\mathbb{N} \cup \{0\})^n$. For $g \in \mathrm{CMO}_p^{-\alpha,q'}$, define a linear functional $L_g$ by

$$L_g(f) = \langle S_\psi(g), S_\varphi(f)\rangle = \sum_Q \langle g, \psi_Q\rangle\langle f, \varphi_Q\rangle \quad \text{for } f \in \mathcal{S}_0. \tag{1.15}$$

We now state our first main result as follows.

**Theorem 1.8** (duality for $\dot{F}_p^{\alpha,q}$). *Suppose that $\alpha \in \mathbb{R}$, $0 < p \le 1$, and $0 < q < \infty$.*

(a) *For $1 < q < \infty$, the dual of $\dot{F}_p^{\alpha,q}$ is $\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$ in the following sense.*

(i) *For $g \in \mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$, the linear functional $L_g$ given by (1.15), defined initially on $\mathcal{S}_0$, extends to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ with $\|L_g\| \le C\|g\|_{\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}}$.*

(ii) *Conversely, every continuous linear functional $L$ on $\dot{F}_p^{\alpha,q}$ satisfies $L = L_g$ for some $g \in \mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$ with $\|g\|_{\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}} \le C\|L\|$.*

(b) *For $0 < q \le 1$, the dual of $\dot{F}_p^{\alpha,q}$ is $CMO_r^{-\alpha+(n/p)-n,\infty}$ (any $r \in \mathbb{R}$) in the following sense.*

(i) *For $g \in \mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$, the linear functional $L_g$ given by (1.15), defined initially on $\mathcal{S}_0$, extends to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ with $\|L_g\| \le C\|g\|_{\mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}}$.*

(ii) *Conversely, every continuous linear functional $L$ on $\dot{F}_p^{\alpha,q}$ satisfies $L = L_g$ for some $g \in \mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$ with $\|g\|_{\mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}} \le C\|L\|$.*

*Remark 1.9.* For $0 < p < 1$ and $0 < q \le 1$, it follows immediately from [2, 3] (Verbitsky [4] corrected a gap of the proof) and definition that $(\dot{F}_p^{\alpha,q})' = \dot{F}_\infty^{-\alpha+(n/p)-n,\infty} = \mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$ (any $r \in \mathbb{R}$). Theorem 1.8 (b) shows a different approach to the duality and includes the case of $p = 1$.

For $p = 1 < q < \infty$, we have $\mathrm{CMO}_{q'-(q'/q)}^{-\alpha,q'} = (\dot{F}_1^{\alpha,q})' = \dot{F}_\infty^{-\alpha,q'}$. For $0 < p < 1 < q < \infty$, $\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'} = (\dot{F}_p^{\alpha,q})' = \dot{F}_\infty^{-\alpha+(n/p)-n,\infty}$, and hence $\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'} = \mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$. That is, each $\mathrm{CMO}_{(q/p)-(q/q')}^{\alpha,q}$ coincides with $\mathrm{CMO}_r^{\alpha+(n/p)-n,\infty}$ for $\alpha, r \in \mathbb{R}$ and $0 < p < 1 < q < \infty$.

*Remark 1.10.* In Remark 1.2 we are aware that $\mathrm{CMO}_r^{\alpha,q}$ generalize BMO by the viewpoint of spaces directly. Choosing $\alpha = 0$ and $q = 2$ in Theorem 1.8, we immediately have $(H^p)' = (\dot{F}_p^{0,2})' = \mathrm{CMO}_{(2/p)-1}^{0,2}$ for $0 < p \le 1$. In particular, $\mathrm{BMO} = \mathrm{CMO}_1^{0,2}$. Once again, we obtain that $\mathrm{CMO}_r^{\alpha,q}$ generalize BMO by the viewpoint of duality. It was also proved in [14] that the dual of the multiparameter product Hardy space is the generalized multiparameter Carleson measure space (cf. [14] for more details).

*Remark 1.11.* For $\alpha, r \in \mathbb{R}$, in order to make each index works, we defined $\mathrm{CMO}_r^{\alpha,\infty}$ to be $\sup_P |P|^{-r} \sup_{Q \subseteq P} |Q|^{-(\alpha/n)-(1/2)} |\langle f, \varphi_Q \rangle|$ in our earlier version and in [7]. In such a situation, for $0 < p, q \leq 1$, the dual of $\dot{F}_p^{\alpha,q}$ would be $\mathrm{CMO}_{(1/p)-1}^{-\alpha,\infty}$. In this paper, however, we follow the referee's suggestion and adopt a more "natural" definition of $\mathrm{CMO}_r^{\alpha,\infty}$ in Definition 1.1, that is, the limit of $\mathrm{CMO}_r^{\alpha,q}$ as $q \to \infty$. The sequence space $c_r^{\alpha,\infty}$ given in Definition 2.1 has a similar story as well.

As applications, we first recall the Haar multipliers introduced in [15, 16]. Given a sequence $\mathbf{t} = \{t_I\}_I$, where the $I$'s are dyadic intervals in $\mathbb{R}$, a Haar multiplier on $L^2(\mathbb{R})$ is a linear operator of the form

$$H_{\mathbf{t}} f(x) := \sum_I t_I \langle f, h_I \rangle h_I(x), \qquad f \in L^2(\mathbb{R}), \tag{1.16}$$

where $h_I$ are the Haar functions corresponding to $I$.

Using Meyer's wavelets, we may generalize the above Haar multiplier to $\mathbb{R}^n$ and obtain a necessary and sufficient condition for the boundedness on Triebel-Lizorkin spaces. Let $\{\psi^i\}$ for $i \in E := \{1, 2, \ldots 2^n - 1\}$ be Meyer's wavelets (cf. [17], [18, pages 71–109]). Then, $\{\psi_Q^i\}$, where $i \in E$ and $Q's$ are dyadic cubes in $\mathbb{R}^n$, is a frame for $\dot{F}_p^{\alpha,q}$ for $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$; that is, $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \sum_{i \in E} \|\{\langle f, \psi_Q^i \rangle\}_Q\|_{\dot{f}_p^{\alpha,q}}$ for $f \in \dot{F}_p^{\alpha,q}$. For $\mathbf{t} = \{t_Q\}_Q$, define a *wavelet multiplier* $\widetilde{T}_{\mathbf{t}}$ on $\mathbb{R}^n$ by

$$\widetilde{T}_{\mathbf{t}}(f) = \sum_{i \in E} \sum_Q |Q|^{-1/2} t_Q \left\langle f, \psi_Q^i \right\rangle \psi_Q^i \tag{1.17}$$

for $f \in \mathcal{S}'/\mathcal{P}$ such that the above summation is well defined.

**Theorem 1.12.** *Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \leq 1$ and $0 < q < \infty$. Then,*

(a) *for $1 < q < \infty$, $\widetilde{T}_{\mathbf{t}}$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ if and only if $\mathbf{t} \in c_{(q'/p)-(q'/q)}^{\beta,q'}$;*

(b) *for $0 < q \leq 1$ and $r \in \mathbb{R}$, $\widetilde{T}_{\mathbf{t}}$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ if and only if $\mathbf{t} \in c_r^{\beta+(n/p)-n,\infty}$, where $c_r^{\alpha,q}$ is given in Definition 2.1.*

We consider another application. Let $\varphi$ and $\psi$ in $\mathcal{S}$ satisfy (1.2) and (3.1). Choose a function $\Phi \in \mathcal{S}$ supported on $[0,1]^n$ and $\int \Phi = 1$. For $\alpha \in \mathbb{R}$ and $g \in \dot{F}_\infty^{\alpha,\infty}$, define the *paraproduct operator* $\Pi_g$ by

$$\Pi_g(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle f, \Phi_Q \rangle \psi_Q. \tag{1.18}$$

Thus, the adjoint operator $\Pi_g^*$ is

$$\Pi_g^*(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle f, \psi_Q \rangle \ \Phi_Q. \tag{1.19}$$

Then, $\Pi_g 1 = g$ and $\Pi_g^* 1 = 0$ since $\langle 1, \Phi_Q \rangle = |Q|^{1/2}$ and $\langle 1, \psi_Q \rangle = 0$. Also, if $g \in \dot{F}_\infty^{0,\infty}$, then both $\Pi_g$ and $\Pi_g^*$ are singular integral operators satisfying the weak boundedness property. Moreover, $\Pi_g$ is a Calderón-Zygmund operator (i.e., $\Pi_g$ is bounded on $L^2(\mathbb{R}^n)$) if and only if $g \in \dot{F}_\infty^{0,2}$ by David-Journé's $T1$ theorem [19] (also see [12, Theorems 5.4 and 5.8]). The authors showed a more general type of paraproduct operators in [12, page 688], which were derived from the discrete Calderón reproducing formula.

**Theorem 1.13.** *Suppose that $\beta \in \mathbb{R}$, $0 < r \le 1$ and $0 < p \le r < q < r/(1-r)$.*

(i) *For $\alpha < 0$, $\Pi_g$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_r^{\alpha+\beta,r}$ if and only if $g \in \mathrm{CMO}_{r(q-p)/p(q-r)}^{\beta,qr/(q-r)}$.*

(ii) *If $\alpha \in \mathbb{R}$ with $\alpha + \beta > 0$ and $g \in \mathrm{CMO}_{r(q-p)/p(q-r)}^{\beta,qr/(q-r)}$, then $\Pi_g^*$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_r^{\alpha+\beta,r}$.*

*Remark 1.14.* When $r = 1$, $0 < p \le 1 < q < \infty$, and $\beta \in \mathbb{R}$, Theorem 1.13 says that $\Pi_g$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ if and only if $g \in \mathrm{CMO}_{(q'/p)-(q'/q)}^{\beta,q'}$ for $\alpha < 0$, and $\Pi_g^*$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ for $\alpha > -\beta$ provided $g \in \mathrm{CMO}_{(q'/p)-(q'/q)}^{\beta,q'}$. In 1995, Youssfi [20] showed that, for $\beta \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le 2$, and $g \in \dot{F}_\infty^{\beta,\infty}$, $\Pi_g$ is bounded from $\dot{F}_p^{0,q}$ into $\dot{F}_p^{\beta,p}$ if and only if $g \in \dot{F}_\infty^{\beta,p}$. The special case of Theorem 1.13(i), $p = r$, generalizes Youssfi's result to $0 < p \le 1$. More precisely, for $\alpha < 0$, $\beta \in \mathbb{R}$, $0 < p \le 1$, and $p < q < p/(1-p)$, $\Pi_g$ is bounded from $\dot{F}_p^{\alpha,q}$ to $\dot{F}_p^{\alpha+\beta,p}$ if and only if $g \in \mathrm{CMO}_1^{\beta,pq/(q-p)} = \dot{F}_\infty^{\beta,pq/(q-p)}$.

The paper is organized as follows. In Section 2, we introduce the discrete version of the generalized Carleson measure spaces $c_r^{\alpha,q}$ and show that the duals of sequence Triebel-Lizorkin spaces $\dot{f}_p^{\alpha,q}$ for $1 < q < \infty$ and $0 < q \le 1$ are $c_{(q'/p)-(q'/q)}^{-\alpha,q'}$ and $c_r^{-\alpha+(n/p)-n,\infty}$ (for any $r \in \mathbb{R}$), respectively. In Section 3, we prove the duals of homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ for $1 < q < \infty$ and $0 < q \le 1$ to be the generalized Carleson measure spaces $\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$ and $\mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty}$ (for any $r \in \mathbb{R}$), respectively. In Section 4, we prove the Plancherel-Pôlya inequalities that give us the independence of the choice of $\varphi$ for the definition of the generalized Carleson measure spaces. In the last section, we show the boundedness of wavelet multipliers and paraproduct operators. Throughout, we use $C$ to denote a universal constant that does not depend on the main variables but may differ from line to line. Also, $Q$ and $P$ always mean the dyadic cubes in $\mathbb{R}^n$, and, for $r > 0$, we denote by $rQ$ the cube concentric with $Q$ whose each edge is $r$ times as long.

## 2. Sequence Spaces

In this section, we introduce sequence spaces $c_r^{\alpha,q}$ and then characterize the duals of $\dot{f}_p^{\alpha,q}$ by means of $c_r^{\alpha,q}$. Let us recall the definition of these sequence spaces $\dot{f}_p^{\alpha,q}$ defined in [2]. For $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, the space $\dot{f}_p^{\alpha,q}$ consists all such sequences $\mathbf{s} = \{s_Q\}_Q$ satisfying

$$\|\mathbf{s}\|_{\dot{f}_p^{\alpha,q}} := \begin{cases} \left\| \left\{ \sum_Q \left( |Q|^{-(\alpha/n)-(1/2)} |s_Q| \chi_Q \right)^q \right\}^{1/q} \right\|_{L^p} < \infty & \text{if } 0 < p < \infty, \\ \sup_P \left\{ |P|^{-1} \int_P \sum_{Q \subseteq P} \left( |Q|^{(-\alpha/n)-(1/2)} \, |s_Q| \chi_Q(x) \right)^q dx \right\}^{1/q} < \infty & \text{if } p = \infty. \end{cases} \tag{2.1}$$

As before, the previous $\ell^q$-norm is modified to the supremum norm for $0 < p < \infty$ and $q = \infty$. For $p = q = \infty$, we adopt the norm

$$\|\mathbf{s}\|_{\dot{f}_\infty^{\alpha,\infty}} := \sup_Q |Q|^{-(\alpha/n)-(1/2)} |s_Q|. \tag{2.2}$$

Note that $\|\mathbf{s}\|^q_{\dot{f}_\infty^{\alpha,q}}$ is equivalent to the Carleson norm of the measure

$$\sum_Q \left(|Q|^{-(\alpha/n)-(1/2)} |s_Q|\right)^q |Q| \delta_{(x_Q,\ell(Q))}, \tag{2.3}$$

where $\delta_{(x,t)}$ is the point mass at $(x,t) \in \mathbb{R}^{n+1}_+$. See [2] for the details.

To study the duals of $\dot{f}_p^{\alpha,q}$, we introduce a *discrete version of the generalized Carleson measure spaces* $c_r^{\alpha,q}$.

*Definition 2.1.* For $\alpha, r \in \mathbb{R}$ and $0 < q \le \infty$, the space $c_r^{\alpha,q}$ is the collection of all sequences $\mathbf{t} = \{t_Q\}_Q$ satisfying $\|\mathbf{t}\|_{c_r^{\alpha,q}} < \infty$, where

$$\|\mathbf{t}\|_{c_r^{\alpha,q}} := \begin{cases} \sup_P |P|^{-r} \int_P \sum_{Q\subseteq P} \left(|Q|^{-(\alpha/n)-(1/2)} |t_Q| \chi_Q(x)\right)^q dx & \text{for } 0 < q < \infty, \\ \sup_P \sup_{Q\subseteq P} |Q|^{-(\alpha/n)-(1/2)} |t_Q| = \sup_Q |Q|^{-(\alpha/n)-(1/2)} |t_Q| & \text{for } q = \infty. \end{cases} \tag{2.4}$$

It is obvious that

$$\|\mathbf{t}\|_{c_r^{\alpha,q}} = \sup_P \left\{ |P|^{-r} \sum_{Q\subseteq P} \left(|Q|^{-(\alpha/n)-(1/2)+(1/q)} |t_Q|\right)^q \right\}^{1/q} \quad \text{for } 0 < q < \infty \tag{2.5}$$

and $\|\mathbf{t}\|_{c_r^{\alpha,\infty}} = \|\mathbf{t}\|_{\dot{f}_\infty^{\alpha,\infty}}$ for $\alpha, r \in \mathbb{R}$. Using embedding theorem, Frazier and Jawerth [2, equation (5.14) and Theorem 5.9] obtained that, for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, the dual of $\dot{f}_p^{\alpha,q}$ is $\dot{f}_\infty^{-\alpha+(n/p)-n,\infty}$ when $0 < p < 1$, and the dual of $\dot{f}_1^{\alpha,q}$ is $\dot{f}_\infty^{-\alpha,q'}$. Note that $c_r^{\alpha,q} = \{0\}$ for $r < 0$ and $0 < q < \infty$. Here we give the dual relationship between sequence spaces $\dot{f}_p^{\alpha,q}$ and $c_r^{\alpha,q}$.

**Theorem 2.2** (duality for $\dot{f}_p^{\alpha,q}$). *Suppose that $\alpha \in \mathbb{R}$, $0 < p \le 1$, and $0 < q < \infty$.*

(a) *For $1 < q < \infty$, the dual of $\dot{f}_p^{\alpha,q}$ is $c^{-\alpha,q'}_{(q'/p)-(q'/q)}$ in the following sense.*

(i) *For $\mathbf{t} = \{t_Q\}_Q \in c^{-\alpha,q'}_{(q'/p)-(q'/q)}$, the linear functional $\ell_{\mathbf{t}}$ on $\dot{f}_p^{\alpha,q}$ given by $\ell_{\mathbf{t}}(\mathbf{s}) = \sum_Q s_Q t_Q$ is continuous with $\|\ell_{\mathbf{t}}\| \le C\|\mathbf{t}\|_{c^{-\alpha,q'}_{(q'/p)-(q'/q)}}$ for $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{\alpha,q}$.*

(ii) *Conversely, every continuous linear functional $\ell$ on $\dot{f}_p^{\alpha,q}$ satisfies $\ell = \ell_{\mathbf{t}}$ for some $\mathbf{t} \in c^{-\alpha,q'}_{(q'/p)-(q'/q)}$ with $\|\mathbf{t}\|_{c^{-\alpha,q'}_{(q'/p)-(q'/q)}} \le C\|\ell\|$.*

(b) *For $0 < q \le 1$, the dual of $\dot{f}_p^{\alpha,q}$ is $c_r^{-\alpha+(n/p)-n,\infty}$ (any $r \in \mathbb{R}$) in the following sense.*

(i) *For* $\mathbf{t} = \{t_Q\}_Q \in c_r^{-\alpha+(n/p)-n,\infty}$, *the linear functional* $\ell_{\mathbf{t}}$ *on* $\dot{f}_p^{\alpha,q}$ *given by* $\ell_{\mathbf{t}}(\mathbf{s}) = \sum_Q s_Q t_Q$ *is continuous with* $\|\ell_{\mathbf{t}}\| \le C\|\mathbf{t}\|_{c_r^{-\alpha+(n/p)-n,\infty}}$ *for* $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{\alpha,q}$.

(ii) *Conversely, every continuous linear functional* $\ell$ *on* $\dot{f}_p^{\alpha,q}$ *satisfies* $\ell = \ell_{\mathbf{t}}$ *for some* $\mathbf{t} \in c_r^{-\alpha+(n/p)-n,\infty}$ *with* $\|\mathbf{t}\|_{c_r^{-\alpha+(n/p)-n,\infty}} \le C\|\ell\|$.

*Remark 2.3.* For $\alpha \in \mathbb{R}$ and $0 < q < \infty$, sequence spaces $c_1^{\alpha,q} = \dot{f}_\infty^{\alpha,q}$ and $c_r^{\alpha,\infty} = \dot{f}_\infty^{\alpha,\infty}$ (for any $r \in \mathbb{R}$) by definitions. Theorem 2.2 shows that $(\dot{f}_1^{\alpha,q})' = \dot{f}_\infty^{-\alpha,q'}$, which gives a different but simpler proof of Frazier-Jawerth's result for the duality of $\dot{f}_1^{\alpha,q}$ (cf. [2, Theorem 5.9]).

*Proof of Theorem 2.2.* For $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{\alpha,q}$ and $\mathbf{t} = \{t_Q\}_Q \in c_r^{-\alpha,q'}$, set $\widetilde{\mathbf{s}} = \{\widetilde{s}_Q\}_Q$ and $\widetilde{\mathbf{t}} = \{\widetilde{t}_Q\}_Q$ to be

$$\widetilde{s}_Q = |Q|^{-\alpha/n} s_Q, \qquad \widetilde{t}_Q = |Q|^{\alpha/n} t_Q. \tag{2.6}$$

Then, $\ell_{\widetilde{\mathbf{t}}}(\widetilde{\mathbf{s}}) = \ell_{\mathbf{t}}(\mathbf{s})$. Also,

$$\|\widetilde{\mathbf{s}}\|_{\dot{f}_p^{0,q}} = \|\mathbf{s}\|_{\dot{f}_p^{\alpha,q}}, \qquad \left\|\widetilde{\mathbf{t}}\right\|_{c_r^{0,q'}} = \|\mathbf{t}\|_{c_r^{-\alpha,q'}}. \tag{2.7}$$

Without loss of generality, we may assume that $\alpha = 0$.

We first consider the case $1 < q < \infty$. Let $\mathbf{t} \in c^{0,q'}_{(q'/p)-(q'/q)}$ and define a linear functional $\ell_{\mathbf{t}}$ on $\dot{f}_p^{0,q}$ by

$$\ell_{\mathbf{t}}(s) = \sum_Q s_Q t_Q \quad \text{for } \mathbf{s} \in \dot{f}_p^{0,q}. \tag{2.8}$$

For $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{0,q}$, let

$$V_q(x) := \left(\sum_Q \left(|Q|^{-1/2}|s_Q|\chi_Q(x)\right)^q\right)^{1/q}. \tag{2.9}$$

For $k \in \mathbb{Z}$, let

$$\begin{aligned}
\Omega_k &: = \left\{x \in \mathbb{R}^n : 2^k < V_q(x) \le 2^{k+1}\right\},\\
\widetilde{\Omega}_k &: = \left\{x \in \mathbb{R}^n : M\chi_{\Omega_k}(x) > \frac{1}{2}\right\},\\
B_k &:= \left\{\text{dyadic } Q : \left|Q \cap \Omega_j\right| > \frac{|Q|}{2},\ \left|Q \cap \Omega_{j+1}\right| \le \frac{|Q|}{2} \text{ for some } j \ge k\right\},
\end{aligned} \tag{2.10}$$

where $M$ is the Hardy-Littlewood maximal function. Then, for each dyadic cube $Q$, there exists exactly a $k \in \mathbb{Z}$ such that $Q \in B_k$. For every $Q \in B_k$, let $\widetilde{Q}$ denote the maximal dyadic cube in $B_k$ containing $Q$. Then all of such $\widetilde{Q}$'s are pairwise disjoint. Thus, by Hölder's inequality for $q$ and the inequality $(a+b)^p \le a^p + b^p$ for $0 < p \le 1$,

$$\begin{aligned}
\left|\sum_Q s_Q t_Q\right| &\le \sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} \sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}} \left(|Q|^{-(1/2)+(1/q)}|s_Q|\right)\left(|Q|^{(1/2)-(1/q)}|t_Q|\right) \\
&\le \left\{\sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} \left(\sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}} \left(|Q|^{-(1/2)+(1/q)}|s_Q|\right)^q\right)^{p/q} \left(\sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}} \left(|Q|^{-(1/2)+(1/q')}|t_Q|\right)^{q'}\right)^{p/q'}\right\}^{1/p} \\
&\le \|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \left\{\sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} \left(\sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}} \left(|Q|^{-(1/2)+(1/q)}|s_Q|\right)^q\right)^{p/q} \left|\widetilde{Q}\right|^{1-(p/q)}\right\}^{1/p}.
\end{aligned} \tag{2.11}$$

Since $\widetilde{Q} \in B_k$ implies $\widetilde{Q} \subseteq \widetilde{\Omega}_k$, the disjointness of $\widetilde{Q}$'s and Hölder's inequality yield

$$\left|\sum_Q s_Q t_Q\right| \le \|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \left\{\sum_{k\in\mathbb{Z}} \left|\widetilde{\Omega}_k\right|^{1-(p/q)} \left(\sum_{Q\in B_k} \left(|Q|^{-(1/2)+(1/q)}|s_Q|\right)^q\right)^{p/q}\right\}^{1/p}. \tag{2.12}$$

We claim that $\sum_{Q\in B_k} (|Q|^{-(1/2)+(1/q)}|s_Q|)^q \le C2^{kq}|\widetilde{\Omega}_k|$ for $k \in \mathbb{Z}$ and $0 < q < \infty$. Assume the claim for the moment. The weak $(1,1)$ boundedness of $M$ gives $|\widetilde{\Omega}_k| \le C|\Omega_k|$, and hence

$$\begin{aligned}
\left|\sum_Q s_Q t_Q\right| &\le C\|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \left(\sum_{k\in\mathbb{Z}} \left|\widetilde{\Omega}_k\right|^{1-(p/q)} \left(2^{kq}\left|\widetilde{\Omega}_k\right|\right)^{p/q}\right)^{1/p} \\
&\le C\|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \left(\sum_{k\in\mathbb{Z}} 2^{kp}|\Omega_k|\right)^{1/p} \\
&\le C\|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \left\|V_q\right\|_{L^p} \\
&= C\|\mathbf{t}\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \|\mathbf{s}\|_{\dot{f}^{0,q}_p}.
\end{aligned} \tag{2.13}$$

To prove the claim, we note that, for $k \in \mathbb{Z}$ and $0 < q < \infty$,

$$\begin{aligned}
2^{q(k+1)}\left|\widetilde{\Omega}_k\right| &\ge \int_{\widetilde{\Omega}_k\setminus\bigcup_{j=k+1}^{\infty}\Omega_j} (V_q(x))^q dx \\
&= \int_{\widetilde{\Omega}_k\setminus\bigcup_{j=k+1}^{\infty}\Omega_j} \sum_Q \left(|Q|^{-1/2}|s_Q|\chi_Q(x)\right)^q dx \\
&\ge \sum_{Q\in B_k} \left(|Q|^{-1/2}|s_Q|\right)^q \left|\left(\widetilde{\Omega}_k \setminus \Omega_j\right)\cap Q\right| \quad \text{for some } j \ge k+1,
\end{aligned} \tag{2.14}$$

which implies

$$2^{q(k+1)}\left|\widetilde{\Omega}_k\right| \geq \frac{1}{2}\sum_{Q\in B_k}\left(|Q|^{-(1/2)+(1/q)}\left|s_Q\right|\right)^q. \tag{2.15}$$

For $0 < q \leq 1$, with a modification, we have

$$\begin{aligned}
\left|\sum_Q s_Q t_Q\right| &\leq \sum_{k\in\mathbb{Z}}\sum_{\widetilde{Q}\in B_k}\sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}}\left(|Q|^{1/2}\left|s_Q\right|\right)\left(|Q|^{-(1/p)+(1/2)}\left|t_Q\right|\right)\left(\frac{|Q|}{\left|\widetilde{Q}\right|}\right)^{(1/p)-1}\left|\widetilde{Q}\right|^{(1/p)-1}\\
&\leq \|\mathbf{t}\|_{c_r^{(n/p)-n,\infty}}\left\{\sum_{k\in\mathbb{Z}}\sum_{\widetilde{Q}\in B_k}\left(\sum_{\substack{Q\subseteq\widetilde{Q}\\ Q\in B_k}}|Q|^{1/2}\left|s_Q\right|\right)^p\left|\widetilde{Q}\right|^{1-p}\right\}^{1/p}\\
&\leq C\|\mathbf{t}\|_{c_r^{(n/p)-n,\infty}}\left(\sum_{k\in\mathbb{Z}}\left|\widetilde{\Omega}_k\right|^{1-p}\left(2^k\left|\widetilde{\Omega}_k\right|\right)^p\right)^{1/p}\\
&\leq C\|\mathbf{t}\|_{c_r^{(n/p)-n,\infty}}\left\|V_q\right\|_{L^p}\\
&\leq C\|\mathbf{t}\|_{c_r^{(n/p)-n,\infty}}\|\mathrm{s}\|_{\dot{f}_p^{0,q}}.
\end{aligned} \tag{2.16}$$

On the other hand, suppose that $\ell$ is a continuous linear functional on $\dot{f}_p^{0,q}$. For each dyadic cube $P$, write $\mathbf{e}^P = \{(\mathbf{e}^P)_Q\}_Q$ to be the sequence defined by

$$\left(\mathbf{e}^P\right)_Q = \begin{cases}1 & \text{if } Q = P,\\ 0 & \text{if } Q \neq P.\end{cases} \tag{2.17}$$

Let $t_P = \ell(\mathbf{e}^P)$ and $\mathbf{t} = \{t_P\}_P$. Then, for $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{0,q}$,

$$\ell(\mathbf{s}) = \sum_Q s_Q t_Q = \ell_{\mathbf{t}}(\mathbf{s}). \tag{2.18}$$

Fix a dyadic cube $P$. For $1 < q < \infty$, let $X$ be the sequence space consisting of $\mathbf{s} = \{s_Q\}_{Q\subseteq P}$, and define a counting measure on dyadic cubes $Q \subseteq P$ by $d\sigma(Q) = |Q|/|P|^{(q'/p)-(q'/q)}$. Then,

$$\begin{aligned}
&\left(\frac{1}{|P|^{(q'/p)-(q'/q)}}\sum_{Q\subseteq P}\left(|Q|^{-(1/2)+(1/q')}\left|t_Q\right|\right)^{q'}\right)^{1/q'}\\
&\quad= \sup_{\|\mathbf{s}\|_{\ell^q(X,d\sigma)}\leq 1}\left|\frac{1}{|P|^{(q'/p)-(q'/q)}}\sum_{Q\subseteq P}|Q|\ s_Q|Q|^{-1/2}|t_Q|\right|\\
&\quad\leq \|\ell\|\sup_{\|\mathbf{s}\|_{\ell^q(X,d\sigma)}\leq 1}\left\|\left\{\frac{s_Q|Q|^{1/2}}{|P|^{(q'/p)-(q'/q)}}\right\}_{Q\subseteq P}\right\|_{\dot{f}_p^{0,q}}.
\end{aligned} \tag{2.19}$$

Note that

$$\left\| \left\{ \frac{s_Q |Q|^{1/2}}{|P|^{(q'/p)-(q'/q)}} \right\}_{Q\subseteq P} \right\|_{\dot{f}_p^{0,q}} \leq \frac{1}{|P|^{(q'/p)-(q'/q)}} \left\{ \left( \sum_{Q\subseteq P} |Q| |s_Q|^q \right)^{p/q} \cdot |P|^{1-(p/q)} \right\}^{1/p} \leq C \|\mathbf{s}\|_{\ell^q(X, d\sigma)}. \tag{2.20}$$

Thus,

$$\left( \frac{1}{|P|^{(q'/p)-(q'/q)}} \sum_{Q\subseteq P} \left( |Q|^{(-1/2)+(1/q')} |t_Q| \right)^{q'} \right)^{1/q'} \leq C\|\ell\|, \tag{2.21}$$

and hence $\mathbf{t} \in c^{0,q'}_{(q'/p)-(q'/q)}$. For $0 < q \leq 1$, consider $\mathbf{e}^P$ defined before. Then, $\|\mathbf{e}^P\|_{\dot{f}_p^{0,q}} = |P|^{-(1/2)+(1/p)}$ and

$$\left( |P|^{(1/2)-(1/p)} |t_P| \right) \left\| \mathbf{e}^P \right\|_{\dot{f}_p^{0,q}} = |t_P| = \left| \ell\left( \mathbf{e}^P \right) \right| \leq \|\ell\| \left\| \mathbf{e}^P \right\|_{\dot{f}_p^{0,q}}. \tag{2.22}$$

Hence, $\|\mathbf{t}\|_{c_r^{(n/p)-n,\infty}} = \sup_P |P|^{(1/2)-(1/p)} |t_P| \leq \|\ell\|$. This completes the proof. □

## 3. Proof of the Main Theorem

Let us recall the $\varphi$-transform identity given by Frazier and Jawerth [2]. Choose a function $\varphi \in \mathcal{S}$ satisfying (1.2). Then there exists a function $\psi \in \mathcal{S}$ satisfying the same conditions as $\varphi$ such that $\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. The *$\varphi$-transform identity* is given by

$$f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q, \tag{3.1}$$

where the identity holds in the sense of $\mathcal{S}'/\mathcal{P}$, $\mathcal{S}_0$, and $\dot{F}_p^{\alpha,q}$-norm.

Define a linear map $S_\varphi$ from $\mathcal{S}'/\mathcal{P}$ into the family of complex sequences by

$$S_\varphi(f) = \{\langle f, \varphi_Q \rangle\}_Q, \tag{3.2}$$

and another linear map $T_\psi$ from the family of complex sequences into $\mathcal{S}'/\mathcal{P}$ by

$$T_\psi\left(\{s_Q\}_Q\right) = \sum_Q s_Q \psi_Q. \tag{3.3}$$

Then, $T_\psi \circ S_\varphi|_{\dot{F}_p^{\alpha,q}}$ is the identity on $\dot{F}_p^{\alpha,q}$ by [2, Theorem 2.2].

**Proposition 3.1.** *Suppose that $\alpha \in \mathbb{R}$ and, $0 < p, q \leq +\infty$, and $\varphi, \psi$ in $\mathcal{S}$ satisfy* (1.2) *and* (3.1). *The linear operators $S_\varphi : \dot{F}_p^{\alpha,q} \mapsto \dot{f}_p^{\alpha,q}$ and $T_\psi : \dot{f}_p^{\alpha,q} \mapsto \dot{F}_p^{\alpha,q}$ defined by* (3.2) *and* (3.3), *respectively, are*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\dot{f}_p^{\alpha,q} \arrow[rrr, dash, "\text{Dual relation by Theorem 2.2(a)}"] \arrow[d, shift left, "T_\varphi"] & & & c^{-\alpha,q'}_{(q'/p)-(q'/q)} \arrow[d, shift left, "T_\varphi"] \\
\dot{F}_p^{\alpha,q} \arrow[rrr, dash, "\text{Dual relation by Theorem 1.8(a)}"] \arrow[u, shift left, "S_\varphi"] & & & \mathrm{CMO}^{-\alpha,q'}_{(q'/p)-(q'/q)} \arrow[u, shift left, "S_\varphi"]
\end{tikzcd}$$

**Figure 1:** Diagram for spaces and maps for $1 < q < \infty$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\dot{f}_p^{\alpha,q} \arrow[rrr, dash, "\text{Dual relation by Theorem 2.2(b)}"] \arrow[d, shift left, "T_\varphi"] & & & c_r^{-\alpha+(n/p)-n,\infty} \arrow[d, shift left, "T_\varphi"] \\
\dot{F}_p^{\alpha,q} \arrow[rrr, dash, "\text{Dual relation by Theorem 1.8(b)}"] \arrow[u, shift left, "S_\varphi"] & & & \mathrm{CMO}_r^{-\alpha+(n/p)-n,\infty} \arrow[u, shift left, "S_\varphi"]
\end{tikzcd}$$

**Figure 2:** Diagram for spaces and maps for $0 < q \leq 1$.

*bounded. Furthermore, $T_\varphi \circ S_\varphi$ is the identity on $\dot{F}_p^{\alpha,q}$. In particular, $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \|S_\varphi(f)\|_{\dot{f}_p^{\alpha,q}}$ and $\dot{F}_p^{\alpha,q}$ can be identified with a complemented subspace of $\dot{f}_p^{\alpha,q}$.*

Figures 1 and 2 illustrate the relationship among $\dot{F}_p^{\alpha,q}$, $\dot{f}_p^{\alpha,q}$, $\mathrm{CMO}_r^{\alpha,q}$, and $c_r^{\alpha,q}$.

One recalls the almost diagonality given by Frazier and Jawerth [2]. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, let $J = n/(\min\{1, p, q\})$. One says that a matrix $A = \{a_{QP}\}_{Q,P}$ is $(\alpha, p, q)$-*almost diagonal* if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P} \frac{|a_{QP}|}{w_{QP}(\varepsilon)} < +\infty, \tag{3.4}$$

where

$$w_{QP}(\varepsilon) = \left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha} \left(1 + \frac{|x_Q - x_P|}{\max(\ell(P), \ell(Q))}\right)^{-J-\varepsilon} \cdot \min\left\{\left(\frac{\ell(Q)}{\ell(P)}\right)^{(n+\varepsilon)/2}, \left(\frac{\ell(P)}{\ell(Q)}\right)^{((n+\varepsilon)/2)+J-n}\right\}. \tag{3.5}$$

**Lemma 3.2.** *For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, an $(\alpha + nr, q, q)$-almost diagonal matrix is bounded on $c_r^{\alpha,q}$. Furthermore, when $r \geq 0$, an $(\alpha + nr, \infty, \infty)$-almost diagonal matrix is bounded on $c_r^{\alpha,\infty}$.*

We postpone the proof of Lemma 3.2 until the end of Section 4.

Let $\alpha, r \in \mathbb{R}$. For $q = \infty$, we have $c_r^{\alpha,\infty} = \dot{f}_\infty^{\alpha,\infty}$ and $\mathrm{CMO}_r^{\alpha,\infty} = \dot{F}_\infty^{\alpha,\infty}$. Thus, $S_\varphi : \mathrm{CMO}_r^{\alpha,\infty} \mapsto c_r^{\alpha,\infty}$ and $T_\psi : c_r^{\alpha,\infty} \mapsto \mathrm{CMO}_r^{\alpha,\infty}$ are bounded by Proposition 3.1. For $0 < q < \infty$ and $f \in \mathrm{CMO}_r^{\alpha,q}$, let $\mathbf{s} = \{s_Q\}_Q = S_\varphi(f)$. Then, the $\varphi$-transform identity (3.1) shows that $f = \sum_Q s_Q \psi_Q$ and $\|f\|_{\mathrm{CMO}_r^{\alpha,q}} = \|S_\varphi(f)\|_{c_r^{\alpha,q}} = \|\mathbf{s}\|_{c_r^{\alpha,q}}$. In particular, $\|f\|_{\mathrm{CMO}_1^{\alpha,q}} = \|S_\varphi(f)\|_{c_1^{\alpha,q}} = \|S_\varphi(f)\|_{\dot{f}_\infty^{\alpha,q}} \approx \|f\|_{\dot{F}_\infty^{\alpha,q}}$. Furthermore, for $\mathbf{s} \in c_r^{\alpha,q}$,

$$\|T_\psi(\mathbf{s})\|_{\mathrm{CMO}_r^{\alpha,q}} = \left\|\sum_P s_P \psi_P\right\|_{\mathrm{CMO}_r^{\alpha,q}} = \left\|\left\{\left\langle \sum_P s_P \psi_P, \varphi_Q \right\rangle\right\}_Q\right\|_{c_r^{\alpha,q}} = \|A\mathbf{s}\|_{c_r^{\alpha,q}}, \tag{3.6}$$

where $A := \{\langle \psi_P, \varphi_Q \rangle\}_{Q,P}$ is $(\alpha + nr, q, q)$-almost diagonal (cf. [2, Lemma 3.6]) and hence $A$ is bounded on $c_r^{\alpha,q}$ by Lemma 3.2. Therefore, $S_\varphi$ is bounded from $\mathrm{CMO}_r^{\alpha,q}$ to $c_r^{\alpha,q}$ and $T_\psi$ is bounded from $c_r^{\alpha,q}$ to $\mathrm{CMO}_r^{\alpha,q}$.

We summarize that $T_\psi \circ S_\varphi|_{\mathrm{CMO}_r^{\alpha,q}}$ is also the identity on $\mathrm{CMO}_r^{\alpha,q}$.

**Proposition 3.3.** *For $(\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ or $(\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times \{\infty\}$, the linear operators $S_\varphi : \mathrm{CMO}_r^{\alpha,q} \mapsto c_r^{\alpha,q}$ and $T_\psi : c_r^{\alpha,q} \mapsto \mathrm{CMO}_r^{\alpha,q}$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\mathrm{CMO}_r^{\alpha,q}$ and $\|f\|_{\mathrm{CMO}_r^{\alpha,q}} = \|S_\varphi f\|_{c_r^{\alpha,q}}$. In particular, $\|f\|_{\mathrm{CMO}_1^{\alpha,q}} = \|S_\varphi(f)\|_{c_1^{\alpha,q}} = \|S_\varphi(f)\|_{\dot{f}_\infty^{\alpha,q}} \approx \|f\|_{\dot{F}_\infty^{\alpha,q}}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, and $\|f\|_{\mathrm{CMO}_r^{\alpha,\infty}} = \|S_\varphi(f)\|_{c_r^{\alpha,\infty}} = \|S_\varphi(f)\|_{\dot{f}_\infty^{\alpha,\infty}} \approx \|f\|_{\dot{F}_\infty^{\alpha,\infty}}$ for $\alpha, r \in \mathbb{R}$.*

Theorem 1.8 can be proved as a consequence of Propositions 3.1–3.3 and a duality result between two sequence spaces.

*Proof of Theorem 1.8.* First let us consider the case for $1 < q < \infty$. Let $g \in \mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}$. Then, by Proposition 3.3, $\|g\|_{\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}} = \|S_\psi(g)\|_{c_{(q'/p)-(q'/q)}^{-\alpha,q'}}$. It follows from Theorem 2.2 that $\ell_{S_\psi(g)}$ is a continuous linear functional on $\dot{f}_p^{\alpha,q}$ and $\|\ell_{S_\psi(g)}\| \approx \|S_\psi(g)\|_{c_{(q'/p)-(q'/q)}^{-\alpha,q'}}$. Hence, for $f \in \mathcal{S}_0$,

$$|L_g(f)| \le C\|S_\psi(g)\|_{c_{(q'/p)-(q'/q)}^{-\alpha,q'}} \|S_\varphi(f)\|_{\dot{f}_p^{\alpha,q}} \le C\|g\|_{\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}} \|f\|_{\dot{F}_p^{\alpha,q}}. \tag{3.7}$$

Since $\mathcal{S}_0$ is dense in $\dot{F}_p^{\alpha,q}$, the functional $L_g$ can be extended to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ satisfying $\|L_g\| \le C\|g\|_{\mathrm{CMO}_{(q'/p)-(q'/q)}^{-\alpha,q'}}$.

Conversely, let $L \in (\dot{F}_p^{\alpha,q})'$, and set $\ell = L \circ T_\psi$ on $\dot{f}_p^{\alpha,q}$. By Proposition 3.1, $\ell \in (\dot{f}_p^{\alpha,q})'$. Thus, by Theorem 2.2, there exists $\mathbf{t} = \{t_Q\}_Q \in c_{(q'/p)-(q'/q)}^{-\alpha,q'}$ such that

$$\ell\Big(\{s_Q\}_Q\Big) = \sum_Q s_Q t_Q \quad \text{for } \{s_Q\}_Q \in \dot{f}_p^{\alpha,q}, \tag{3.8}$$

and $\|\mathbf{t}\|_{c_{(q'/p)-(q'/q)}^{-\alpha,q'}} \approx \|\ell\| \le C\|L\|$. For $f \in \dot{F}_p^{\alpha,q}$, we have

$$\ell \circ S_\varphi(f) = L \circ T_\psi \circ S_\varphi(f) = L(f). \tag{3.9}$$

So, for $f \in \mathcal{S}_0$ and letting $g = T_\psi(\mathbf{t}) = \sum_Q t_Q \psi_Q$,

$$L(f) = \ell \circ S_\varphi(f) = \sum_Q \langle f, \varphi_Q \rangle t_Q = \langle \mathbf{t}, S_\varphi(f) \rangle. \tag{3.10}$$

It follows from [2, equations (2.7)-(2.8)] that $\langle g, f \rangle = \langle S_\psi(g), S_\varphi(f) \rangle$ and $\langle \mathbf{t}, S_\varphi(f) \rangle = \langle T_\psi(\mathbf{t}), f \rangle$ for $f \in \mathcal{S}_0$ and $g \in \mathcal{S}'/\mathcal{P}$. This shows that $L(f) = \langle T_\psi(\mathbf{t}), f \rangle = L_g(f)$ for $f \in \mathcal{S}_0$. Proposition 3.3 and Theorem 2.2 give

$$\|g\|_{\mathrm{CMO}^{-\alpha,q'}_{(q'/p)-(q'/q)}} \le C\|\mathbf{t}\|_{c^{-\alpha,q'}_{(q'/p)-(q'/q)}} \le C\|L\|. \tag{3.11}$$

A similar argument gives the desired result for $0 < q \le 1$ with a slight modification, and hence the proof is finished. □

*Remark 3.4.* As pointed out by one of the referees, Yang and Yuan [8, Theorem 1] show that if $\tau > 1/p$ and $0 < p, q < \infty$, then $\dot{F}^{\alpha,\tau}_{p,q} = \dot{F}^{\alpha+n\tau-(n/p),\infty}_{\infty}$, where the definition of $\dot{F}^{\alpha,\tau}_{p,q}$ is given in Remark 1.3. Thus, for $0 < p < 1$ and $1 < q < \infty$,

$$\left(\dot{F}^{\alpha,q}_{p}\right)' = \dot{F}^{-\alpha+(n/p)-n,\infty}_{\infty} = \dot{F}^{-\alpha,(1/p)-(1/q)}_{q',q'} = \mathrm{CMO}^{-\alpha,q'}_{(q'/p)-(q'/q)}, \tag{3.12}$$

which demonstrates a different approach to the duality.

## 4. Proofs of the Plancherel-Pôlya Inequalities

In this section we demonstrate the Plancherel-Pôlya inequalities.

*Proof of Theorem 1.4.* Without loss of generality, we may assume that $\alpha = 0$. By (3.1), we rewrite $\widetilde{\phi}_j * f(u)$ as

$$\begin{aligned}\widetilde{\phi}_j * f(u) &= \sum_Q \langle f, \varphi_Q \rangle \int \widetilde{\phi}_j(u-x)\psi_Q(x)\,dx \\ &= \sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} |Q| \langle f, \varphi_k(\cdot - x_Q) \rangle \int \widetilde{\phi}_j(u-x)\psi_k(x-x_Q)\,dx.\end{aligned} \tag{4.1}$$

Using the inequality [2, page 151, equation (B.5)]

$$\left| \int \widetilde{\phi}_j(u-x)\psi_k(x-x_Q)\,dx \right| \le C 2^{-K|j-k|} \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |u - x_Q|\right)^{n+1}}, \tag{4.2}$$

where $j \wedge k = \min\{j,k\}$ and $K > 1 + nr$, we obtain

$$\left|\widetilde{\phi}_j * f(u)\right| \le C \sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} 2^{-K|j-k|}|Q| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |u - x_Q|\right)^{n+1}} \left|\widetilde{\varphi}_k * f(x_Q)\right|. \tag{4.3}$$

Thus, for $\ell(Q') = 2^{-j}$,

$$\begin{aligned}\left(\sup_{u\in Q'}\left|\widetilde{\phi}_j * f(u)\right|\right)^q &\le C\left(\sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} 2^{-K|j-k|}|Q| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |x_{Q'} - x_Q|\right)^{n+1}} \left|\widetilde{\varphi}_k * f(x_Q)\right|\right)^q \\ &\le C\sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} 2^{-K|j-k|}|Q| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |x_{Q'} - x_Q|\right)^{n+1}} \left|\widetilde{\varphi}_k * f(x_Q)\right|^q,\end{aligned} \tag{4.4}$$

where the last inequality is followed by Hölder's inequality and

$$\sum_{\substack{Q \\ \ell(Q)=2^{-k}}} |Q| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |x_{Q'} - x_Q|\right)^{n+1}} \le C. \tag{4.5}$$

Denote $T_Q$ by

$$T_Q := \inf_{u\in Q}\left|\widetilde{\varphi}_k * f(u)\right|^q. \tag{4.6}$$

Since $x_Q$ can be replaced by any point in $Q$ in the last inequality,

$$\left(\sup_{u\in Q'}\left|\widetilde{\phi}_j * f(u)\right|\right)^q \le C\sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} 2^{-K|j-k|}|Q| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |x_{Q'} - x_Q|\right)^{n+1}} T_Q. \tag{4.7}$$

Given a dyadic cube $P$ with $\ell(P) = 2^{-k_0}$, the above estimates yield

$$\begin{aligned}&\sum_{j=k_0}^{\infty} \sum_{\substack{Q'\subseteq P \\ \ell(Q')=2^{-j}}} \left(\sup_{u\in Q'}\left|\widetilde{\phi}_j * f(u)\right|\right)^q |Q'| \\ &\quad\le C\sum_{j=k_0}^{\infty} \sum_{\substack{Q'\subseteq P \\ \ell(Q')=2^{-j}}} \sum_{k\in\mathbb{Z}} \sum_{\substack{Q \\ \ell(Q)=2^{-k}}} 2^{-K|j-k|}|Q'| \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)} + |x_{Q'} - x_Q|\right)^{n+1}} T_Q|Q| \\ &\quad:= CA_1 + CA_2,\end{aligned} \tag{4.8}$$

where

$$A_1=\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k\ge k_0}\sum_{\substack{Q\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}|Q'|\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{Q'}-x_Q|\right)^{n+1}}T_Q|Q|,$$
$$A_2=\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k< k_0}\sum_{\substack{Q\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}|Q'|\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{Q'}-x_Q|\right)^{n+1}}T_Q|Q|. \tag{4.9}$$

Then, $A_1$ can be further decomposed as

$$\begin{aligned}A_1&=\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k\ge k_0}\sum_{\substack{Q\subseteq 3P\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}|Q'|\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{Q'}-x_Q|\right)^{n+1}}T_Q|Q|\\&\quad+\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k\ge k_0}\sum_{\substack{Q\cap 3P=\emptyset\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}|Q'|\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{Q'}-x_Q|\right)^{n+1}}T_Q|Q|\\&:=A_{11}+A_{12}.\end{aligned}\tag{4.10}$$

There are $3^n$ dyadic cubes in $3P$ with the same side length as $P$, so

$$\sum_{\substack{Q\subseteq 3P\\ \ell(Q)\le\ell(P)}}T_Q|Q|\le 3^n\sup_{\substack{P'\subseteq 3P\\ \ell(P')=\ell(P)}}\sum_{\substack{Q\subseteq P'\\ \ell(Q)\le\ell(P')}}T_Q|Q|. \tag{4.11}$$

Thus,

$$\begin{aligned}|P|^{-r}A_{11}&\le C|P|^{-r}\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k\ge k_0}\sum_{\substack{Q\subseteq 3P\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}|Q'|\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{Q'}-x_P|\right)^{n+1}}T_Q|Q|\\&\le C\sup_{P'}|P'|^{-r}\sum_{k=-\log_2\ell(P')}^{\infty}\sum_{\substack{Q\subseteq P'\\ \ell(Q)=2^{-k}}}\inf_{u\in Q}|\tilde{\varphi}_k*f(u)|^q|Q|.\end{aligned}\tag{4.12}$$

Next we decompose the set of dyadic cubes $\{Q:Q\cap 3P=\emptyset,\ \ell(Q)=\ell(P)\}$ into $\{B_i\}_{i\in\mathbb{N}}$ according to the distance between each $Q$ and $P$. Namely, for each $i\in\mathbb{N}$,

$$B_i:=\left\{P':P'\cap 3P=\emptyset,\ \ell(P)=\ell(P'),\ 2^{i-k_0}\le|y_{P'}-y_P|<2^{i-k_0+1}\right\}, \tag{4.13}$$

where $y_Q$ denotes the center of $Q$. Then, we obtain

$$|P|^{-r}A_{12} \le C\sum_{i=1}^{\infty}\sum_{P'\in B_i}\left|P'\right|^{-r}\sum_{j=k_0}^{\infty}\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}\sum_{k\ge k_0}\sum_{\substack{Q\subseteq P'\\ \ell(Q)=2^{-k}}}2^{-K|j-k|}\left|Q'\right| \times \frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+|x_{P'}-x_P|\right)^{n+1}}T_Q|Q|. \tag{4.14}$$

Since $\sum_{\substack{Q'\subseteq P\\ \ell(Q')=2^{-j}}}|Q'| = |P|$ for each $j \ge k_0$ and $|x_{P'}-x_P| \approx 2^{i-k_0}$ for $P' \in B_i$, the right-hand side of (4.14) is dominated by

$$C\sum_{i=1}^{\infty}\sum_{P'\in B_i}|P|\frac{2^{-k_0}}{2^{(i-k_0)(n+1)}}\sum_{k\ge k_0}\left(\sum_{j=k_0}^{\infty}2^{k_0-(j\wedge k)-|k-j|}\right)\left(\left|P'\right|^{-r}\sum_{\substack{Q\subseteq P'\\ \ell(Q)=2^{-k}}}T_Q|Q|\right). \tag{4.15}$$

There are at most $2^{(i+2)n}$ cubes in $B_i$, and hence

$$\begin{aligned}|P|^{-r}A_{12} &\le C\left\{\sup_{P'}\left|P'\right|^{-r}\sum_{k\ge k_0}\sum_{\substack{Q\subseteq P'\\ \ell(Q)=2^{-k}}}T_Q|Q|\right\}\sum_{i=1}^{\infty}|P|\frac{2^{-k_0}}{2^{(i-k_0)(n+1)}}2^{in}\\ &= C\sup_{P'}\left|P'\right|^{-r}\sum_{k=-\log_2\ell(P')}^{\infty}\sum_{\substack{Q\subseteq P'\\ \ell(Q)=2^{-k}}}\inf_{u\in Q}\left|\tilde{\varphi}_k * f(u)\right|^q|Q|.\end{aligned} \tag{4.16}$$

To estimate $A_2$, for $i \in \mathbb{N}$ and $k < k_0$, set

$$E_{i,k} := \left\{Q : \ell(Q) = 2^{-k},\quad x_Q \in 2^iP \setminus 2^{i-1}P\right\}. \tag{4.17}$$

Then, $|x_Q - x_P| \approx 2^{i-k_0}$ for $Q \in E_{i,k}$ and

$$A_2 = \sum_{j=k_0}^{\infty}\sum_{k<k_0}\sum_{i=1}^{\infty}\sum_{Q\in E_{i,k}}\frac{2^{-K|j-k|}|P|}{|Q|^{-r}}\frac{2^{-(j\wedge k)}}{\left(2^{-(j\wedge k)}+\left|x_Q-x_P\right|\right)^{n+1}}|Q|^{-r}T_Q|Q|. \tag{4.18}$$

Since, for $Q \in E_{i,k}$,

$$|Q|^{-r}T_Q|Q| \le \sup_{P'}\left|P'\right|^{-r}\sum_{m=-\log_2\ell(P')}^{\infty}\sum_{\substack{Q'\subseteq P'\\ \ell(Q')=2^{-m}}}T_{Q'}\left|Q'\right| \tag{4.19}$$

and the number of dyadic cubes contained in $E_{i,k}$ is at most $2^{(i+k-k_0)n}$,

$$\begin{aligned}
|P|^{-r} A_2 &\leq C\left\{\sup_{P'} |P'|^{-r} \sum_{m=-\log_2 \ell(P')}^{\infty} \sum_{\substack{Q' \subseteq P' \\ \ell(Q')=2^{-m}}} T_{Q'} |Q'|\right\} \\
&\quad \times \sum_{j=k_0}^{\infty} \sum_{k<k_0} \sum_{i=1}^{\infty} 2^{(k_0-k)nr} 2^{K(k-j)} \frac{2^{-k(n+1)}}{2^{(i-k_0)(n+1)}} 2^{(i+k-k_0)n} \\
&= C \sup_{P'} |P'|^{-r} \sum_{m=-\log_2 \ell(P')}^{\infty} \sum_{\substack{Q' \subseteq P' \\ \ell(Q')=2^{-m}}} \inf_{u \in Q'} |\tilde{\varphi}_m * f(u)|^q |Q'|,
\end{aligned} \tag{4.20}$$

where the condition $K > 1 + nr$ is used in the last equality. Combining the estimates of $A_1$ and $A_2$, we prove Theorem 1.4. □

By modifying the proof above, we may easily show Theorem 1.5. Detailed verifications are left to the reader.

We now return to show Lemma 3.2.

*Proof of Lemma 3.2.* For $r < 0$, $c_r^{\alpha,q} = \{0\}$, and hence the result holds. For $r = 0$, $c_0^{\alpha,q} = \dot{f}_q^{\alpha,q}$, and so the matrix is bounded by [2, Theorem 3.3]. To complete the proof, it suffices to show the boundedness of $(\alpha + nr, q, q)$-almost diagonal matrices for the case $r > 0$.

We may assume that $\alpha = 0$ since the case implies the general case. The proof is similar to the proof of Theorem 1.4. Here, we only outline the proof. First let us consider the case for $q > 1$. Let $A = \{a_{QP}\}_{Q,P}$ be an $(nr, q, q)$-almost diagonal matrix. Then, for $\ell(Q) = 2^{-k}$,

$$\begin{aligned}
\left|(A\mathbf{s})_Q\right| &\leq C \sum_{j\in\mathbb{Z}} \sum_{\ell(P)=2^{-j}} 2^{(j-k)(nr+((n+\varepsilon)/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\varepsilon} |s_P|, \\
\left(|Q|^{-1/2} \left|(A\mathbf{s})_Q\right|\right)^q &\leq C \sum_{j\in\mathbb{Z}} \sum_{\ell(P)=2^{-j}} 2^{(j-k)(nr+(\varepsilon/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\varepsilon} \left(|P|^{-1/2} |s_P|\right)^q
\end{aligned} \tag{4.21}$$

due to Hölder's inequality. Given a dyadic cube $R$ with $\ell(R) = 2^{-\delta}$,

$$\sum_{k\geq\delta} \sum_{\substack{Q\subseteq R \\ \ell(Q)=2^{-k}}} \left(|Q|^{-1/2} \left|(A\mathbf{s})_Q\right|\right)^q |Q| \leq CI + CII, \tag{4.22}$$

where

$$\begin{aligned}
I &= \sum_{k\geq\delta} \sum_{\substack{Q\subseteq R \\ \ell(Q)=2^{-k}}} \sum_{j\geq\delta} \sum_{\ell(P)=2^{-j}} 2^{(j-k)(nr+n+(\varepsilon/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\varepsilon} \left(|P|^{-1/2} |s_P|\right)^q |P|, \\
II &= \sum_{k\geq\delta} \sum_{\substack{Q\subseteq R \\ \ell(Q)=2^{-k}}} \sum_{j<\delta} \sum_{\ell(P)=2^{-j}} 2^{(j-k)(nr+n+(\varepsilon/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\varepsilon} \left(|P|^{-1/2} |s_P|\right)^q |P|.
\end{aligned} \tag{4.23}$$

Then, $I$ can be further decomposed as

$$\begin{aligned} I = &\sum_{k\geq\delta}\sum_{\substack{Q\subseteq R\\ \ell(Q)=2^{-k}}}\sum_{j\geq\delta}\sum_{\substack{P\subseteq 3R\\ \ell(P)=2^{-j}}} 2^{(j-k)(nr+n+(\varepsilon/2))}\left(1+2^j\left|x_Q-x_P\right|\right)^{-n-\varepsilon}\left(|P|^{-1/2}|s_P|\right)^q|P| \\ &+\sum_{k\geq\delta}\sum_{\substack{Q\subseteq R\\ \ell(Q)=2^{-k}}}\sum_{j\geq\delta}\sum_{\substack{P\cap 3R=\emptyset\\ \ell(P)=2^{-j}}} 2^{(j-k)(nr+n+(\varepsilon/2))}\left(1+2^j\left|x_Q-x_P\right|\right)^{-n-\varepsilon}\left(|P|^{-1/2}|s_P|\right)^q|P| \\ &:= I_{11}+I_{12}. \end{aligned} \tag{4.24}$$

The same argument showed in the proof of Theorem 1.4 for the term $A_1$ gives us

$$|R|^{-r}I \leq C\|\mathbf{s}\|^q_{c_r^{0,q}}. \tag{4.25}$$

To estimate $II$, for $i\in\mathbb{N}$ and $j<\delta$, let

$$E_{i,j} := \left\{Q : \ell(Q)=2^{-j},\quad x_Q\in 2^iR\setminus 2^{i-1}R\right\}. \tag{4.26}$$

Then, using the same argument as Theorem 1.4 for $A_2$, we have

$$|R|^{-r}II \leq C\|s\|^q_{c_r^{0,q}}. \tag{4.27}$$

Both estimates for $I$ and $II$ show the desired result for $q>1$.

When $q\leq 1$, we modify the previous proof by replacing Hölder's inequality with $q$-triangle inequality to get the result.

When $q=\infty$ and $r\geq 0$, the space $c_r^{\alpha,\infty} = \dot{f}_\infty^{\alpha,\infty}$, and hence an $(\alpha+nr,\infty,\infty)$-almost diagonal matrix is bounded on $c_r^{\alpha,\infty}$ by Proposition 5.3. □

*Remark 4.1.* Note that $c_1^{\alpha,q} = \dot{f}_\infty^{\alpha,q}$. By a duality argument and [2, Theorem 3.3 and page 81], one can show that the $(\alpha+n,q,q)$-almost diagonal matrix is bounded on $\dot{f}_\infty^{\alpha,q}$. When $q>1$ and $r>1$, we can prove Lemma 3.2 by duality in Theorem 2.2. Let $A=\{a_{QP}\}_{Q,P}$ be an $(nr,q,q)$-almost diagonal matrix. Also define the transpose of $A$ by $A'=\{\overline{a_{PQ}}\}_{Q,P}$. For $q>1$ and $r>1$, let $p=(q+q')/(q'r+q)$. Then, $p<1$. Since $A$ is $(nr,q,q)$-almost diagonal, $A'$ is $(0,p,q')$-almost diagonal by a calculation for a different value of $\varepsilon$. Thus, by Theorem 2.2 (a) and Proposition 5.3, $A'$ is bounded on $c_r^{0,q}$.

## 5. Applications

We define another wavelet multiplier on $\mathbb{R}^n$ by using $\varphi$-transform identity as follows. Let $\varphi$ and $\psi$ in $\mathcal{S}$ satisfy (1.2) and (3.1). For a sequence $\mathbf{t}=\{t_Q\}_Q$, where the $Q's$ are dyadic cubes in $\mathbb{R}^n$, define the *wavelet multiplier* $T_\mathbf{t}$ by

$$T_\mathbf{t}(f) = \sum_Q |Q|^{-1/2}t_Q\langle f,\varphi_Q\rangle\psi_Q \tag{5.1}$$

for $f \in \mathcal{S}'/\mathcal{P}$ such that the above summation is well defined. Thus, we have the following characterization.

**Theorem 5.1.** *Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \le 1$, and $0 < q < \infty$. Then,*

(a) *for $1 < q < \infty$, $T_{\mathbf{t}}$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ if $\mathbf{t} \in c^{\beta,q'}_{(q'/p)-(q'/q)}$,*

(b) *for $0 < q \le 1$ and $r \in \mathbb{R}$, $T_{\mathbf{t}}$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$ if $\mathbf{t} \in c_r^{\beta+(n/p)-n,\infty}$.*

*Proof.* We show the case $\alpha = 0$ only, which implies the general case by (2.7). For $\beta \in \mathbb{R}$, $0 < p \le 1$, and $1 < q < \infty$, let $f \in \dot{F}_p^{0,q}$ and $\mathbf{t} \in c^{\beta,q'}_{(q'/p)-(q'/q)}$. It follows from Theorem 2.2 and Proposition 3.1 that

$$\begin{aligned}
\|T_{\mathbf{t}}(f)\|_{\dot{F}_1^{\beta,1}} &\le C \left\| \left\{ |Q|^{-1/2} t_Q \langle f, \varphi_Q \rangle \right\}_Q \right\|_{\dot{f}_1^{\beta,1}} \\
&= C \sum_Q \left( |Q|^{-\beta/n} |t_Q| \right) |\langle f, \varphi_Q \rangle| \\
&\le C \left\| \{ \langle f, \varphi_Q \rangle \}_Q \right\|_{\dot{f}_p^{0,q}} \left\| \left\{ |Q|^{-\beta/n} t_Q \right\}_Q \right\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \\
&\le C \|f\|_{\dot{F}_p^{0,q}} \|t\|_{c^{\beta,q'}_{(q'/p)-(q'/q)}}.
\end{aligned} \tag{5.2}$$

This shows that $T_{\mathbf{t}}$ is bounded from $\dot{F}_p^{0,q}$ into $\dot{F}_1^{\beta,1}$ and $\|T_{\mathbf{t}}\| \le C \|\mathbf{t}\|_{c^{\beta,q'}_{(q'/p)-(q'/q)}}$. A similar argument yields the boundedness of $T_{\mathbf{t}}$ for the case $0 < q \le 1$. □

In order to prove Theorem 1.12, we demonstrate a similar result in sequence spaces first. For a sequence $\mathbf{t} = \{t_Q\}_Q$, define $D_{\mathbf{t}}$ by

$$D_{\mathbf{t}}(\mathbf{s}) = \left\{ |Q|^{-1/2} t_Q s_Q \right\}_Q \quad \text{for } \mathbf{s} = \{s_Q\}_Q \text{ with finitely many nonzero terms.} \tag{5.3}$$

**Theorem 5.2.** *Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \le 1$, and $0 < q < \infty$. Then,*

(a) *for $1 < q < \infty$, $D_{\mathbf{t}}$ is extendible to be bounded from $\dot{f}_p^{\alpha,q}$ into $\dot{f}_1^{\alpha+\beta,1}$ if and only if $\mathbf{t} \in c^{\beta,q'}_{(q'/p)-(q'/q)}$,*

(b) *for $0 < q \le 1$ and $r \in \mathbb{R}$, $D_{\mathbf{t}}$ is extendible to be bounded from $\dot{f}_p^{\alpha,q}$ into $\dot{f}_1^{\alpha+\beta,1}$ if and only if $\mathbf{t} \in c_r^{\beta+(n/p)-n,\infty}$.*

*Proof.* We still assume that $\alpha = 0$. For $\beta \in \mathbb{R}$, $0 < p \le 1$, and $1 < q < \infty$, let $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_p^{0,q}$ and $\mathbf{t} = \{t_Q\}_Q \in c^{\beta,q'}_{(q'/p)-(q'/q)}$. It follows from Theorem 2.2 that

$$\begin{aligned} \|D_{\mathbf{t}}(\mathbf{s})\|_{\dot{f}_1^{\beta,1}} &= \sum_Q \left(|Q|^{-\beta/n}|t_Q|\right)|s_Q| \\ &\le C\|\mathbf{s}\|_{\dot{f}_p^{0,q}} \left\| \left\{ |Q|^{-\beta/n} t_Q \right\}_Q \right\|_{c^{0,q'}_{(q'/p)-(q'/q)}} \\ &= C\|\mathbf{s}\|_{\dot{f}_p^{0,q}} \|\mathbf{t}\|_{c^{\beta,q'}_{(q'/p)-(q'/q)}}. \end{aligned} \tag{5.4}$$

Conversely, suppose that $D_{\mathbf{t}}$ maps from $\dot{f}_p^{0,q}$ into $\dot{f}_1^{\beta,1}$ boundedly. For $\mathbf{t} = \{t_Q\}_Q$, let $\widetilde{\mathbf{t}} = \{|Q|^{-\beta/n} t_Q\}_Q$. Define a linear functional $\ell_{\widetilde{\mathbf{t}}}$ by

$$\ell_{\widetilde{\mathbf{t}}}(\mathbf{s}) = \sum_Q s_Q \widetilde{t}_Q \quad \text{for } \mathbf{s} = \{s_Q\}_Q \text{ with finitely many nonzero terms.} \tag{5.5}$$

Then,

$$|\ell_{\widetilde{\mathbf{t}}}(\mathbf{s})| \le \sum_Q \left(|Q|^{-\beta/n}|t_Q|\right)|s_Q| = \|D_{\mathbf{t}}(\mathbf{s})\|_{\dot{f}_1^{\beta,1}}. \tag{5.6}$$

The assumption shows that $\ell_{\widetilde{\mathbf{t}}}$ is a continuous linear functional on $\dot{f}_p^{0,q}$. Using Theorem 2.2, we have $\widetilde{\mathbf{t}} \in c^{0,q'}_{(q'/p)-(q'/q)}$, and hence $\mathbf{t} \in c^{\beta,q'}_{(q'/p)-(q'/q)}$.

For $0 < q \le 1$, a similar argument gives the desired result of (b). □

*Proof of Theorem 1.12.* The "if" part follows from Theorem 5.1. To show the "only if" part, define $\widetilde{T}^i_{\mathbf{t}}$ by

$$\widetilde{T}^i_{\mathbf{t}}(f) = \sum_Q |Q|^{-1/2} t_Q \left\langle f, \varphi^i_Q \right\rangle \varphi^i_Q. \tag{5.7}$$

The boundedness of $\widetilde{T}_{\mathbf{t}}$ says that $\widetilde{T}^i_{\mathbf{t}}$ is bounded from $\dot{F}_p^{\alpha,q}$ into $\dot{F}_1^{\alpha+\beta,1}$. Clearly,

$$S_{\psi^i} \circ \widetilde{T}^i_{\mathbf{t}} \circ T_{\psi^i}(\mathbf{s}) = D_{\mathbf{t}}(\mathbf{s}) \quad \text{for } \mathbf{s} \in \dot{f}_p^{\alpha,q}. \tag{5.8}$$

It follows from Proposition 3.1 that $D_{\mathbf{t}}$ is bounded from $\dot{f}_p^{\alpha,q}$ into $\dot{f}_1^{\alpha+\beta,1}$, and hence $\mathbf{t} \in c^{\beta,q'}_{(q'/p)-(q'/q)}$ for $1 < q < \infty$ and $\mathbf{t} \in c_r^{\beta+(n/p)-n,\infty}$ for $0 < q \le 1$ and $r \in \mathbb{R}$ by Theorem 5.2. □

In order to study the boundedness of the paraproduct operators acting on Triebel-Lizorkin spaces, we need more results described as follows.

**Proposition 5.3** ([2, pages 54 and 81]). *For $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, an $(\alpha, p, q)$-almost diagonal matrix is bounded on $\dot{f}_p^{\alpha,q}$.*

**Lemma 5.4.** *Define a matrix by* $G = \{\langle \psi_P, \Phi_Q\rangle\}_{Q,P}$. *Then, for* $\alpha < 0$ *and* $0 < p, q \le +\infty$, $G$ *is* $(\alpha, p, q)$*-almost diagonal and hence is bounded on* $\dot{f}_p^{\alpha,q}$.

*Proof.* For $\ell(P) \le \ell(Q)$, since $\int x^\gamma \psi_P(x)dx = 0$ for all $\gamma$, by [2, page 150, Lemma B.1], we have

$$\left|\langle \psi_P, \Phi_Q\rangle\right| \le C\left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha}\left(1+\frac{|x_Q - x_P|}{\ell(Q)}\right)^{-J-\varepsilon}\left(\frac{\ell(P)}{\ell(Q)}\right)^{((n+\varepsilon)/2)+J-n} \tag{5.9}$$

for $\varepsilon > 0$ and $\alpha < J - n + (\varepsilon/2)$, where $J = n/\min\{1, p, q\}$ and $C$ is independent of $P$ and $Q$. For $\ell(Q) < \ell(P)$, by [2, page 152, Lemma B.2], we obtain

$$\begin{aligned}\left|\langle \psi_P, \Phi_Q\rangle\right| &\le C\left(1+\frac{|x_Q - x_P|}{\ell(P)}\right)^{-J-\varepsilon}\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\\ &= C\left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha}\left(1+\frac{|x_Q - x_P|}{\ell(P)}\right)^{-J-\varepsilon}\left(\frac{\ell(Q)}{\ell(P)}\right)^{(n-2\alpha)/2}.\end{aligned} \tag{5.10}$$

Choosing $\varepsilon = -2\alpha$, we obtain the result. □

We now can prove Theorem 1.13.

*Proof of Theorem 1.13.* To simplify notations, let $q_0 = qr/(q-r)$ and $(1/p_0) = (1/p) - (1/q) + (1/q_0')$ . The requirement $p \le r < q < r/(1-r)$ guarantees that $p_0 \le 1 \le q_0$. Now assume that $g \in \mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q_0')}$ and $f \in \dot{F}_p^{\alpha,q}$. To prove part (i), by (3.1) we rewrite $\Pi_g(f)$ as

$$\begin{aligned}\Pi_g(f) &= \sum_Q \langle g, \varphi_Q\rangle |Q|^{-1/2}\left\langle \sum_P \langle f, \varphi_P\rangle \psi_P, \Phi_Q\right\rangle \varphi_Q\\ &= \sum_Q \langle g, \varphi_Q\rangle |Q|^{-1/2}(G\mathbf{s})_Q \varphi_Q,\end{aligned} \tag{5.11}$$

where $\mathbf{s} = \{\langle f, \varphi_p\rangle\}_P$. Proposition 3.1 and Theorem 2.2 give

$$\begin{aligned}\|\Pi_g(f)\|^r_{\dot{F}_r^{\alpha+\beta,r}} &\le C\left\|\left\{|Q|^{-1/2}\ \langle g, \varphi_Q\rangle\ (G\mathbf{s})_Q\right\}_Q\right\|^r_{\dot{f}_r^{\alpha+\beta,r}}\\ &= C\sum_Q\left(|Q|^{-(\beta/n)-(1/2)+(1/2r)}\left|\langle g, \varphi_Q\rangle\right|\right)^r\cdot\left(|Q|^{-(\alpha/n)-(1/2)+(1/2r)}\left|(G\mathbf{s})_Q\right|\right)^r\\ &\le C\left\|\left\{\left(|Q|^{-(\beta/n)-(1/2)+(1/2r)}\left|\langle g, \varphi_Q\rangle\right|\right)^r\right\}_Q\right\|_{c^{0,(q/r)'}_{((q/r)'/(p/r))-((q/r)'/(q/r))}}\\ &\quad\times\left\|\left\{\left(|Q|^{-(\alpha/n)-(1/2)+(1/2r)}\left|(G\mathbf{s})_Q\right|\right)^r\right\}_Q\right\|_{\dot{f}_{p/r}^{0,q/r}}.\end{aligned} \tag{5.12}$$

It is clear that

$$\begin{aligned}
&\left\|\left\{\left(|Q|^{-(\beta/n)-(1/2)+(1/2r)}|\langle g,\varphi_Q\rangle|\right)^r\right\}_Q\right\|_{c^{0,(q/r)'}_{((q/r)'/(p/r))-((q/r)'/(q/r))}}\\
&\quad=\sup_P\left\{|P|^{-r(q/r)'((1/p)-(1/q))}\int_P\sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g,\varphi_Q\rangle|\chi_Q(x)\right)^{r(q/r)'}dx\right\}^{1/(q/r)'}\\
&\quad=\left\|\{\langle g,\varphi_Q\rangle\}_Q\right\|^r_{c^{\beta,q_0}_{(q_0/p_0)-(q_0/p'_0)}}
\end{aligned}\tag{5.13}$$

and

$$\begin{aligned}
&\left\|\left\{\left(|Q|^{-(\alpha/n)-(1/2)+(1/2r)}(G\mathbf{s})_Q\right)^r\right\}_Q\right\|_{\dot f^{0,(q/r)}_{p/r}}\\
&=\left\|\left(\sum_Q\left(|Q|^{-(\alpha/n)-(1/2)}\left|(G\mathbf{s})_Q\right|\right)^q\chi_Q(x)\right)^{r/q}\right\|_{L^{p/r}}\\
&=\|G\mathbf{s}\|^r_{\dot f^{\alpha,q}_p}.
\end{aligned}\tag{5.14}$$

Hence, by Propositions 3.1 and 3.3, and Lemma 5.4,

$$\begin{aligned}
\left\|\Pi_g(f)\right\|_{\dot F^{\alpha+\beta,r}_r}&\le C\left\|\{\langle g,\varphi_Q\rangle\}_Q\right\|_{c^{\beta,q_0}_{(q_0/p_0)-(q_0/q'_0)}}\|G\mathbf{s}\|_{\dot f^{\alpha,q}_p}\\
&\le C\|g\|_{\mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q'_0)}}\|\mathbf{s}\|_{\dot f^{\alpha,q}_p}\\
&\le C\|g\|_{\mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q'_0)}}\|f\|_{\dot F^{\alpha,q}_p}.
\end{aligned}\tag{5.15}$$

Next suppose that $\Pi_g$ is bounded from $\dot F^{\alpha,q}_p$ into $\dot F^{\alpha+\beta,r}_r$. Without lost of generality, we may assume that $\alpha=0$. A computation yields

$$\begin{aligned}
&\left(|P|^{-q_0((1/p_0)+(1/q_0)\,-1)}\int_P\sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g,\varphi_Q\rangle|\chi_Q(x)\right)^{q_0}dx\right)^{1/q_0}\\
&\quad=|P|^{-(1/p)+(1/q)}\left(\int_P\sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g,\varphi_Q\rangle|\chi_Q(x)\right)^{qr/(q-r)}dx\right)^{(q-r)/qr}\\
&\quad\le C|P|^{-(1/p)}\left(\int_P\sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g,\varphi_Q\rangle|\chi_Q(x)\right)^{r}dx\right)^{1/r}.
\end{aligned}\tag{5.16}$$

Fix an integer $N > (n/p) - n$. Choose a function $\theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\theta(x) = 1$ on $[0,1]^n$, $\theta(x) = 0$ if $x \notin 3[0,1]^n$ and $\int x^\gamma \theta(x)dx = 0$ for all multi-indices $\gamma$ with $|\gamma| \le N$. By the molecular theory [2, page 56], it follows that $\theta \in \dot{F}_p^{0,q}$. For each dyadic cube $P$, define $\theta^P$ by

$$\theta^P(x) = \theta\left(\frac{x - x_P}{\ell(P)}\right). \tag{5.17}$$

Then, $\langle \theta^P, \Phi_Q \rangle = \int \Phi_Q(x)dx = |Q|^{1/2}$ for all dyadic cubes $Q \subseteq P$ and $\|\theta^P\|_{\dot{F}_p^{0,q}} = C|P|^{1/p}$ by the translation invariance and the dilation properties of $\dot{F}_p^{0,q}$. By Proposition 3.1,

$$\begin{aligned}\left\|\Pi_g(\theta^P)\right\|_{\dot{F}_r^{\beta,r}} &\approx \left\|\left\{\langle g, \varphi_Q\rangle |Q|^{-1/2}\left\langle \theta^P, \Phi_Q\right\rangle\right\}_Q\right\|_{\dot{f}_r^{\beta,r}} \\ &\ge \left(\int_P \sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g, \varphi_Q\rangle|\chi_Q(x)\right)^r dx\right)^{1/r},\end{aligned} \tag{5.18}$$

and hence, by the boundedness of $\Pi_g$,

$$\left(|P|^{-q_0((1/p_0)+(1/q_0)-1)}\int_P \sum_{Q\subseteq P}\left(|Q|^{-(\beta/n)-(1/2)}|\langle g, \varphi_Q\rangle|\chi_Q(x)\right)^{q_0} dx\right)^{1/q_0} \le C. \tag{5.19}$$

Taking the supremum on $P$, we show that $g \in \mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q_0')}$.

To prove part (ii), assume that $g \in \mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q_0')}$ and $f \in \dot{F}_p^{\alpha,q}$. Let $\mathbf{t} = \{\langle g, \varphi_Q\rangle\}_Q$ and $\mathbf{s} = \{\langle f, \psi_Q\rangle\}_Q$. By Proposition 3.1,

$$\left\|\Pi_g^*(f)\right\|_{\dot{F}_r^{\alpha+\beta,r}} \approx \left\|\left\{\sum_P |P|^{-1/2}\langle g, \varphi_P\rangle\langle \Phi_P, \varphi_Q\rangle\langle f, \psi_P\rangle\right\}_Q\right\|_{\dot{f}_r^{\alpha+\beta,r}} = \left\|\widetilde{G}D_{\mathbf{t}}\mathbf{s}\right\|_{\dot{f}_r^{\alpha+\beta,r}}, \tag{5.20}$$

where $\widetilde{G} := \{\langle \Phi_P, \varphi_Q\rangle\}_{Q,P}$ is the transpose of $\{\langle \varphi_P, \Phi_Q\rangle\}_{Q,P}$. Since $\alpha + \beta > 0$, by Lemma 5.4, $\widetilde{G}$ is $(\alpha+\beta, r, r)$-almost diagonal and hence is bounded on $\dot{f}_r^{\alpha+\beta,r}$. Following the same argument as the proof of part (i), we get

$$\begin{aligned}\left\|\Pi_g^*(f)\right\|^r_{\dot{F}_r^{\alpha+\beta,r}} &\le C\|D_{\mathbf{t}}\mathbf{s}\|^r_{\dot{f}_r^{\alpha+\beta,r}} \\ &= C\sum_Q\left(|Q|^{-(\beta/n)-(1/2)+(1/2r)}|\langle g, \varphi_Q\rangle|\right)^r \cdot \left(|Q|^{-(\alpha/n)-(1/2)+(1/2r)}|\langle f, \psi_Q\rangle|\right)^r \\ &\le C\|g\|^r_{\mathrm{CMO}^{\beta,q_0}_{(q_0/p_0)-(q_0/q_0')}}\|f\|^r_{\dot{f}_p^{\alpha,q}},\end{aligned} \tag{5.21}$$

which completes the proof. □

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