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Research Article

Approximate Analytic Solution for the KdV and Burger Equations with the Homotopy Analysis Method

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The homotopy analysis method (HAM) is applied to obtain the approximate analytic solution of the Korteweg-de Vries (KdV) and Burgers equations. The homotopy analysis method (HAM) is an analytic technique which provides us with a new way to obtain series solutions of such nonlinear problems. HAM contains the auxiliary parameter $\hbar$, which provides us with a straightforward way to adjust and control the convergence region of the series solution. The resulted HAM solution at 8th-order and 14th-order approximation is then compared with that of the exact soliton solutions of KdV and Burgers equations, respectively, and shown to be in excellent agreement.

1. Introduction

It is difficult to solve nonlinear problems, especially by analytic technique. The homotopy analysis method (HAM) [1, 2] is an analytic technique for nonlinear problems, which was first introduced by Liao in 1992. This method has been successfully applied to many nonlinear problems in engineering and science, such as the magnetohydrodynamics flows of non-Newtonian fluids over a stretching sheet [3], boundary layer flows over an impermeable stretched plate [4], nonlinear model of combined convective and radiative cooling of a spherical body [5], exponentially decaying boundary layers [6], and unsteady boundary layer flows over a stretching flat plate [7]. Thus the validity, effectiveness, and flexibility of the HAM are verified via all of these successful applications. Also, many types of nonlinear problems were solved with HAM by others [8–22].

The Korteweg-de Vries equation (KdV equation) describes the theory of water wave in shallow channels, such as canal. It is an important mathematical model in nonlinear wave's theory and nonlinear optics. The same examples are widely used in solid-state physics, fluid physics, plasma physics, and quantum field theory.

The Burgers equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. The first steady-state solution of Burgers equation was given by Bateman [23] in 1915. Although, the equation gets its name from the immense research of Burgers [24] beginning in 1939. The study of the general properties of the Burgers equation can be used as a model for any nonlinear wave diffusion problem subject to destruction [25]. Depending on the problem being modeled, this destruction may result from elasticity, gas dynamics, heat conduction, chemical reaction, or other resource.

In this paper, we employ the homotopy analysis method to obtain the solutions of the Korteweg-de Vries (KdV) and Burgers equations so as to provide us a new analytic approach for nonlinear problems.

2. Basic Ideas of Homotopy Analysis Method (HAM)

Consider a nonlinear equation in a general form:

$$\mathcal{N}[u(r,t)] = 0, \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator, $u(r,t)$ is unknown function. Let $u_0(r,t)$ denote an initial guess of the exact solution $u(r,t)$, $\hbar \neq 0$ an auxiliary parameter $\mathcal{H}(r,t) \neq 0$ an auxiliary function, and ℓ an auxiliary linear operator, $Q \in [0,1]$ as an embedding parameter by means of homotopy analysis method, we construct the so-called *zeroth-order deformation* equation

$$(1-Q)\ell\left[\phi(r,t;Q) - u_0(r,t)\right] = Q\hbar\mathcal{H}(r,t)\mathcal{N}\left[\phi(r,t;Q)\right]. \tag{2.2}$$

It is very significant that one has great freedom to choose auxiliary objects in HAM. Clearly, when $Q = 0, 1$ it holds that

$$\phi(r,t;0) = u_0(r,t), \qquad \phi(r,t;1) = u(r,t), \tag{2.3}$$

respectively. Then as long as Q increase from 0 to 1, the solution $\phi(r,t;Q)$ varies from initial guess $u_0(r,t)$ to the exact solution $u(r,t)$.

Liao [2] by Taylor theorem expanded $\emptyset(r,t;Q)$ in a power series of Q as follow:

$$\phi(r,t;Q) = \phi(r,t;0) + \sum_{m=1}^{\infty} u_m(r,t)Q^m, \tag{2.4}$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;Q)}{\partial Q^m}\bigg|_{Q=0}. \tag{2.5}$$

The convergence of the series (2.4) depends upon the auxiliary parameter $\hbar$, auxiliary function $\mathscr{H}(r,t)$, initial guess $u_0(r,t)$, and auxiliary linear operator ℓ. If they were chosen properly, the series (2.4) is convergence at $Q = 1$ one has

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{2.6}$$

According to definition (2.5), the governing equation can be inferred from the *zeroth-order deformation* equation (2.2). Define the vector

$$\overrightarrow{u_n(r,t)} = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\}. \tag{2.7}$$

Differentiating the zero-order deformation equation (2.2) m-times with respect to Q and dividing them by $m!$ and finally setting $Q = 0$ we obtain the so-called *mth-order deformation* equation

$$\ell\left[u_m(r,t) - \chi_m u_{m-1}(r,t)\right] = \hbar\mathscr{H}(r,t)\mathcal{R}_m(u_{m-1}, r, t), \tag{2.8}$$

where

$$\chi_m = \begin{cases} o, & m \leq 1, \\ 1, & m > 1, \end{cases}$$
$$\mathcal{R}_m(u_{m-1}, r, t) = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial Q^{m-1}}\mathcal{N}\left[\sum_{m=0}^{\infty} u_m(r,t)Q^m\right]\right\}\bigg|_{Q=0}. \tag{2.9}$$

Theorem 2.1 (Liao [2]). *As long as the series* (2.6) *is convergent, it is convergent to exact solution of* (2.1).

Note that homotopy analysis method contains the auxiliary parameter $\hbar$, which provide us with that control and adjustment of the convergence of the series solution (2.6).

3. Exact Solution

The Korteweg-de Vries equation (KdV equation) describes the theory of water wave in shallow channels, such as canal. It is a nonlinear equation which governed by

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \in R, \tag{3.1}$$

subject to

$$u(x,0) = f(x). \tag{3.2}$$

We will suppose that the solution $u(x,t)$ with its derivative, tends to zero [26, 27] when $|x| \to \infty$.

In 2001, Wazwaz [28] provided an exact solution

$$u(x,t) = -\frac{k^2}{2}\operatorname{sech}^2\frac{k}{2}\left(x - k^2 t\right), \tag{3.3}$$

or equivalently

$$u(x,t) = -2k^2\frac{e^{k(x-k^2t)}}{\left(1+e^{k(x-k^2t)}\right)^2}. \tag{3.4}$$

The Burgers equation is describe by

$$u_t + uu_x - u_{xx} = 0, \quad x \in R, \tag{3.5}$$

subject to

$$u(x,0) = f(x). \tag{3.6}$$

The exact solution of this equation is [29]

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}\left(x - \frac{1}{2}t\right). \tag{3.7}$$

4. HAM Solution

4.1. The KdV Equation

For HAM solution of KdV equation we choose

$$u_0(x,t) = -2\frac{e^x}{(1+e^x)^2} \tag{4.1}$$

as the initial guess and

$$\ell[u(x,t;Q)] = \frac{\partial u(x,t;Q)}{\partial t} \tag{4.2}$$

as the auxiliary linear operator satisfying

$$\ell[c] = 0, \tag{4.3}$$

where c is a constant.

We consider auxiliary function

$$\mathscr{H}(r,t) = 1, \tag{4.4}$$

zeroth-order deformation problem

$$\begin{gathered}(1-Q)\ell[u(x,t;Q) - u_0(x,t)] = Q\hbar\mathcal{N}[u(x,t;Q)],\\ u_0(x,t) = -2\frac{e^x}{(1+e^x)^2},\\ \mathcal{N}[u(x,t;Q)] = \frac{\partial u(x,t;Q)}{\partial t} - 6u(x,t;Q)\frac{\partial u(x,t;Q)}{\partial x} + \frac{\partial^3 u(x,t;Q)}{\partial x^3},\end{gathered} \tag{4.5}$$

mth-order deformation problem

$$\ell[u_m(x,t) - \chi_m u_m(x,t)] = Q\hbar\left[\frac{\partial u_{m-1}}{\partial t} - 6\sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} + \frac{\partial^3 u_{m-1}}{\partial x^3}\right], \tag{4.6}$$

$$u_m(x,0) = 0, \quad (m \geq 1). \tag{4.7}$$

We can use software Mathematica for solving the set of linear equation (4.6) with condition (4.7). It is found that the solution in a series form is given by

$$\begin{aligned} u(x,t) = &-2\frac{e^x}{(1+e^x)^2}\\ &+\frac{2e^x(-1+e^x)\hbar t\mathrm{Log}[e]\left(12e^x + \mathrm{Log}[e]^2 - 10e^x\mathrm{Log}[e]^2 + e^{2x}\mathrm{Log}[e]^2\right)}{(1+e^x)^5} + \cdots. \end{aligned} \tag{4.8}$$

The analytical solution given by (4.8) contains the auxiliary parameter $\hbar$, which influences the convergence region and rate of approximation for the HAM solution. In Figure 1, the $\hbar$-curves are plotted for $u(x,t), \ddot{u}(x,t), \dddot{u}(x,t)$ when $x = t = 0.01$ at 8th-order approximation.

As pointed out by Liao [2], the valid region of $\hbar$ is a horizontal line segment. It is clear that the valid region for this case is $-1.15 < \hbar < -0.6$. According to Theorem 2.1, the solution series (4.8) must be exact solution, as long as it is convergent. In this case, for $-1 < t < 1$ and $\hbar = -1$, the exact solution and HAM solution are the same, as shown in Figure 2. The obtained numerical results are summarized in Table 1.

In Figure 3, we study the diagrams of the results obtained by HAM for $\hbar = -0.5, \hbar = -0.75$, and $\hbar = -1$ in comparison with the exact solution (3.1); we can see the best value for $\hbar$ in this case is $\hbar = -1$.

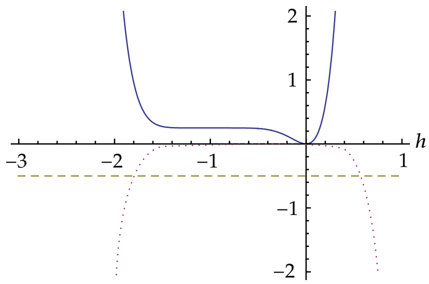


Figure 1: The $\hbar$-curve of 8th-order approximation, dashed point: $u(0.01, 0.01)$; solid line: $\dot{u}(0.01, 0.01)$; dashed line: $\ddot{u}(0.01, 0.01)$.

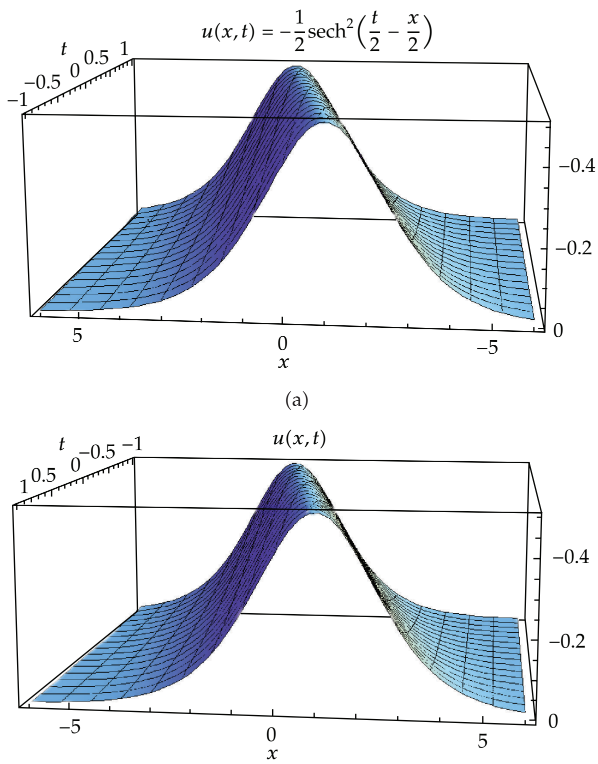


Figure 2: Comparison of the exact solution with the HAM solution of $u(x, t)$, when $\hbar = -1$: (a) exact solution, (b) HAM solution.

4.2. The Burgers Equation

In this section, for HAM solution of the Burgers equation we choose

$$u_0(x, t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x) \tag{4.9}$$

Table 1: Comparison of the HAM solution with exact solution when $\hbar = -1$ and $t = 0, 0.25, 0.5, 0.75, 1$, respectively.

t	x	Exact solution	HAM solution	Absolute error
0.05	−6	−0.004693564364865844	−0.0046935643648653635	$4.80518402845575 \times 10^{-16}$
	−3	−0.08634570715015544	−0.08634570715015137	$4.06619182768963 \times 10^{-15}$
	2	−0.21807963830331248	−0.21807963830330718	$5.30131494258512 \times 10^{-15}$
	5	−0.01396823161345761	−0.013968231613456493	$1.11716191852906 \times 10^{-15}$
0.25	−6	−0.003846044713671389	−0.0038460447137292153	$5.782613970994888 \times 10^{-14}$
	−3	−0.07186718166243183	−0.07186718165065627	$1.17755666328989 \times 10^{-11}$
	2	−0.2522584503864486	−0.25225845037331035	$1.31382682511116 \times 10^{-11}$
	5	−0.017007824315900595	−0.017007824315379085	$5.21509918582907 \times 10^{-13}$
0.5	−6	−0.0029978573890808136	−0.002997857417138099	$2.805728522778383 \times 10^{-11}$
	−3	−0.056906053378639285	−0.05690604775947111	$5.619168172432687 \times 10^{-9}$
	2	−0.29829291297695726	−0.2982929041406657	$8.836291531810758 \times 10^{-9}$
	5	−0.02173245972250237	−0.02173245944445047	$2.78051900948206 \times 10^{-10}$
0.75	−6	−0.002336284005515009	−0.0023362850215908897	$1.016075880551359 \times 10^{-9}$
	−3	−0.044899022133116494	−0.04489882076408693	$2.013690295621373 \times 10^{-7}$
	2	−0.3462100376919465	−0.3462095739849941	$4.637069524471293 \times 10^{-7}$
	5	−0.027731694033315973	−0.02773168287526794	$1.115804803414333 \times 10^{-8}$
1	−6	−0.0018204295917628886	−0.001820442360243653	$1.276848076445583 \times 10^{-8}$
	−3	−0.03532791434845481	−0.03532541242658223	$2.50192187 \times 10^{-7}$
	2	−0.3932319186044759	−0.3932238664829637	$8.052121512225341 \times 10^{-7}$
	5	−0.03532556752567461	−0.03532541242658223	$1.550990923818163 \times 10^{-7}$

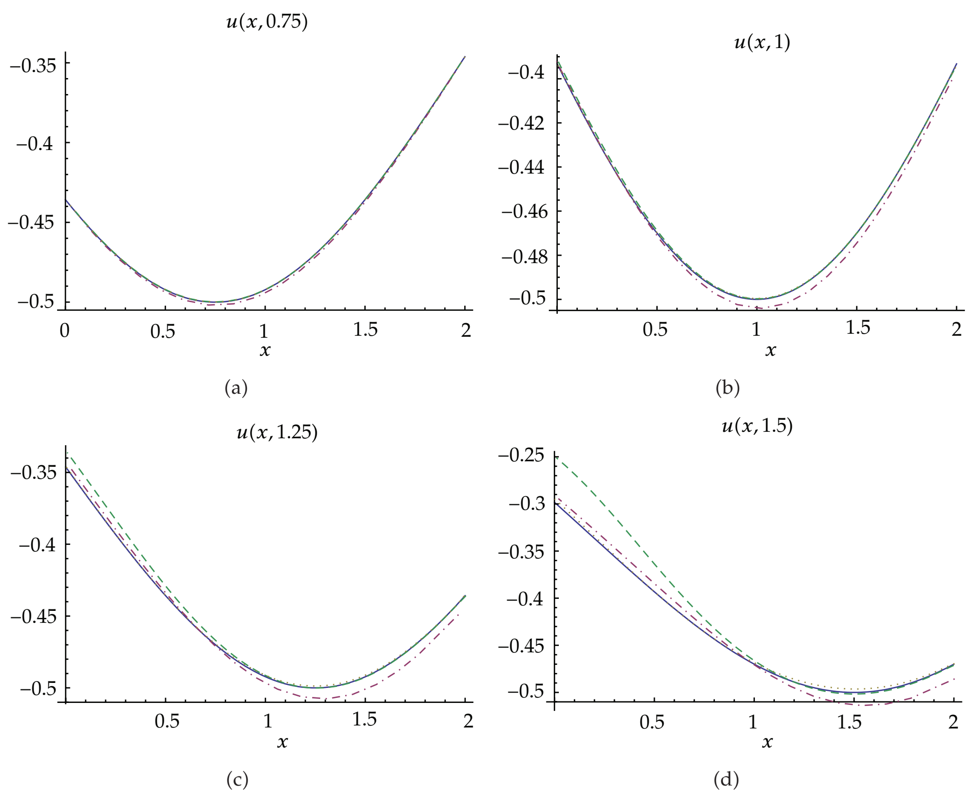


Figure 3: The results obtained by 8th-order approximation for $h = [-0.5; -0.75; -1]$. Solid line: exact solution; dashing-large for $h = -0.5$; dashing-medium for $h = -0.75$; dashing-tiny for $h = -1$.

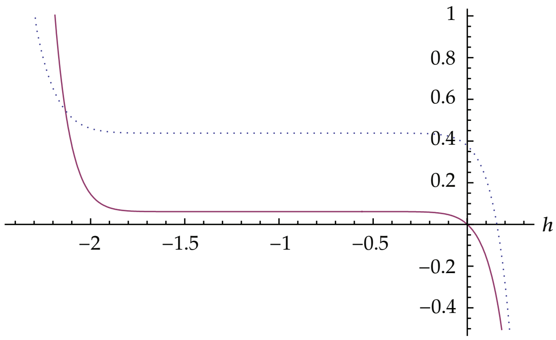


Figure 4: The $\hbar$-curve of at 14th-order approximation solid line $u(1,1)$; dotted line $\dot{u}(1,1)$.

as the initial guess and

$$\ell[u(x,t;Q)] = \frac{\partial u(x,t;Q)}{\partial t}. \tag{4.10}$$

as the auxiliary linear operator satisfying

$$\ell[c] = 0, \tag{4.11}$$

where c is a constant.

We consider auxiliary function

$$\mathscr{H}(r,t) = 1, \tag{4.12}$$

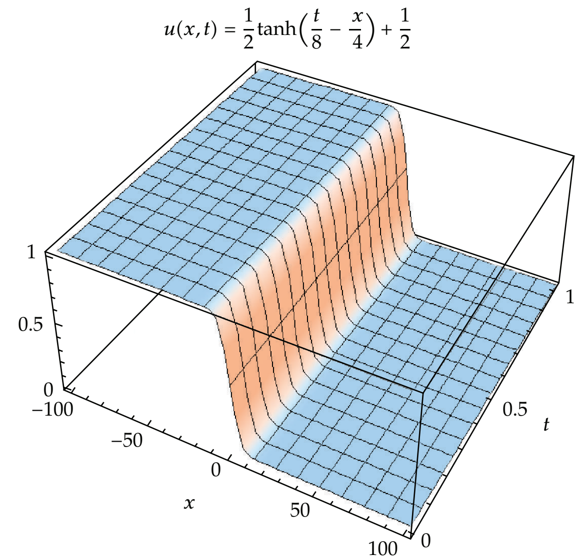
zeroth-order deformation problem

$$\begin{gathered}(1-Q)\ell[u(x,t;Q) - u_0(x,t)] = Q\hbar\mathcal{N}[u(x,t;Q)],\\ u_0(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x),\\ \mathcal{N}[u(x,t;Q)] = \frac{\partial u(x,t;Q)}{\partial t} + u(x,t;Q)\frac{\partial u(x,t;Q)}{\partial x} - \frac{\partial^2 u(x,t;Q)}{\partial x^2}.\end{gathered} \tag{4.13}$$

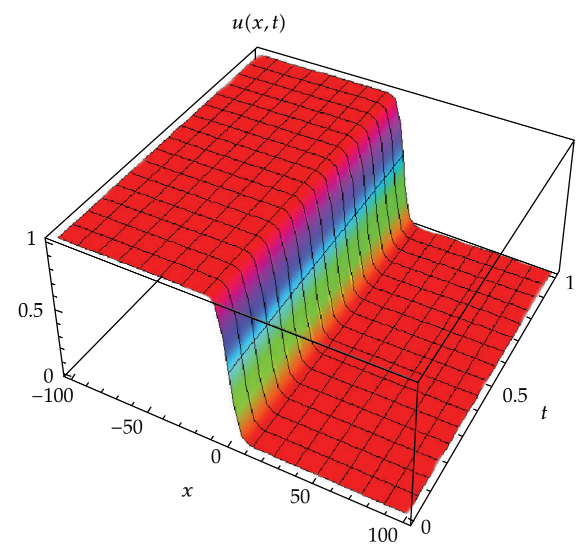
mth-order deformation problem

$$\ell\left[u_m(x,t) - \chi_m u_m(x,t)\right] = Q\hbar\left[\frac{\partial u_{m-1}}{\partial t} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x^2}\right], \tag{4.14}$$

$$u_m(x,0) = 0, \quad (m \geq 1). \tag{4.15}$$



(a)



(b)

Figure 5: Comparison of the exact solution with the HAM solution of $u(x,t)$, when $\hbar = -0.5$: (a) exact solution, (b) HAM solution.

We can use software Mathematica for solving the set of linear equation (4.14) with condition (4.15). It is found that the solution in a series form is given by

$$u(x,t) = \frac{1}{2} - \frac{-1 + e^{x/2}}{2\left(1 + e^{x/2}\right)} - \frac{e^{x/2} h t \operatorname{Log}[e]\left(2 - \operatorname{Log}[e] + e^{x/2}\operatorname{Log}[e]\right)}{4\left(1 + e^{x/2}\right)^3} + \cdots . \tag{4.16}$$

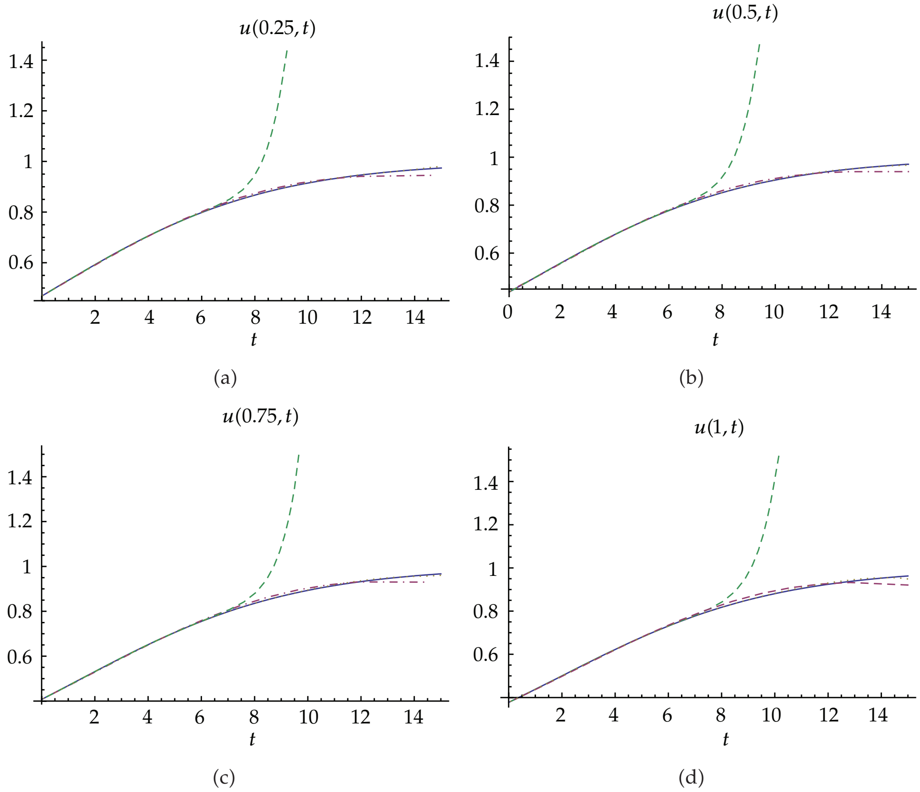


Figure 6: The results obtained by 14th-order approximation of HAM for various h. Solid line: exact solution; dashing-tiny: $h = -0.5$; dashing-large: $h = -0.3$; dashing-small: $h = -1$, when $0 \leq t \leq 15$.

The analytical solution given by (4.16) contains the auxiliary parameter $\hbar$, which influences the convergence region and the rate of approximation for the HAM solution. In Figure 4, the $\hbar$-curve is plotted for $u(x,t), \dot{u}(x,t)$, when $x = t = 1$ at 14th-order approximation.

It is clear that the valid region for this case is $-1.72 < \hbar < -0.3$. According to Theorem 2.1, the solution series (4.16) must be exact solution, as long as it is convergent. In this case, for $0 < t < 1$ and $\hbar = -0.5$, the exact solution and HAM solution are the same, as shown in Figure 5. The obtained numerical results are summarized in Table 2.

In Figure 6, we study the diagrams of the results obtained by HAM for $\hbar = -0.3, \hbar = -0.5$ and $\hbar = -1$ in comparison with the exact solution (3.5); we can see the best value for $\hbar$ in this case is $\hbar = -0.5$.

5. Conclusion

In this paper, the homotopy analysis method (HAM) [2] is applied to obtain the solitary solution of the KdV and Burger equations. HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. So, these examples show the flexibility and potential of the homotopy analysis method for complicated nonlinear problems in engineering.

Table 2: Comparison of the HAM solution with exact solution when $\hbar = -0.5$ and $t = 0, 0.25, 0.5, 0.75, 1$, respectively.

t	x	Exact solution	HAM solution	Absolute error
0	$-\infty$	1.	1.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	-100	1.	1.	0.
	-50	0.99999999998611	0.9999999999	$2.220446049 \times 10^{-16}$
	-20	0.99995460229829	0.9999546021	$1.670014126 \times 10^{-10}$
	0	0.5	0.5	0
	20	0.00004539770170	0.0000453978	$1.670015792 \times 10^{-10}$
	50	$1.3887835 \times 10^{-11}$	1.388×10^{-11}	$1.110223024 \times 10^{-16}$
	100	0.	0.	0.
	200	0.	0.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	∞	0.	0.	0.
0.25	$-\infty$	1.	1.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	-100	1.	1.	0.
	-50	0.9999999999999	0.99999999998	$4.1400216588 \times 10^{-13}$
	-20	0.9999559989777	0.99995735253	$1.3535540239 \times 10^{-6}$
	0	0.5078118671525	0.51561991572	$7.8080485704 \times 10^{-3}$
	20	0.0000468387124	0.00004832563	$1.4869257015 \times 10^{-6}$
	50	$1.4328641 \times 10^{-11}$	1.47836×10^{-11}	$4.549923954 \times 10^{-13}$
	100	0.	1.24393×10^{-23}	$1.243936778 \times 10^{-23}$
	200	0.	2.39924×10^{-45}	$2.399242872 \times 10^{-45}$
	300	0	0	0
0.5	$-\infty$	1.	1.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	-100	1.	1.	0.
	-50	0.999999999986	0.99999999998	$7.902567489 \times 10^{-13}$
	-20	0.999957352689	0.99995993630	$2.583618091 \times 10^{-6}$
	0	0.515619921465	0.53120937337	$1.558945190 \times 10^{-3}$
	20	0.000048325461	0.00005144221	$3.116752215 \times 10^{-6}$
	50	1.478351×10^{-11}	1.57371×10^{-11}	$9.535413278 \times 10^{-13}$
	100	0.	2.56810×10^{-23}	$2.568100586 \times 10^{-23}$
	200	0.	4.95322×10^{-45}	$4.953223614 \times 10^{-45}$
	300	0.	9.55352×10^{-67}	$9.553529294 \times 10^{-67}$
0.75	$-\infty$	1.	1.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	-100	1.	1.	0.
	-50	0.999999999987	0.99999999998	$1.131539306 \times 10^{-12}$
	-20	0.99995866475	0.99996236355	$3.698797104 \times 10^{-12}$
	0	0.523420357539	0.54673815197	$2.331779443 \times 10^{-2}$
	20	0.000049859400	0.00005475976	$4.900369371 \times 10^{-6}$
	50	1.525279×10^{-11}	1.67520×10^{-11}	$1.499261451 \times 10^{-12}$
	100	5.551115×10^{-17}	3.97766×10^{-23}	$5.551111145 \times 10^{-17}$
	200	0.	7.67192×10^{-45}	$7.671921963 \times 10^{-45}$
	300	0.	1.47972×10^{-66}	$1.479721832 \times 10^{-66}$
1	$-\infty$	1.	1.	0.
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	-100	1.	1.	0.
	-50	0.999999999987744	0.9999999999	$1.440070285 \times 10^{-12}$
	-20	0.9999599364569929	0.9999646437	$4.707292264 \times 10^{-6}$

Table 2: Continued.

t	x	Exact solution	HAM solution	Absolute error
	0	0.5312093848251956	0.5621765008	$3.096711597 \times 10^{-2}$
	20	0.000051442026872461355	0.0000582912	$6.849238788 \times 10^{-6}$
	50	$1.5737022796002 \times 10^{-11}$	1.7832×10^{-11}	$2.095452595 \times 10^{-12}$
	100	0.	5.4781×10^{-23}	$5.478139792 \times 10^{-23}$
	200	0.	1.0565×10^{-44}	$1.056596129 \times 10^{-44}$
	300	0.	2.0379×10^{-66}	$2.037909623 \times 10^{-66}$

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