Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 847958, 15 pages
doi:10.1155/2012/847958

*Research Article*

# A Minimum Problem for Finite Sets of Real Numbers with Nonnegative Sum

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Received 6 February 2012; Accepted 2 March 2012

Academic Editor: Yonghong Yao



Let $n$ and $r$ be two integers such that $0 < r \leq n$; we denote by $\gamma(n,r)[\eta(n,r)]$ the minimum [maximum] number of the nonnegative partial sums of a sum $\sum_{1=1}^{n} a_i \geq 0$, where $a_1, \ldots, a_n$ are $n$ real numbers arbitrarily chosen in such a way that $r$ of them are nonnegative and the remaining $n-r$ are negative. We study the following two problems: $(P1)$ *which are the values of* $\gamma(n,r)$ *and* $\eta(n,r)$ *for each* $n$ *and* $r$, $0 < r \leq n$? $(P2)$ *if* $q$ *is an integer such that* $\gamma(n,r) \leq q \leq \eta(n,r)$, *can we find* $n$*real numbers* $a_1, \ldots, a_n$, *such that* $r$ *of them are nonnegative and the remaining* $n-r$ *are negative with* $\sum_{1=1}^{n} a_i \geq 0$, *such that the number of the nonnegative sums formed from these numbers is exactly* $q$?

## 1. Introduction

In [1] Manickam and Miklós raised several interesting extremal combinatorial sum problems, two of which will be described below. Let $n$ and $r$ be two integers such that $0 < r \leq n$; we denote by $\gamma(n,r)[\eta(n,r)]$ the minimum [maximum] number of the nonnegative partial sums of a sum $\sum_{1=1}^{n} a_i \geq 0$, when $a_1, \ldots, a_n$ are $n$ real numbers arbitrarily chosen in such a way that $r$ of them are nonnegative and the remaining $n-r$ are negative. Put $A(n) = \min\{\gamma(n,r) : 0 < r \leq n\}$. In [1] the authors answered the following question.

$(Q1)$ *Which is the value of* $A(n)$?

In [1, Theorem 1] they found that $A(n) = 2^{n-1}$. On the other side, from the proof of Theorem 1 of [1] also it follows that $\gamma(n,r) \geq 2^{n-1}$ for each $r$ and that $\gamma(n,1) \leq 2^{n-1}$; therefore $\gamma(n,1) = 2^{n-1}$ (since $2^{n-1} = A(n) \leq \gamma(n,1) \leq 2^{n-1}$). It is natural to set then the following problem which is a refinement of $(Q1)$.

$(P1)$ *Which are the values of* $\gamma(n,r)$ *and* $\eta(n,r)$ *for each* $n$ *and* $r$ *with* $0 < r \leq n$*?*

In the first part of this paper we solve the problem $(P1)$, and we prove (see Theorem 4.1) that $\gamma(n,r) = 2^{n-1}$ and $\eta(n,r) = 2^n - 2^{n-r}$ for each positive integer $r \leq n$.

A further question that the authors raised in [1] is the following.

$(Q2)$ *"We do not know what is the range of the possible numbers of the nonnegative partial sums of a nonnegative* $n$*-element sum. The minimum is* $2^{n-1}$ *as it was proven and the maximum is obviously* $2^n - 1$ *but we do not know which are the numbers between them for which we can find reals* $a_1, \dots, a_n$ *with* $\sum_{1=1}^{n} a_i \geq 0$ *such that the number of the nonnegative sums formed from these numbers is equal to that number."*

The following problem is a natural refinement of (Q2).

$(P2)$ *If* $q$ *is an integer such that* $\gamma(n,r) \leq q \leq \eta(n,r)$*, can we find* $n$ *real numbers* $a_1, \dots, a_n$*, such that* $r$ *of them are nonnegative and the remaining* $n-r$ *are negative with* $\sum_{1=1}^{n} a_i \geq 0$*, such that the number of the nonnegative sums formed from these numbers is exactly* $q$*?*

In the latter part of this paper (see Theorem 4.5) we give a partial solution to the problem $(P2)$.

To be more precise in the formulation of the problems that we study and to better underline the links with some interesting problems raised in [1], it will be convenient-identify a finite set of real-numbers with an appropriate real-valued function. Let then $n$ and $r$ be two fixed integers such that $0 < r \leq n$ and let $I_n = \{1, 2, \dots, n\}$ (we call $I_n$ the index set). We denote by $W(n,r)$, the set of all the functions $f : I_n \to \mathbb{R}$ such that $\sum_{x \in I_n} f(x) \geq 0$ and $|\{x \in I_n : f(x) \geq 0\}| = r$. If $f \in W(n,r)$, we set $\alpha(f) = |\{Y \subseteq I_n : \sum_{y \in Y} f(y) \geq 0\}|$. It is easy to observe that $\gamma(n,r) = \min\{\alpha(f) : f \in W(n,r)\}$ and $\eta(n,r) = \max\{\alpha(f) : f \in W(n,r)\}$. We can reformulate the problem $(P2)$ in an equivalent way using the functions terminology instead of the sets terminology.

$(P2)$ *If* $q$ *is an integer such that* $\gamma(n,r) \leq q \leq \eta(n,r)$*, does there exist a function* $f \in W(n,r)$ *with the property that* $\alpha(f) = q$*?*

To solve the problem $(P1)$ and (partially) $(P2)$, we use some abstract results on a particular class of lattices introduced in [2, 3]. In this paper we substantially continue the research project started in [4], which is the attempt to solve some extremal sum problems as started in [1] and further studied in [3, 5–11].

## 2. A Partial Order on the Subsets of $I_n$

Of course if we take two functions $f, g \in W(n,r)$ such that $f(I_n) = g(I_n)$, then $\alpha(f) = \alpha(g)$. This implies that if we define on $W(n,r)$ the equivalence relation $f \sim g$ iff $f(I_n) = g(I_n)$ and we denote by $[f]$ the equivalence class of a function $f \in W(n,r)$, then the definition $\beta([f]) = \alpha(f)$ is well placed. It is also clear that it holds $\gamma(n,r) = \min\{\beta([f]) : [f] \in W(n,r)/\sim\}$ and $\eta(n,r) = \max\{\beta([f]) : [f] \in W(n,r)/\sim\}$. Now, when we take an equivalence class $[f] \in W(n,r)/\sim$, there is a unique $f^* \in [f]$ such that

$$f^*(r) \geq \cdots \geq f^*(1) \geq 0 > f^*(r+1) \geq \cdots \geq f^*(n). \tag{2.1}$$

We can then identify the quotient set $W(n,r)/\sim$ with the subset of all the functions $f^* \in W(n,r)$ that satisfy the condition (2.1). This simple remark conducts us to rename the indexes of $I_n$ as follows: $\widetilde{r}$ instead of $r, \dots, \widetilde{1}$ instead of 1, $\overline{1}$ instead of $r+1, \dots, \overline{n-r}$ instead of $n$.

Therefore, if we set $I(n,r) = \{\widetilde{1},\ldots,\widetilde{r},\overline{1},\ldots,\overline{n-r}\}$, we can identify the quotient set $W(n,r)/\sim$ with the set of all the functions $f : I(n,r) \to \mathbb{R}$ that satisfy the following two conditions:

$$\begin{gathered} f(\widetilde{r}) + \cdots + f\left(\widetilde{1}\right) + f\left(\overline{1}\right) + \cdots + f\left(\overline{n-r}\right) \geq 0, \\ f(\widetilde{r}) \geq \cdots \geq f\left(\widetilde{1}\right) \geq 0 > f\left(\overline{1}\right) \geq \cdots \geq f\left(\overline{n-r}\right). \end{gathered} \tag{2.2}$$

Now, if a generic function $f : I(n,r) \to \mathbb{R}$ that satisfies (2.2) is given, we are interested to find all the subsets $Y \subseteq I(n,r)$ such that $\sum_{y\in Y} f(y) \geq 0$. This goal becomes then easier if we can have an appropriate partial order $\sqsubseteq$ on the power set $\mathcal{P}(I(n,r))$ "compatible" with the total order of the partial sums inducted by $f$, that is, a partial order $\sqsubseteq$ that satisfies the following *monotonicity property*: if $Y, Z \in \mathcal{P}(I(n,r))$, then $\sum_{z\in Z} f(x) \leq \sum_{y\in Y} f(y)$ whenever $Z \sqsubseteq Y$. To have such a partial order $\sqsubseteq$ on $\mathcal{P}(I(n,r))$ that has the monotonicity property we must introduce a new formal symbol that we denote by $0^{\S}$. We add this new symbol to the index set $I(n,r)$, so we set $A(n,r) = I(n,r) \cup \{0^{\S}\}$. We introduce on $A(n,r)$ the following total order:

$$\overline{n-r} \prec \cdots \prec \overline{2} \prec \overline{1} \prec 0^{\S} \prec \widetilde{1} \prec \widetilde{2} \prec \cdots \prec \widetilde{r}. \tag{2.3}$$

If $i, j \in A(n,r)$, then we write: $i \preceq j$ for $i = j$ or $i \prec j$. We denote by $S(n,r)$ the set of all the formal expressions $i_1 \cdots i_r \mid j_1 \cdots j_{n-r}$ (hereafter called *strings*) that satisfy the following properties:

(i) $i_1, \ldots, i_r \in \{\widetilde{1}, \ldots, \widetilde{r},\ 0^{\S}\}$,

(ii) $j_1, \ldots, j_{n-r} \in \{\overline{1}, \ldots, \overline{n-r}, 0^{\S}\}$,

(iii) $i_1 \succeq \cdots \succeq i_r \succeq 0^{\S} \succeq j_1 \succeq \cdots \succeq j_{n-r}$,

(iv) the unique element which can be repeated is $0^{\S}$.

In the sequel we often use the lowercase letters $u, w, z, \ldots$ to denote a generic string in $S(n,r)$. Moreover to make smoother reading, in the numerical examples the formal symbols which appear in a string will be written without $\widetilde{}$ $\overline{}$ and $^{\S}$; in such way the vertical bar | will indicate that the symbols on the left of | are in $\{\widetilde{1}, \ldots, \widetilde{r}, 0^{\S}\}$ and the symbols on the right of | are elements in $\{0^{\S}, \overline{1}, \ldots, \overline{n-r}\}$. For example, if $n = 3$ and $r = 2$, then $A(3,2) = \{\widetilde{2} \succ \widetilde{1} \succ 0^{\S} \succ \overline{1}\}$ and $S(3,2) = \{21 \mid 0,\ 21 \mid 1,\ 10 \mid 0,\ 20 \mid 0,\ 10 \mid 1,\ 20 \mid 1,\ 00 \mid 1,\ 00 \mid 0\}$.

Note that there is a natural bijective setcorrespondence $* : w \in S(n,r) \mapsto w^* \in \mathcal{P}(I(n,r))$ between $S(n,r)$ and $\mathcal{P}(I(n,r))$ defined as follows: if $w = i_1 \cdots i_r \mid j_1 \cdots j_{n-r} \in S(n,r)$, then $w^*$ is the subset of $I(n,r)$ made with the elements $i_k$ and $j_l$ such that $i_k \neq 0^{\S}$ and $j_l \neq 0^{\S}$. For example, if $w = 4310 \mid 013 \in S(7,4)$, then $w^* = \{\widetilde{1}, \widetilde{3}, \widetilde{4}, \overline{1}, \overline{3}\}$. In particular, if $w = 0 \cdots 0 \mid 0 \cdots 0$, then $w^* = \emptyset$.

Now, if $v = i_1 \cdots i_r \mid j_1 \cdots j_{n-r}$ and $w = i'_1 \cdots i'_r \mid j'_1 \cdots j'_{n-r}$ are two strings in $S(n,r)$, we define: $v \sqsubseteq w$ iff $i_1 \preceq i'_1, \ldots, i_r \preceq i'_r, j_1 \preceq j'_r, \ldots, j_{n-r} \preceq j'_{n-r}$.

It is easily seen that

(1) $(S(n,r),\sqsubseteq)$ is a finite distributive (hence also graded) lattice with minimum element $0\cdots 0 \mid 12\cdots(n-r)$ and maximum element $r(r-1)\cdots 21 \mid 0\cdots 0$;

(2) $(S(n,r),\sqsubseteq)$ has the following unary complementary operation $c$:

$$\left(p_1\cdots p_k 0\cdots 0 \mid 0\cdots 0 q_1\cdots q_l\right)^c = p'_1\cdots p'_{r-k}0\cdots 0 \mid 0\cdots 0 q'_1\cdots q'_{n-r-l}, \tag{2.4}$$

where $\{p'_1,\ldots,p'_{r-k}\}$ is the usual complement of $\{p_1,\ldots,p_k\}$ in $\{\widetilde{1},\ldots,\widetilde{r}\}$ and $\{q'_1,\ldots,q'_{n-r-l}\}$ is the usual complement of $\{q_1,\ldots,q_l\}$ in $\{\overline{1},\ldots,\overline{n-r}\}$ (e.g., in $S(7,4)$, we have that $(4310 \mid 001)^c = 2000 \mid 023$).

Since we have the formal necessity to consider functions $f$ defined on the extended set $A(n,r)$ instead of on the indexes set $I(n,r)$, then we will put $f(0^{\S}) = 0$. Precisely we can identify the quotient set $W(n,r)/\sim$ with the set $WF(n,r)$, defined by

$$\begin{aligned} WF(n,r) = \Big\{ f : A(n,r) \to \mathbb{R} : f(\widetilde{r}) \ge \cdots \ge f\left(\widetilde{1}\right) \ge f\left(0^{\S}\right) \\ = 0 > f\left(\overline{1}\right) \ge \cdots \ge f\left(\overline{n-r}\right),\ f(\widetilde{r}) + \cdots + f\left(\widetilde{1}\right) + f\left(\overline{1}\right) + \cdots + f\left(\overline{n-r}\right) \ge 0 \Big\}. \end{aligned} \tag{2.5}$$

We call an element of $WF(n,r)$ a $(n,r)$*-weight function,* and if $f \in WF(n,r)$, we will continue to set $\alpha(f) := |\{Y \subseteq I(n,r) : \sum_{y\in Y} f(y) \ge 0\}|$. Therefore, with these last notations we have that $\gamma(n,r) = \min\{\alpha(f) : f \in WF(n,r)\}$, $\eta(n,r) = \max\{\alpha(f) : f \in WF(n,r)\}$ and the question $(P2)$ becomes equivalent to the following.

$(P2)$ *If $q$ is an integer such that $\gamma(n,r) \le q \le \eta(n,r)$, does there exist a function $f \in WF(n,r)$ with the property that $\alpha(f) = q$?*

## 3. Boolean Maps Induct by Weight Functions

We denote by $\mathbf{2}$ the boolean lattice composed of a chain with 2 elements that we denote $N$ (the minimum element) and $P$ (the maximum element). A *Boolean map* (briefly BM) *on* $S(n,r)$ is a map $A :\ \mathrm{dom}(A) \subseteq S(n,r) \to \mathbf{2}$; in particular if $\mathrm{dom}(A) = S(n,r)$,we also say that $A$ is a *Boolean total map* (briefly BTM) on $S(n,r)$. If $A$ is BM on $S(n,r)$, we set $S^+_A(n,r) = \{w \in \mathrm{dom}(A) : A(w) = P\}$.

If $f \in WF(n,r)$, the *sum function* $\Sigma_f : S(n,r) \to \mathbb{R}$ induced by $f$ on $S(n,r)$ is the function that associates to $w = i_r\cdots i_1 \mid j_1\cdots j_{n-r} \in S(n,r)$ the real number $\Sigma_f(w) = f(i_1) + \cdots + f(i_r) + f(j_1) + \cdots + f(j_{n-r})$, and therefore we can associate to $f \in WF(n,r)$ the map $A_f : S(n,r) \to \mathbf{2}$ setting:

$$A_f(w) = \begin{cases} P & \text{if } \Sigma_f(w) \ge 0,\ w \ne 0\cdots 0 \mid 0\cdots 0, \\ N & \text{if } w = 0\cdots 0 \mid 0\cdots 0, \\ N & \text{if } \Sigma_f(w) < 0. \end{cases} \tag{3.1}$$

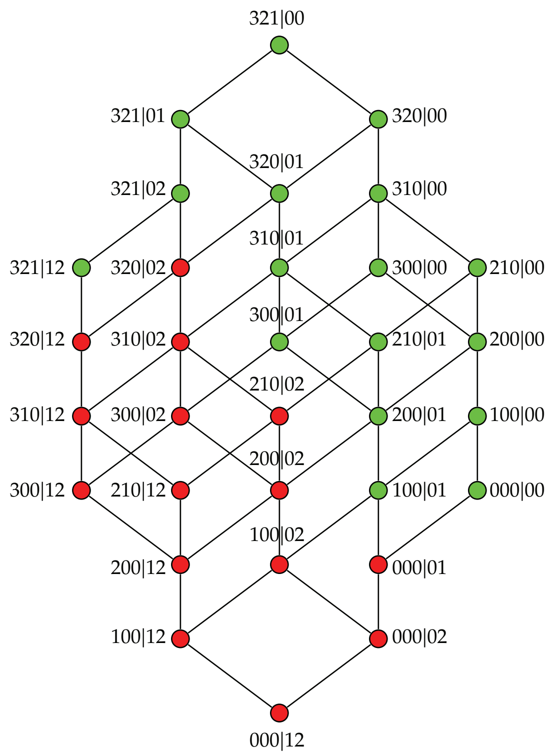


**Figure** 1

Let us note that $|S^+_{A_f}(n,r)| = |\{w \in S(n,r) : A_f(w) = P\}| = \alpha(f)$, and so $\gamma(n,r) = \min\{|S^+_{A_f}(n,r)| : f \in WF(n,r)\}$, $\eta(n,r) = \max\{|S^+_{A_f}(n,r)| : f \in WF(n,r)\}$.

Our goal is now to underline that some properties of such maps simplify the study of our problems. It is easy to observe that the map $A_f$ has the following properties.

(i) $A_f$ is orderpreserving.

(ii) If $w \in S(n,r)$ is such that $A_f(w) = N$, then $A_f(w^c) = P$,

(iii) $A_f(10\cdots0 \mid 0\cdots0) = P$, $A(0\cdots0 \mid 0\cdots0) = N$, and $A_f(r\cdots21 \mid 12\cdots(n-r)) = P$.

*Example 3.1.* Let $f$ be the following $(5,3)$-weight function:

$$\begin{array}{cccccc} & \tilde{3} & \tilde{2} & \tilde{1} & \overline{1} & \overline{2} \\ f : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 1 & 1 & 0.9 & -0.8 & -2.1 \end{array} \tag{3.2}$$

We represent the map $A_f$ defined on $S(5,3)$ by using the Hasse diagram of this lattice, on which we color green the nodes where the map $A_f$ assumes value $P$ and red the nodes where it assumes value $N$ (See Figure 1).

Note that if we have a generic Boolean total map $A : S(n,r) \rightarrow \mathbf{2}$ which has the properties (i), (ii), and (iii) of $A_f$, that is, the following:

(BM1) $A$ is orderpreserving,

(BM2) if $w \in S(n,r)$ is such that $A(w) = N$, then $A(w^c) = P$,

(BM3) $A(10\cdots 0 \mid 0\cdots 0) = P$, $A(0\cdots 0 \mid 0\cdots 0) = N$, and $A(r\cdots 21 \mid 12\cdots(n-r)) = P$,

in general there does not exist a function $f \in WF(n,r)$ such that $A_f = A$ (see [3] for a counterexample).

We denote by $\mathcal{W}_+(S(n,r),\mathbf{2})$ the set of all the maps $A : S(n,r) \rightarrow \mathbf{2}$ which satisfy (BM1) and (BM2) and by $\mathcal{W}_+(n,r)$ the subset of all the maps in $\mathcal{W}_+(S(n,r),\mathbf{2})$ which satisfy also (BM3). We also set $\gamma^*(n,r) := \min\{|S_A^+(n,r) : A \in \mathcal{W}_+(n,r)\}$ and $\eta^*(n,r) := \max\{|S_A^+(n,r)| : A \in \mathcal{W}_+(n,r)\}$. Let us observe that $\gamma^*(n,r) \le \gamma(n,r) \le \eta(n,r) \le \eta^*(n,r)$. A natural question raises at this point.

$(Q)$: *if $q$ is an integer such that $\gamma^*(n,r) \le q \le \eta^*(n,r)$, does there exist a map $A \in \mathcal{W}_+(n,r)$ with the property that $|S_A^+(n,r)| = q$?*

The question $(Q)$ is the analogue of $(P2)$ expressed in terms of Boolean total maps on $S(n,r)$ instead of $(n,r)$-weight functions, and if we are able to respond to $(Q)$, we provide also a partial answer to $(P2)$. In Section 4 we give an affirmative answer to the question $(Q)$, and also we give a constructive method to build the map $A$.

## 4. Main Results

In the sequel of this paper we adopt the classical terminology and notations usually used in the context of the partially ordered sets (see [12–14] for the general aspects on this subject). If $Z \subseteq S(n,r)$, we will set $\downarrow Z = \{x \in S(n,r) : \exists\, z \in Z \text{ such that } x \sqsubseteq z\}$, $\uparrow Z = \{x \in S(n,r) : \exists\, z \in Z \text{ such that } z \sqsubseteq x\}$. In particular, if $z \in S(n,r)$, we will set $\downarrow z = \downarrow \{z\} = \{x \in S(n,r) : z \sqsupseteq x\}$, $\uparrow z = \uparrow \{z\} = \{x \in S(n,r) : z \sqsubseteq x\}$. $Z$ is called a *downset* of $S(n,r)$ if for $z \in Z$ and $x \in S(n,r)$ with $z \sqsupseteq x$, then $x \in Z$. $Z$ is called an *upset* of $S(n,r)$ if for $z \in Z$ and $x \in S(n,r)$ with $z \sqsubseteq x$, then $x \in Z$. $\downarrow Z$ is the smallest down-set of $S(n,r)$ which contains $Z$, and $Z$ is a downset of $S(n,r)$ if and only if $Z = \downarrow Z$. Similarly, $\uparrow Z$ is the smallest up-set of $S(n,r)$ which contains $Z$, and $Z$ is an upset in $S(n,r)$ if and only if $Z = \uparrow Z$.

**Theorem 4.1.** *If $n$ and $r$ are two integers such that $0 < r < n$, then $\gamma(n,r) = \gamma^*(n,r) = 2^{n-1}$ and $\eta(n,r) = \eta^*(n,r) = 2^n - 2^{n-r}$.*

*Proof.* Assume that $0 < r < n$. We denote by $S_1(n,r)$ the sublattice of $S(n,r)$ of all the strings $w$ of the form $w = i_1\cdots i_r \mid j_1\cdots j_{n-r-1}(n-r)$, with $j_1\cdots j_{n-r-1} \in \{0^{\S},\overline{1},\ldots,\overline{n-r-1}\}$ and by $S_2(n,r)$ the sublattice of $S(n,r)$ of all the strings $w$ of the form $w = i_1\cdots i_r \mid 0 j_2\cdots j_{n-r}$, with $j_2 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$. It is clear that $S(n,r) = S_1(n,r) \dot{\bigcup} S_2(n,r)$ and $S_1(n,r) \cong S_2(n,r) \cong S(n-1,r)$. We consider now the following further sublattices of $S(n,r)$:

$$\begin{aligned}
S_1^{+}(n,r) &:= \{ w \in S_1(n,r) : w = r(r-1)\cdots 21 \mid j_1 \cdots j_{n-r-1}(n-r) \}, \\
S_1^{\pm}(n,r) &:= \left\{ w \in S_1(n,r) : w = i_1 \cdots i_{r-1}0 \mid j_1 \cdots j_{n-r-1}(n-r), i_1 \succ 0^{\S} \right\}, \\
S_1^{-}(n,r) &:= \{ w \in S_1(n,r) : w = 0 \cdots 0 \mid j_1 \cdots j_{n-r-1}(n-r) \}, \\
S_2^{+}(n,r) &:= \{ w \in S_2(n,r) : w = r(r-1)\cdots 21 \mid 0 j_2 \cdots j_{n-r} \}, \\
S_2^{\pm}(n,r) &:= \left\{ w \in S_2(n,r) : w = i_1 \cdots i_{r-1}0 \mid 0 j_2 \cdots j_{n-r}, i_1 \succ 0^{\S} \right\}, \\
S_2^{-}(n,r) &:= \{ w \in S_2(n,r) : w = 0 \cdots 0 \mid 0 j_2 \cdots j_{n-r} \}.
\end{aligned} \tag{4.1}$$

It occurs immediately that $S_i(n,r) = S_i^{+}(n,r) \dot{\bigcup} S_i^{\pm}(n,r) \dot{\bigcup} S_i^{-}(n,r)$, for $i = 1,2$ and $S_i^{\pm}(n,r)$ is a distributive sublattice of $S_i(n,r)$ with $2^{n-1} - 2 \cdot 2^{n-r-1} = 2^{n-1} - 2^{n-r}$ elements, for $i = 1,2$.

Now we consider the following $(n,r)$-weight function $f : A(n,r) \to \mathbb{R}$:

$$f : \begin{array}{cccccccc}
\tilde{r} & \cdots & \tilde{1} & 0^{\S} & \overline{1} & \cdots & \overline{(n-r-1)} & \overline{(n-r)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(n-r) & \cdots & (n-r) & 0 & -1 & \cdots & -1 & (n-r)(1-r)-1
\end{array} \tag{4.2}$$

Then it follows that $\Sigma_f : S(n,r) \to \mathbb{R}$ is such that

$$\Sigma_f(w) \begin{cases} \geq 0 & \text{if } w \in S_2^{\pm}(n,r), \\ < 0 & \text{if } w \in S_1^{\pm}(n,r). \end{cases} \tag{4.3}$$

It means that the boolean map $A_f \in \mathcal{W}_{+}(n,r)$ is such that

$$A_f(w) \begin{cases} P & \text{if } w \in S_2^{\pm}(n,r), \\ N & \text{if } w \in S_1^{\pm}(n,r). \end{cases} \tag{4.4}$$

This shows that $|S^{+}_{A_f}(n,r)| = |S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r) \dot{\bigcup} S_2^{\pm}(n,r)| = 2^{n-r-1} + 2^{n-r-1} + 2^{n-1} - 2 \cdot 2^{n-r-1} = 2^{n-1}$.

In [1, Theorem 1] it has been proved that $\gamma(n,1) = 2^{n-1}$ and $\gamma(n,r) \geq 2^{n-1}$. Since $\gamma(n,r) \geq \gamma^{*}(n,r)$, using a technique similar to that used in the proof Theorem 1 of [1], it easily follows that $\gamma^{*}(n,r) \geq 2^{n-1}$. As shown above, it results that $|S^{+}_{A_f}(n,r)| = 2^{n-1}$. Hence $2^{n-1} \geq \gamma(n,r) \geq \gamma^{*}(n,r) \geq 2^{n-1}$, that is, $\gamma(n,r) = \gamma^{*}(n,r) = 2^{n-1}$. This proves; the first part of theorem; it remains to prove the latter part.

We consider now the following $(n,r)$-positive weight function $g : A(n,r) \to \mathbb{R}$:

$$g : \begin{array}{cccccccc} \widetilde{r} & \cdots & \widetilde{1} & 0^{\S} & \overline{1} & \cdots & \overline{(n-r-1)} & \overline{(n-r)} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & \cdots & 1 & 0 & \dfrac{-1}{n-r} & \cdots & \dfrac{-1}{n-r} & \dfrac{-1}{n-r} \end{array} \tag{4.5}$$

It results then that the sum $\Sigma_g : S(n,r) \to \mathbb{R}$ is such that $\Sigma_g(w) \geq 0$ if $w \in S_1^{\pm}(n,r)\dot{\bigcup}S_2^{\pm}(n,r)$; that is, the boolean map $A_g \in \mathcal{W}_+(n,r)$ is such that $A_g(w) = P$ if $w \in S_1^{\pm}(n,r)\dot{\bigcup}S_2^{\pm}(n,r)$. This shows that $|S^+_{A_g}(n,r)\ | = |S_1^{\pm}(n,r)\dot{\bigcup}S_2^{\pm}(n,r)\dot{\bigcup}S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r)| = (2^{n-1} - 2^{n-r}) + (2^{n-1} - 2^{n-r}) + 2^{n-r-1} + 2^{n-r-1} = 2^n - 2^{n-r}$.

On the other hand, it is clear that for any $A \in \mathcal{W}_+(n,r)$ it results $A(w) = P$ for each $w \in S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r)$ and $A(w) = N$ for each $w \in S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r)$. The number $(2^n - 2^{n-r})$ is the biggest number of values $P$ that a boolean map $A \in \mathcal{W}_+(n,r)$ can assume. Hence, since $\eta(n,r) \leq \eta^*(n,r)$, we have $2^n - 2^{n-r} = |S^+_{A_g}(n,r)| \leq \eta(n,r) \leq \eta^*(n,r) \leq 2^n - 2^{n-r}$; that is be; $\eta(n,r) = \eta^*(n,r) = 2^n - 2^{n-r}$. This concludes the proof of Theorem 4.1. □

To better visualize the previous result, we give a numerical example on a specific Hasse diagram. Let $n = 6$ and $r = 2$ and let $f$ be as given in previous theorem; that is

$$f : \begin{array}{ccccccc} \widetilde{2} & \widetilde{1} & 0^{\S} & \overline{1} & \overline{2} & \overline{3} & \overline{4} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 4 & 0 & -1 & -1 & -1 & -5 \end{array} \tag{4.6}$$

In Figure 2 we have shown the Hasse diagram of the lattice $S(6,2)$, where $S_1^{+}(n,r)$ is black, $S_1^{\pm}(n,r)$ is violet, $S_1^{-}(n,r)$ is red, $S_2^{+}(n,r)$ is blue, $S_2^{\pm}(n,r)$ is brown, $S_2^{-}(n,r)$ is green. Therefore $A_f$ assume the following values.

(i) The blue, black, and brown nodes correspond to values $P$ of $A_f$,

(ii) The violet, red, and green nodes correspond to values $N$ of $A_f$.

First to give the proof of Theorem 4.5 we need to introduce some useful results and the concept of basis in $S(n,r)$. In the following first lemma we show some properties of the sublattices of $S(n,r)$.

**Lemma 4.2.** *Here the following properties hold, where* $\theta = 00\cdots0 \mid 0\cdots0$ *and* $\Theta = r\cdots21 \mid 12\cdots(n-r)$.

(i) $\uparrow \Theta = S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r)$.

(ii) $\downarrow \theta = S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r)$.

(iii) $\uparrow S_2^{\pm}(n,r) \subseteq S_2^{\pm}(n,r)\dot{\bigcup}S_2^{+}(n,r)$.

(iv) $\downarrow S_1^{\pm}(n,r) \subseteq S_1^{\pm}(n,r)\dot{\bigcup}S_1^{-}(n,r)$.

(v) $((S_1^{\pm}(n,r))^c = S_2^{\pm}(n,r)$.

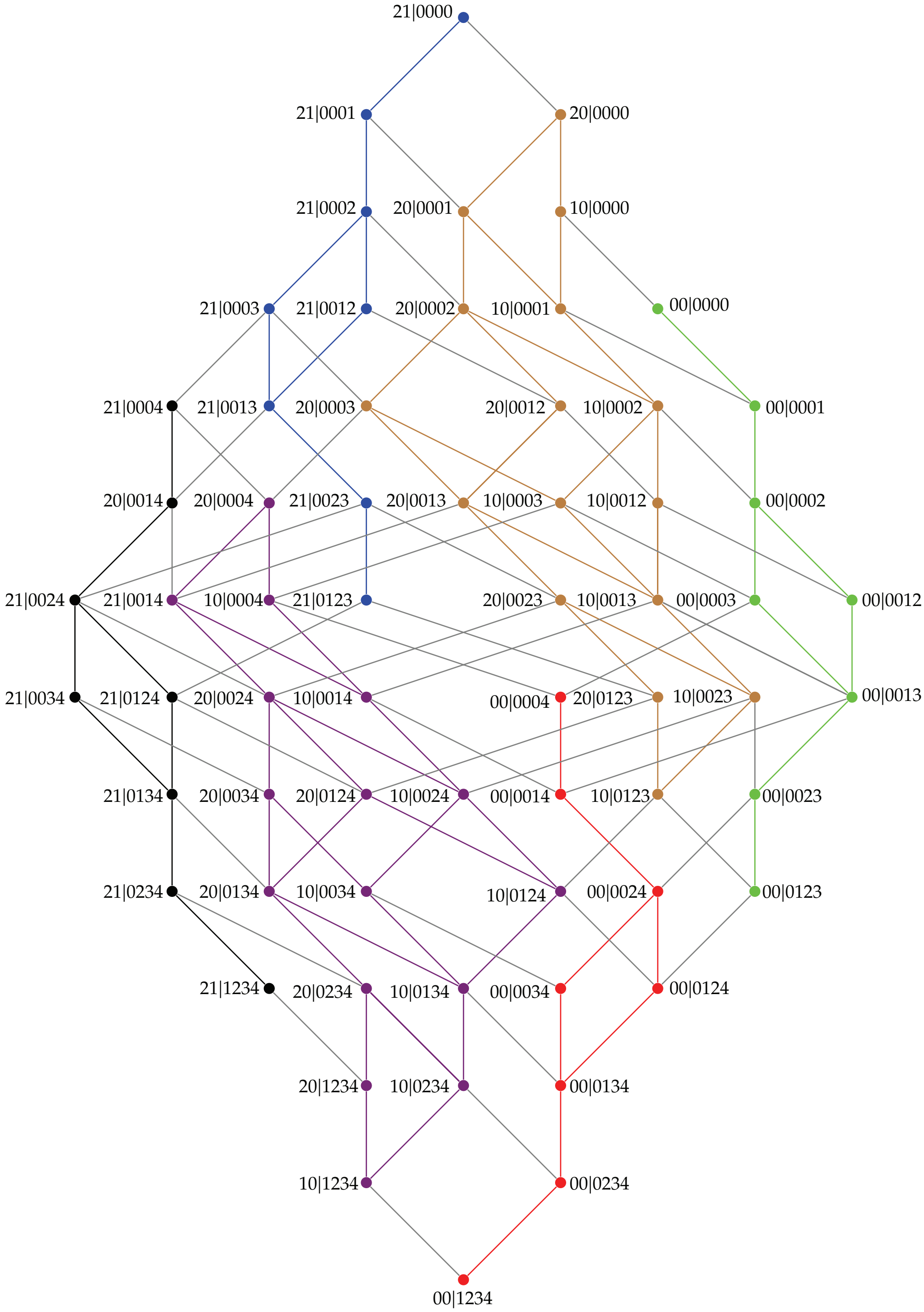

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**Figure** 2

*Proof.* (i) If $w \in (\uparrow \Theta)$, then $\Theta \sqsubseteq w$; that is, it has the form $w = r \cdots 1 \mid j_1 \cdots j_{n-r}$, where $j_1 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r}\}$; therefore $w \in S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r)$. If $w \in S_1^{+}(n,r)$, it has the form $w = r \cdots 1 \mid j_1 \cdots j_{n-r-1}(n-r)$, where $j_1 \cdots j_{n-r-1} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$; if $w \in S_2^{+}(n,r)$, it has the form $w = r \cdots 1 \mid 0 j_2 \cdots j_{n-r}$, where $j_2 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$. In both cases it results that $\Theta \sqsubseteq w$, that is, $w \in (\uparrow \Theta)$.

(ii) It is analogue to (i).

(iii) The minimum of the sublattice $S_2^{\pm}(n,r)$ is $\alpha = 10 \cdots 0 \mid 01 \cdots (n-r-1)$; since $\uparrow S_2^{\pm}(n,r) \subseteq \uparrow \alpha$, it is sufficient to show that $\uparrow \alpha = S_2^{\pm}(n,r) \bigcup S_2^{+}(n,r)$. The inclusion $S_2^{\pm}(n,r) \bigcup S_2^{+}(n,r) \subseteq \uparrow \alpha$ follows by the definition of $S_2^{\pm}(n,r)$ and of $S_2^{+}(n,r)$. On the other side, if $w \in \uparrow \alpha$, it follows that $\alpha \sqsubseteq w$; that is, $w = i_1 \cdots i_r \mid 0 j_2 \cdots j_{n-r}$, with $i_1 \succ 0^{\S}$ and $j_2 \cdots j_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$. Therefore $w \in S_2^{\pm}(n,r) \bigcup S_2^{+}(n,r)$, and this proves the other inclusion.

(iv) Let us consider the maximum of the sublattice $S_1^{\pm}(n,r)$; that is $t_1 = r(r-1) \cdots 20 \mid 0 \cdots 0(n-r)$. Since $\downarrow S_1^{\pm}(n,r) \subseteq \downarrow \beta$, it is sufficient to show that $\downarrow t_1 = S_1^{\pm}(n,r) \bigcup S_1^{-}(n,r)$; this proof is similar to (iii).

(v) If $w \in S_1^{\pm}(n,r)$; it has the form $w = i_1 \cdots i_{r-1} 0 \mid j_1 \cdots j_{n-r-1}(n-r)$, with $i_1 \succ 0^{\S}$ and $j_1 \cdots j_{n-r-1} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$; therefore its complement has the form $w^c = i'_1 \cdots i'_{r-1} 0 \mid 0 j'_2 \cdots j'_{n-r}$, with $i'_1 \succ 0^{\S}$ and $j'_2 \cdots j'_{n-r} \in \{0^{\S}, \overline{1}, \ldots, \overline{n-r-1}\}$; hence $w^c \in S_2^{\pm}(n,r)$ . This shows that $(S_1^{\pm}(n,r))^c \subseteq S_2^{\pm}(n,r)$.

Now, if $w \in S_2^{\pm}(n,r)$, we write $w$ in the form $(w^c)^c$; then $w^c \in (S_2^{\pm}(n,r))^c \subseteq S_1^{\pm}(n,r)$; therefore $w \in (S_1^{\pm}(n,r))^c$. This shows that $S_2^{\pm}(n,r) \subseteq (S_1^{\pm}(n,r))^c$. □

At this point let us recall the definition of basis for $S(n,r)$, as given in [3] in a more general context. In the same way as an antichain uniquely determines a Boolean order-preserving map, a basis uniquely determines a Boolean map that has the properties (BM1) and (BM2) (see [3] for details). Hence the concept of basis will be fundamental in the sequel of this proof.

*Definition 4.3.* A basis for $S(n,r)$ is an ordered couple $\langle Y_+ \mid Y_- \rangle$, where $Y_+$ and $Y_-$ are two disjoint antichains of $S(n,r)$ such that

(B1) $(\downarrow Y_+) \bigcap (Y_-^c) = \emptyset$,

(B2) $((\uparrow Y_+) \bigcup \uparrow (Y_-^c)) \bigcap \downarrow Y_- = \emptyset$,

(B3) $S(n,r) = ((\uparrow Y_+) \bigcup \uparrow (Y_-^c)) \bigcup \downarrow Y_-$.

In the proof of Theorem 4.5 we will construct explicitly such a basis.

We will also use the following result that was proved in [3].

**Lemma 4.4.** *Let $\langle W_+ \mid W_- \rangle$ be a basis for $S(n,r)$. Then the map*

$$A(x) \begin{cases} P & \text{if } x \in \uparrow (W_+) \bigcup \uparrow (W_-^c), \\ N & \text{if } x \in \downarrow W_- \end{cases} \tag{4.7}$$

*is such that $A \in \mathcal{W}_+(S(n,r), \mathbf{2})$.*

**Theorem 4.5.** *If $q$ is a fixed integer with $2^{n-1} \leq q \leq 2^n - 2^{n-r}$, then there exists a boolean map $A_q \in \mathcal{W}_+(n,r)$ such that $|S^+_{A_q}(n,r)| = q$.*

*Proof.* Let $q$ be a fixed integer such that $2^{n-1} \leq q \leq 2^n - 2^{n-r}$. We determine a specific Boolean total map $A \in \mathcal{W}_+(n,r)$ such that $|S^+_A(n,r)| = q$. We proceed as follows.

The case $r = 1$ is proved in the previous Theorem 4.1. Let us assume then $r > 1$. Since $S^{\pm}_1(n,r)$ is a finite distributive sublattice of graded lattice $S(n,r)$, also $S^{\pm}_1(n,r)$ is a graded lattice. We denote by $R$ the rank of $S^{\pm}_1(n,r)$ and with $\rho_1$ its rank function. Note that the bottom of $S^{\pm}_1(n,r)$ is $b_1 = 10\cdots 0 \mid 1\cdots(n-r-1)(n-r)$ and the top is $t_1 = r(r-1)\cdots 20 \mid 0\cdots 0(n-r)$.

We write $q$ in the form $q = 2^{n-1} + p$ with $0 \leq p \leq 2^n - 2^{n-r} - 2^{n-1} = 2^{n-1} - 2^{n-r} = |S^{\pm}_1(n,r)|$. We will build a map $A \in W_+(n,r)$ such that $|S^+_A(n,r)| = 2^{n-1} + p$.

If $p = 0$, we take $A = A_f$, with $f$ as in Theorem 4.1, and hence we have $|S^+_A(n,r)| = 2^{n-1}$.

If $p = 2^{n-1} - 2^{n-r}$, we take $A = A_g$, with $g$ as in Theorem 4.1, and we have $|S^+_A(n,r)| = 2^n - 2^{n-r}$.

Therefore we can assume that $0 < p < 2^{n-1} - 2^{n-r}$.

If $0 \leq i \leq R$, we denote by $\mathfrak{R}_i$ the set of elements $w \in S^{\pm}_1(n,r)$ such that $\rho_1(w) = R - i$, and we also set $\beta_i := |\mathfrak{R}_i|$ to simplify the notation. We write each $\mathfrak{R}_i$ in the following form: $\mathfrak{R}_i = \{v_{i1}, \ldots, v_{i\beta_i}\}$ . If $0 \leq l \leq R$, we set $\mathfrak{U}_l := \dot{\bigcup}_{i=0,\ldots,l} \mathfrak{R}_i$.

If $0 \leq l \leq R-2$, we set $\mathfrak{B}_l := \dot{\bigcup}_{i=l+2,\ldots,R} \mathfrak{R}_i$ and $\mathfrak{B}_{R-1} := \mathfrak{B}_R := \emptyset$. We can then write $p$ in the form $p = \sum_{i=0}^{k} |\mathfrak{R}_i| + s = |\mathfrak{U}_k| + s$, for some $0 \leq s < |\mathfrak{R}_{k+1}|$ and some $0 \leq k \leq R-1$. Depending on the previous number $s$ we partition $\mathfrak{R}_{k+1}$ into the following two disjoint subsets: $\mathfrak{R}_{k+1} = \{v_{(k+1)1}, \ldots, v_{(k+1)s}\} \dot{\bigcup} \{v_{(k+1)(s+1)}, \ldots, v_{(k+1)\beta_{k+1}}\}$, where the first subset is considered empty if $s = 0$.

In the sequel, to avoid an overload of notations, we write simply $v_i$ instead of $v_{(k+1)i}$, for $i = 1, \ldots, \beta_{k+1}$.

Let us note that $S^{\pm}_1(n,r) = \mathfrak{U}_k \dot{\bigcup} \mathfrak{R}_{k+1} \dot{\bigcup} \mathfrak{B}_k$.

We define now the map $A : S(n,r) \to \mathbf{2}$:

$$A(w) = \begin{cases} P & \text{if } w \in S^{\pm}_2(n,r) \dot{\bigcup} S^+_1(n,r) \dot{\bigcup} S^+_2(n,r), \\ P & \text{if } w \in \mathfrak{U}_k \dot{\bigcup} \{v_1, \ldots, v_s\}, \\ N & \text{if } w \in \mathfrak{B}_k \dot{\bigcup} \{v_{s+1}, \ldots, v_{\beta_{k+1}}\}, \\ N & \text{if } w \in S^-_1(n,r) \dot{\bigcup} S^-_2(n,r). \end{cases} \tag{4.8}$$

Let us observe that $|S^+_A(n,r)| = |S^{\pm}_2(n,r) \dot{\bigcup} S^+_1(n,r) \dot{\bigcup} S^+_2(n,r)| + |\mathfrak{U}_k \dot{\bigcup} \{v_1, \ldots, v_s\}| = (2^{n-1} - 2^{n-r}) + 2^{n-r-1} + 2^{n-r-1} + |\mathfrak{U}_k| + s = 2^{n-1} + p = q$.

Therefore, if we show that $A \in \mathcal{W}_+(n,r)$, the theorem is proved.

We write $\mathfrak{R}_k$ in the following way: $\mathfrak{R}_k = \{w_1, \ldots, w_t\} \dot{\bigcup} \{w_{t+1}, \ldots, w_{\beta_k}\}$, where $\{w_1, \ldots, w_t\} = \mathfrak{R}_k \bigcap \uparrow \{v_1, \ldots, v_s\}$ and $\{w_{t+1}, \ldots, w_{\beta_k}\} = \mathfrak{R}_k \setminus \{w_1, \ldots, w_t\}$. Analogously $\mathfrak{R}_{k+2} = \{z_1, \ldots, z_q\} \dot{\bigcup} \{z_{q+1}, \ldots, z_{\beta_{k+2}}\}$, where $\{z_{q+1}, \ldots, z_{\beta_{k+2}}\} = \mathfrak{R}_{k+2} \bigcap \downarrow \{v_{s+1}, \ldots, v_{\beta_{k+1}}\}$ and $\{z_1, \ldots, z_q\} = \mathfrak{R}_{k+2} \setminus \{z_{q+1}, \ldots, z_{\beta_{k+2}}\}$.

We can see a picture of this partition of the sublattice $S^{\pm}_1(n,r)$ in Figure 3.

Depending on $s$ and $k$, we build now a particular basis for $S(n,r)$.

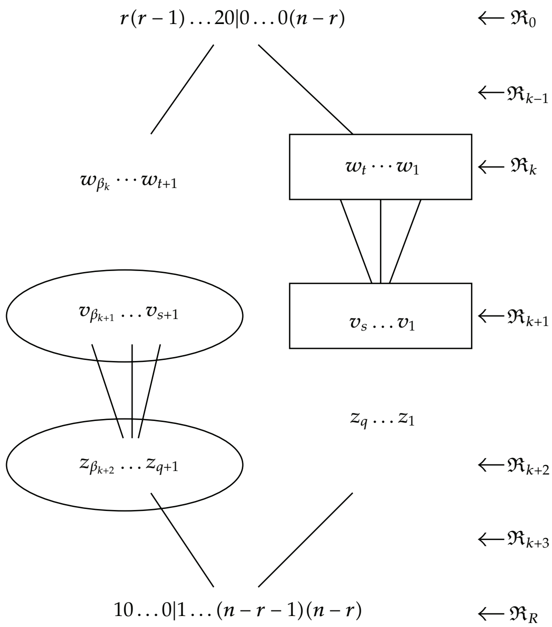


**Figure** 3

To such aim, let us consider the minimum $\alpha = 10\cdots 0 \mid 01\cdots(n-r-1)$ of $S_2^{\pm}(n,r)$ and the subsets $T_+ := \{v_1,\dots,v_s,w_{t+1},\dots,w_{\beta_k}\}$ and $T_- := \{v_{s+1},\dots,v_{\beta_{k+1}},z_1,\dots,z_q\}$. Let us distinguish two cases:

(a1) $\alpha \in\uparrow T_+$,

(a2) $\alpha \notin\uparrow T_+$.

We define two different couples of subsets as follows:

In the case (a1) we set $Y_+ := T_+$ and $Y_- := T_-\dot{\bigcup}\{\theta\}$; in the case (a2) we set $Y_+ := T_+\bigcup\{\alpha\}$ and $Y_- := T_-\dot{\bigcup}\{\theta\}$.

*Step 1.* $\langle Y_+ \mid Y_-\rangle$ is a couple of two disjoint antichains of $S(n,r)$.

In both cases (a1) and (a2) it is obvious that $Y_+\bigcap Y_- = \emptyset$.

*Case (a1)*

The elements $\{v_1,\dots,v_s\}$ are not comparable between them because they have all rank $R-(k+1)$ and analogously for the elements $\{w_{t+1},\dots,w_{\beta_k}\}$ that have all rank $R-k$. Let now $v\in\{v_1,\dots,v_s\}$ and let $w\in\{w_{t+1},\dots,w_{\beta_k}\}$; then $w\notin\downarrow v$ because $\rho_1(v)<\rho_1(w)$, and $w\notin\uparrow v$ because $\{w_{t+1},\dots,w_{\beta_k}\}\bigcap\uparrow\{v_1,\dots,v_s\}=\emptyset$ by construction. For the elements in $Y_-$ different from $\theta$, we can proceed as for $Y_+$. On the other side, we can observe that $\theta$ is not comparable with none of the elements $v_{s+1},\dots,v_{\beta_{k+1}},z_1,\dots,z_q$ since these are all in $S_1^{\pm}(n,r)$ while $\theta\in S_2^{-}(n,r)$. Thus $Y_+$ is an antichain.

*Case (a2)*

In this case we only must show that $\alpha$ is not comparable with none of the elements $v_1, \ldots, v_s, w_{t+1}, \ldots, w_{\beta_k}$. At first from the fact that $\alpha \notin\uparrow T_+$ it follows $v_i \notin\downarrow \alpha$ for each $i = 1, \ldots, s$, and $w_j \notin\downarrow \alpha$ for each $j = t+1, \ldots, \beta_k$. Moreover, the elements $v_1, \ldots, v_s$ and $w_{t+1}, \ldots, w_{\beta_k}$ are all in $S_1^{\pm}(n,r)$; hence they have the form $i_1 \cdots i_{r-1}0 \mid j_1 \cdots j_{n-r-1}(n-r)$, with $i_1 \succ 0^{\S}$, while $\alpha = 10\cdots 0 \mid 01 \cdots (n-r-1)$, so $\alpha \notin\downarrow v_i$ and $\alpha \notin\downarrow w_j$ for each $i = 1, \ldots, s$ and each $j = t+1, \ldots, \beta_k$.

*Step 2.* $\langle Y_+ \mid Y_- \rangle$ is a basis for $S(n,r)$.
We must see that (B1), (B2), and (B3) hold in both cases (a1) and (a2).

*Case (a1)*

(B1) Let us begin to observe that $T_-^c = \{v_{s+1}^c, \ldots, v_{\beta_{k+1}}^c, z_1^c, \ldots, z_q^c\} \subseteq (S_1^{\pm}(n,r))^c = S_2^{\pm}(n,r)$. Since $\theta^c = \Theta$, we have then $Y_-^c = T_-^c \dot{\bigcup} \{\theta^c\} \subseteq S_2^{\pm}(n,r) \dot{\bigcup} \{\Theta\} \subseteq S_2^{\pm}(n,r) \dot{\bigcup} S_1^{+}(n,r)$ . On the other hand, since $\downarrow Y_+ \subseteq\downarrow S_1^{\pm}(n,r)$, by Lemma 4.2(iv) we have also that $\downarrow Y_+ \subseteq S_1^{\pm}(n,r) \dot{\bigcup} S_1^{-}(n,r)$. Hence $(\downarrow Y_+) \bigcap (Y_-^c) = \emptyset$. This proves (B1).

(B2) We show at first that $\uparrow Y_-^c \bigcap \downarrow Y_- = \emptyset$. Since $Y_-^c \subseteq S_2^{\pm}(n,r) \dot{\bigcup} \{\Theta\}$, we have that $\uparrow Y_-^c \subseteq\uparrow (S_2^{\pm}(n,r)) \bigcup \uparrow \Theta$. By Lemma 4.2(i) and (iii) we have then $\uparrow Y_-^c \subseteq S_2^{\pm}(n,r) \dot{\bigcup} S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r)$. On the other side, since $\downarrow T_- \subseteq\downarrow S_1^{\pm}(n,r)$, by Lemma 4.2(iv) it follows that $\downarrow T_- \subseteq S_1^{\pm}(n,r) \dot{\bigcup} S_1^{-}(n,r)$. By Lemma 4.2(ii), we have $\downarrow \theta = S_1^{-}(n,r) \dot{\bigcup} S_2^{-}(n,r)$. Hence it holds $\downarrow Y_- \subseteq S_1^{\pm}(n,r) \dot{\bigcup} S_1^{-}(n,r) \dot{\bigcup} S_2^{-}(n,r)$. This proves that $\uparrow Y_-^c \bigcap \downarrow Y_- = \emptyset$. We show now that also $\uparrow Y_+ \bigcap \downarrow Y_- = \emptyset$; we proceed by contradiction. Let us suppose that there exists an element $z \in\uparrow Y_+ \bigcap \downarrow Y_-$, then there are two elements $w_+ \in Y_+$ and $w_- \in Y_-$ such that $w_+ \sqsubseteq z \sqsubseteq w_-$, hence $w_+ \sqsubseteq w_-$. We will distinguish the following five cases, and in each of them we will find a contradiction.

(a) $w_+ \in \{v_1, \ldots, v_s\}$ and $w_- \in \{v_{s+1}, \ldots, v_{\beta_{k+1}}\}$. In this case $w_+$ and $w_-$ are two distinct elements having both rank $R-(k+1)$ and such that $w_+ \sqsubseteq w_-$; it is not possible.

(b) $w_+ \in \{v_1, \ldots, v_s\}$ and $w_- \in \{z_1, \ldots, z_q\}$. In this case $w_+$ has rank $R-(k+1)$ while $w_-$ has rank $R-(k+2) < R-(k+1)$, and this contradicts the condition $w_+ \sqsubseteq w_-$.

(c) $w_+ \in \{w_{t+1}, \ldots, w_{\beta_k}\}$ and $w_- \in \{v_{s+1}, \ldots, v_{\beta_{k+1}}\}$. This case is similar to the previous because $w_+$ has rank $R-k$ while $w_-$ has rank $R-(k+1)$.

(d) $w_+ \in \{w_{t+1}, \ldots, w_{\beta_k}\}$ and $w_- \in \{z_1, \ldots, z_q\}$. This is similar to the previous because $w_+$ has rank $R-k$ while $w_-$ has rank $R-(k+2)$.

(e) $w_+ \in \{v_1, \ldots, v_s, w_{t+1}, \ldots, w_{\beta_k}\}$ and $w_- = \theta$. In this case the condition $w_+ \sqsubseteq w_-$ implies that $w_+ \in\downarrow \theta$; since $\downarrow \theta = S_1^{-}(n,r) \dot{\bigcup} S_2^{-}(n,r)$ this is not possible since $w_+ \in S_1^{\pm}(n,r)$.

(B3) Since $\alpha$ is the minimum of $S_2^{\pm}(n,r)$ we have that $S_2^{\pm}(n,r) \subseteq\uparrow \alpha$, moreover, $\alpha \in\uparrow Y_+$; therefore $S_2^{\pm}(n,r) \subseteq\uparrow \alpha \subseteq\uparrow Y_+$. Since $\Theta = \theta^c \in Y_-^c$, it follows that $\uparrow \Theta \subseteq\uparrow Y_-^c$. By Lemma 4.2(i) we have then that $(S_1^{+}(n,r) \dot{\bigcup} S_2^{+}(n,r)) =\uparrow \Theta \subseteq\uparrow Y_-^c$. By Lemma 4.2(ii) we also have that $(S_1^{-}(n,r) \dot{\bigcup} S_2^{-}(n,r)) =\downarrow \theta \subseteq\downarrow Y_-$.

To complete the proof of (B3) let us observe that $S_1^{\pm}(n,r) = (\bigcup_{i=0}^{k} \mathfrak{R}_i)\dot{\bigcup}\mathfrak{R}_{k+1}\dot{\bigcup}(\bigcup_{j=k+2}^{R} \mathfrak{R}_j)$, where $(\bigcup_{i=0}^{k} \mathfrak{R}_i) \subseteq \uparrow \mathfrak{R}_k \subseteq \uparrow Y_+$, $(\bigcup_{j=k+2}^{R} \mathfrak{R}_j) \subseteq \downarrow \mathfrak{R}_{k+2} \subseteq \downarrow Y_-$.

Moreover, since $\mathfrak{R}_{k+1} = \{v_1, \dots, v_s\}\dot{\bigcup}\{v_{s+1}, \dots, v_{\beta_{k+1}}\}$, with $\{v_1, \dots, v_s\} \subseteq Y_+ \subseteq \uparrow Y_+$ and $\{v_{s+1}, \dots, v_{\beta_{k+1}}\} \subseteq Y_- \subseteq \downarrow Y_-$, then $\mathfrak{R}_{k+1} \subseteq (\uparrow Y_+ \bigcup \downarrow Y_-)$. This shows that $S_1^{\pm}(n,r) \subseteq (\uparrow Y_+ \bigcup \downarrow Y_-)$. Since the six sublattices $S_i^{+}(n,r)$, $S_i^{\pm}(n,r)$ and $S_i^{-}(n,r)$ for $i = 1,2$ are a partition of $S(n,r)$, the property (B3) is proved.

*Case (a2)*

(B1) We begin to show that $\alpha \notin Y_-^c$. In fact, it holds that $\alpha = (r(r-1)\cdots 20 \mid 0\cdots 0(n-r))^c$; suppose that $\alpha \in Y_-^c$, and this shows that we obtain a contradiction. By $\alpha \in Y_-^c$ it follows $\alpha^c = t_1 \in Y_-$. But $t_1$ is the top of $S_1^{\pm}(n,r)$, so $Y_- = t_1$, since $Y_-$ is antichain. This means that $\mathfrak{R}_{k+1} = \mathfrak{R}_0$ and this is not possible, of course. Since $Y_+ = T_+\dot{\bigcup}\{\alpha\}$, we have $(\downarrow Y_+) \bigcap Y_-^c = ((\downarrow T_+) \bigcap Y_-^c) \bigcup((\downarrow \alpha) \bigcap Y_-^c)$, and, moreover, as in the proof of (B1) in the case (a1), we also have that $(\downarrow T_+) \bigcap Y_-^c = \emptyset$; therefore, to prove (B1) it is sufficient to show that $(\downarrow \alpha) \bigcap Y_-^c = \emptyset$; As in the proof of (B1) in the case (a1), we have $Y_-^c \subseteq S_2^{\pm}(n,r)\dot{\bigcup}\{\Theta\}$. Since $\alpha$ is the minimum of $S_2^{\pm}(n,r)$ and $\Theta \notin\downarrow \alpha$, it follows that the unique element of $\downarrow \alpha$ that can belong to $Y_-^c$ is $\alpha$, but we have shown before that this is not true.

(B2) As in the previous case we have $(\uparrow Y_-^c) \bigcap \downarrow Y_- = \emptyset$; moreover $(\uparrow Y_+) \bigcap(\downarrow Y_-) = ((\uparrow T_+) \bigcap(\uparrow \alpha)) \bigcap(\downarrow Y_-)((\uparrow T_+) \bigcap(\downarrow Y_-)) \bigcup((\uparrow \alpha) \bigcap(\downarrow Y_-))$. As in the case (a1) we have $((\uparrow T_+) \bigcap(\downarrow Y_-)) = \emptyset$; therefore, to prove (B2) also in the case (a2), it is sufficient to show that $((\uparrow \alpha) \bigcap(\downarrow Y_-)) = \emptyset$. As in the proof of (B2) in the case (a1) it results that $\downarrow Y_- \subseteq S_2^{\pm}(n,r)\dot{\bigcup}S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r)$, and, by definition of $\alpha$, it is easy to observe that $(\uparrow \alpha) \bigcap(S_1^{\pm}(n,r)\dot{\bigcup}S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r)) = \emptyset$. Hence $((\uparrow \alpha) \bigcap(\downarrow Y_-)) = \emptyset$.

(B3) It is identical to that of case (a1).

*Step 3.* The map A defined in (4.8) is such that $A \in \mathcal{W}_+(n,r)$.

Since we have proved that$\langle Y_+ \mid Y_- \rangle$ is a basis for $S(n,r)$, by Lemma 4.4 it follows that the map $A \in \mathcal{W}_+(S(n,r), \mathbf{2})$ if the two following identities hold:

$$(\uparrow Y_+) \bigcup(\uparrow Y_-^c) = S_2^{\pm}(n,r)\dot{\bigcup}S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r)\dot{\bigcup}\mathfrak{U}_k\dot{\bigcup}\{v_1, \dots, v_s\}, \tag{4.9}$$

$$\downarrow Y_- = \mathfrak{B}_k\dot{\bigcup}\{v_{s+1}, \dots, v_{\beta_{k+1}}\}\dot{\bigcup}S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r). \tag{4.10}$$

We prove at first (4.10). By definition of $\mathfrak{B}_k$ and $T_-$ it easy to observe that $\mathfrak{B}_k\dot{\bigcup}\{v_{s+1}, \dots, v_{\beta_{k+1}}\} \subseteq\downarrow T_-$, and, moreover, by Lemma 4.2(ii) we also have that $\downarrow \theta = S_1^{-}(n,r) \bigcup S_2^{-}(n,r)$; hence, since $\downarrow Y_- =\downarrow \theta \bigcup \downarrow T_-$, it results that $\mathfrak{B}_k\dot{\bigcup}\{v_{s+1}, \dots, v_{\beta_{k+1}}\} \bigcup S_1^{-}(n,r)\dot{\bigcup}S_2^{-}(n,r) \subseteq\downarrow Y_-$. On the other hand, by Lemma 4.2(iv) we have that $\downarrow T_- \subseteq S_1^{\pm}(n,r)\dot{\bigcup}S_1^{-}(n,r)$ because $T_-$ is a subset of $S_1^{\pm}(n,r)$. At this point let us note that the elements of $\downarrow T_-$ that are also in $S_1^{\pm}(n,r)$ must belong necessarily to the subset $\mathfrak{B}_k\dot{\bigcup} \{v_{s+1}, \dots, v_{\beta_{k+1}}\}$. This proves the other inclusion and hence (4.10).

To prove now (4.9) we must distinguish the cases (a1) and (a2). We set $\Delta := S_2^{\pm}(n,r)\dot{\bigcup}S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r) \bigcup \mathfrak{U}_k\dot{\bigcup}\{v_1, \dots, v_s\}$. Let us first examine the case (a1). Since $\alpha$ is the minimum of $S_2^{\pm}(n,r)$, we have $S_2^{\pm}(n,r) \subseteq\uparrow \alpha \subseteq\uparrow Y_+$. Moreover, since $Y_- = T_- \bigcup\{\theta\}$, it follows that $\uparrow Y_-^c \supseteq\uparrow \theta^c =\uparrow (\Theta) = S_1^{+}(n,r)\dot{\bigcup}S_2^{+}(n,r)$ by lemma 3.2(i). Finally, since $\mathfrak{U}_k\dot{\bigcup}\{v_1, \dots, v_s\} \subseteq\uparrow T_+ =\uparrow Y_+$, the inclusion $\supseteq$ in (4.9) is proved. To prove the other inclusion

$\subseteq$ we begin to observe that $\uparrow Y_+ \bigcap (S_1^-(n,r) \bigcup S_1^-(n,r)) = \emptyset$; therefore the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r) \dot{\bigcup} S_1^+(n,r) \dot{\bigcup} S_2^+(n,r)$ must be necessarily in $S_1^{\pm}(n,r)$, and such elements, by definition of $T_+$, must be necessarily in $\mathfrak{U}_k \dot{\bigcup} \{v_1, \ldots, v_s\}$. This proves that $\uparrow Y_+ \subseteq \Delta$, For $\uparrow (Y_-^c)$, we have $\uparrow (Y_-^c) = \uparrow (T_-^c \bigcup \{\theta\}^c) = (\uparrow T_-^c) \bigcup (\uparrow \Theta)$ , where $\uparrow (\Theta) = S_1^+(n,r) \dot{\bigcup} S_2^+(n,r)$ by Lemma 4.2(i), and, since $T_- \subseteq S_1^{\pm}(n,r)$, also $T_-^c \subseteq (S_1^{\pm}(n,r))^c = S_2^{\pm}(n,r)$, by Lemma 4.2(v). Therefore $\uparrow T_-^c \subseteq \uparrow S_2^{\pm}(n,r) \subseteq S_2^{\pm}(n,r) \bigcup S_2^+(n,r)$ by Lemma 4.2(iii). This shows that $\uparrow (Y_-^c) \subseteq S_2^{\pm}(n,r) \bigcup S_2^+(n,r) \bigcup S_1^+(n,r) \subseteq \Delta$, hence, the inclusion $\subseteq$. The proof of (4.9) in the case (a1) is therefore complete.

Finally, to prove (4.9) in the case (a2), it easy to observe that the only difference with respect to case (a1) is when we must show that $\uparrow Y_+ \subseteq \Delta$. In fact, in the case (a2) it results that $\alpha \notin \uparrow T_+$ and $Y_+ = T_+ \bigcup \{\alpha\}$, while $Y_-$ is the same in both cases (a1) and (a2). Therefore, in the case (a2), the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r) \dot{\bigcup} S_1^+(n,r) \dot{\bigcup} S_2^+(n,r)$ must be in $(\uparrow T_+) \bigcap S_1^{\pm}(n,r)$ or in $(\uparrow \alpha) \bigcap S_1^{\pm}(n,r)$. As in the case (a1) we have $(\uparrow T_+) \bigcap S_1^{\pm}(n,r) = \mathfrak{U}_k \dot{\bigcup} \{v_1, \ldots, v_s\}$, and, since $\alpha$ is the minimum of $S_2^{\pm}(n,r)$, it results that $\uparrow \alpha = \uparrow S_2^{\pm}(n,r) \subseteq S_2^{\pm}(n,r) \bigcup S_2^+(n,r)$ by Lemma 4.2(iii); hence $(\uparrow \alpha) \bigcap S_1^{\pm}(n,r) = \emptyset$. Therefore, also in the case (a2), the elements of $\uparrow Y_+$ that are not in $S_2^{\pm}(n,r) \dot{\bigcup} S_1^+(n,r) \dot{\bigcup} S_2^+(n,r)$ must be necessarily in $\mathfrak{U}_k \dot{\bigcup} \{v_1, \ldots, v_s\}$. The other parts of the proof are same as in case (a1). Hence we have proved the identities (4.9) and (4.10). By Lemma 4.4 it follows then that the map $A \in \mathcal{W}_+(S(n,r), \mathbf{2})$. Finally, by definition of $A$, we have obviously $A(\theta) = N$, $A(\xi_1) = P$, and $A(\Theta) = P$. This shows that $A \in \mathcal{W}_+(n,r)$. The proof is complete. $\square$

To conclude we emphasize the elegant symmetry of the induced partitions on $S(n,r)$ from the boolean total maps $A_q$'s constructed in the proof of the Theorem 4.5.

## References


[1] N. Manickam and D. Miklós, “On the number of nonnegative partial sums of a nonnegative sum,” *Colloquia Mathematica Societatis Janos Bolyai*, vol. 52, pp. 385–392, 1987.

[2] C. Bisi and G. Chiaselotti, “A class of lattices and boolean functions related to a,” *Manickam-Miklös-Singhi Conjecture*. In press.

[3] C. Bisi and G. Chiaselotti, “Extension results for boolean maps and a class of systems of linear inequalities,” http://arxiv.org/abs/1012.5486.

[4] G. Chiaselotti, “On a problem concerning the weight functions,” *European Journal of Combinatorics*, vol. 23, no. 1, pp. 15–22, 2002.

[5] A. Bhattacharya, “On a conjecture of Manickam and Singhi,” *Discrete Mathematics*, vol. 272, no. 2-3, pp. 259–261, 2003.

[6] A. Bhattacharya, *Some problems in combinatorics*, Ph.D. thesis, Indian Institute of Technology, Bombay, India, 2004.

[7] T. Bier and N. Manickam, “The first distribution invariant of the Johnson-scheme,” *Southeast Asian Bulletin of Mathematics*, vol. 11, no. 1, pp. 61–68, 1987.

[8] G. Chiaselotti, G. Infante, and G. Marino, “New results related to a conjecture of Manickam and Singhi,” *European Journal of Combinatorics*, vol. 29, no. 2, pp. 361–368, 2008.

[9] G. Marino and G. Chiaselotti, “A method to count the positive 3-subsets in a set of real numbers with non-negative sum,” *European Journal of Combinatorics*, vol. 23, no. 5, pp. 619–629, 2002.

[10] N. Manickam, “Distribution invariants of association schemes,” *Congressus Numerantium*, vol. 61, pp. 121–131, 1988.

[11] N. Manickam, “First distributed sets in the association scheme of bilinear forms,” *Colloquia Mathematica Societatis Janos Bolyai*, vol. 60, pp. 465–468, 1992.

[12] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, New York, NY, USA, 2nd edition, 2002.

[13] P. Erdos, C. Ko, and R. Rado, “Intersection theorems for systems of finite sets,” *Quarterly Journal of Mathematics*, vol. 2, no. 12, pp. 313–320, 1961.

[14] R. P. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, Cambridge, UK, 1997.