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Research Article

Positive Solutions of an Initial Value Problem for Nonlinear Fractional Differential Equations

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We investigate the existence and multiplicity of positive solutions for the nonlinear fractional differential equation initial value problem $D_{0+}^{\alpha}u(t) + D_{0+}^{\beta}u(t) = f(t, u(t))$, $u(0) = 0$, $0 < t < 1$, where $0 < \beta < \alpha < 1$, D_{0+}^{α} is the standard Riemann-Liouville differentiation and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. By using some fixed-point results on cones, some existence and multiplicity results of positive solutions are obtained.

1. Introduction

Fractional differential equations have been subjected to an intense debate during the last few years (see, e.g., [1–5] and the references therein). This trend is due to the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering [5–15]. The fractional differential equations started to be used extensively in studying the dynamical systems possessing memory effect. Comprehensive treatment of the fractional equations techniques such as Laplace and Fourier transform method, method of Green function, Mellin transform, and some numerical techniques are given in [5, 7, 9] and the references therein. In classical approach, linear initial fractional differential equations are solved by special functions [9, 16]. In some papers, for nonlinear problems, techniques of functional analysis such as fixed point theory, the Banach contraction principle, and Leray-Schauder theory are applied for solving such kind of the problems (see, e.g., [17–19] and the references therein). The existence of nonlinear fractional differential equations of one time fractional derivative is considered in [6, 7, 9, 20]. Also, the existence and multiplicity of positive solutions to nonlinear Dirichlet problem

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \ u(0) = u(1) = 0, \ 1 < \alpha \le 2, \ \alpha \in \mathbb{R}, \tag{1.1}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and D_{0+}^{α} is the Riemann-Liouville differentiation, have been reviewed by some authors (see e.g., [18–21] and the references therein).

In this paper, by using some fixed-point results, we investigate the existence and multiplicity of positive solutions for the nonlinear fractional differential equation initial value problem

$$D_{0+}^{\alpha}u(t) + D_{0+}^{\beta}u(t) = f(t, u(t)), \quad u(0) = 0, \ 0 < t < 1, \tag{1.2}$$

where $0 < \beta < \alpha < 1$, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Now, we present some necessary notions. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by $I^{\alpha}f(t) := (1/\Gamma(\alpha))\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau$ [20]. Also, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by $D^{\alpha}f(t) := (1/\Gamma(n-\alpha))(d/dt)^n \int_0^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau$, where $n = [\alpha] + 1$ and the right side is pointwise defined on $(0,\infty)$ ([20]). The formula of Laplace transform for the Riemann-Liouville derivative is defined by

$$L\{D^{\alpha}f(t); s\} = s^{\alpha}\tilde{f}(s)\sum_{k=0}^{m-1}\left[D^k I^{m-\alpha}\right]f(0^+)s^{m-k-1} \tag{1.3}$$

when the limiting values $f^{(k)}(0^+)$ are finite and $m - 1 < \alpha < m$. This formula simplifies to $L\{D^{\alpha}f(t); s\} = s^{\alpha}\tilde{f}(s)$ [21]. Also, two-parametric Mittag-Leffler function is defined by $E_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} z^k/\Gamma(k\alpha + \beta)$ for $\alpha > 0$ and $\beta > 0$ [21]. Analytic properties and asymptotical expansion of this function are given in [9]. For example, if $\alpha < 2$, $\pi\alpha/2 < \mu < \min(\pi, \pi\alpha)$, $\beta \in \mathbb{R}$ and c_3 is a real constant, then $|E_{\alpha,\beta}(z)| \le c_3/(1+|z|)$, whenever $|z| \ge 0$ and $\mu \le |\arg z| \le \pi$. Also, by using the formula for integration of the Mittag-Leffler function term by term, we have (see [9])

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})dt = z^{\beta}E_{\alpha,\beta+1}(\lambda t^{\alpha}). \tag{$*$}$$

Let P be a cone in a Banach space E. The map $\theta : P \to [0,\infty]$ is said to be a nonnegative continuous concave functional whenever θ is continuous and $\theta(tx + (1-t)y) \ge t\theta(x) + (1-t)\theta(y)$ for all $x, y \in P$ and $0 \le t \le 1$ [20]. We need the following fixed point theorems for obtaining our results.

Lemma 1.1 (see [22]). *Let E be a Banach space, P a cone in E, and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 1.2 (see [23]). *Let P be a cone in a real Banach space E, c, b, and d positive real numbers, $P_c = \{x \in P : \|x\| \leq c\}$, θ a nonnegative concave functional on P such that $\theta(x) \leq \|x\|$ for all $x \in \overline{P_c}$ and*

$$P(\theta, b, d) = \{x \in P : b \leq \theta(x),\ \|x\| \leq d\}. \tag{1.4}$$

Suppose that $A : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that

(c_1) *$\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset$, and for some $x \in P(\theta, b, d)$ we have $\theta(Ax) > b$,*

(c_2) *$\|Ax\| < a$ for all x with $\|x\| \leq a$,*

(c_3) *$\theta(Ax) > b$ for all $x \in P(\theta, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 such that $\|x_1\| < a$, $b < \theta(x_2)$, $a < \|x_3\|$ with $\theta(x_3) < b$.

Note that the condition (c_1) implies (c_3) whenever $d = c$.

2. Main Results

As we know, there is an integral form of the solution for the following equation:

$$D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t)), \quad u(0) = 0,\ 0 < t < 1, \tag{2.1}$$

Suppose that the functions u and f are continuous on $[0,1]$. Then $u(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau$ is a solution for (2.1), where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}$ is the two-parameter function of the Mittag-Leffler type (see [9]). Now, we give an equivalent solution for (2.1). In fact, if we apply the Laplace transform to (2.1), then by using a calculation and finding the inverse Laplace transform we get that $u(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) * f(t, u(t))$ is an equivalent solution for (2.1). In this way, note that

$$D^{\alpha}u(t) + D^{\beta}u(t) = \left(D^{\alpha}G(t) + D^{\beta}G(t)\right) * f(t, u(t)), \tag{2.2}$$

where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$. But, we have

$$\begin{aligned} D^{\alpha}G(t) + D^{\beta}G(t) &= t^{-1}E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) + t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}\left(-t^{\alpha-\beta}\right) \\ &= E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) - E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) - \frac{1}{t}\frac{1}{\Gamma(\alpha-\beta)}. \end{aligned} \tag{2.3}$$

Since $\lim_{t\to 0}(1/t)(1/\Gamma(\alpha-\beta)) = \delta(t)$, we get $D^{\alpha}G(t) + D^{\beta}G(t) = \delta(t)$ and so

$$D^{\alpha}u(t) + D^{\beta}u(t) = \delta(t) * f(t, u(t)) = f(t, u(t)). \tag{2.4}$$

Now, we establish some results on existence and multiplicity of positive solutions for the problem (2.1). Let $E = (C[0,1], \|\cdot\|_{\infty})$ be endowed via the order $u \leq v$ if and only if $u(t) \leq v(t)$

for all $t \in [0,1]$. Consider the cone $P = \{u \in E \mid u(t) \geq 0\}$ and the nonnegative continuous concave functional $\theta(u) = \inf_{1/2<t<1}|u(t)|$. Now, we give our first result.

Lemma 2.1. *Define $T : P \to P$ by $Tu(t) := \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau$, where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}(z)$ is the two-parameter function of the Mittag-Leffler type. Then T is completely continuous.*

Proof. Since the mappings G and f are nonnegative and continuous, it is easy to see that T is continuous. Now, we show that T is a relatively compact operator. This implies that T is completely continuous. Let $\Omega \subset P$ be a bounded subset. Then there exists a positive constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$. Put $L = \sup_{0\leq t\leq 1}|f(t, u(t))| + 1$. Then, for each $u \in \Omega$, we have

$$\begin{aligned} |Tu(t)| &= \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) f(\tau, u(\tau)) d\tau\right| \\ &\leq L\left|-t^{\alpha} E_{\alpha-\beta,\alpha+1}\left(-t^{\alpha-\beta}\right)\right| \leq L\left|\frac{-t^{\alpha}}{1+\left|-t^{\alpha-\beta}\right|}\right| \leq Lt^{\alpha} \leq L, \end{aligned} \tag{2.5}$$

where $0 < \alpha < 1$ and $t \in [0,1]$. Thus, T is uniformly bounded. Now, we show that T is equicontinuous. Let $t, \tau \in [0,1]$ and $t_1 \leq t_2$. Thus,

$$\begin{aligned} &|Tu(t_1) - Tu(t_2)| \\ &= \left|\int_0^{t_1} G(t_1-\tau) f(\tau, u(\tau)) d\tau - \int_0^{t_2} G(t_2-\tau) f(\tau, u(\tau)) d\tau\right| \\ &= \left|\int_0^{t_1} \left(G(t_1-\tau) f(\tau, u(\tau)) - G(t_2-\tau) f(\tau, u(\tau))\right) d\tau + \int_{t_2}^{t_1} G(t_2-\tau) f(\tau, u(\tau)) d\tau\right| \\ &\leq \left|\int_0^{t_1} \left[G(t_1-\tau) f(\tau, u(\tau)) - G(t_2-\tau) f(\tau, u(\tau))\right] d\tau\right| + \left|\int_{t_2}^{t_1} G(t_2-\tau) f(\tau, u(\tau)) d\tau\right|. \end{aligned} \tag{2.6}$$

Now, by using the formula for integration of the Mittag-Leffler function term by term given in $(*)$, we obtain that

$$\begin{aligned} &|Tu(t_1) - Tu(t_2)| \\ &\leq \|f\|\left[\left(\frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} + \frac{(t_2-t_1)^{\alpha}}{1+\left|-(t_2-t_1)^{\alpha-\beta}\right|}\right)\right. \\ &\qquad \left. + \left(\frac{t_2^{\alpha}}{1+\left|-t_2^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} - \frac{(t_2-t_1)^{\alpha}}{1+\left|-(t_2-t_1)^{\alpha-\beta}\right|}\right)\right] \\ &= \|f\|\left[\frac{t_2^{\alpha}}{1+\left|-t_2^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|}\right] \leq \|f\|\left[\frac{\left(t_2^{\alpha}-t_1^{\alpha}\right) - t_2^{\alpha}\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right) + t_2^{\alpha-\beta}\left(t_2^{\alpha}-t_1^{\alpha}\right)}{\left(1+\left|-t_1^{\alpha-\beta}\right|\right)\left(1+\left|-t_2^{\alpha-\beta}\right|\right)}\right]. \end{aligned} \tag{2.7}$$

Thus, by using the formula $t_2^s - t_1^s = (t_2 - t_1)/(t_2^{s-1} + \cdots + t_1^{s-1})$, we obtain a common factor $(t_1 - t_2)$. This implies that small changes of u cause small changes of Tu. that is, T is equicontinuous. Now by using the Arzela-Ascoli theorem, we get that T is a relatively compact operator. □

Theorem 2.2. *Suppose that in the problem* (1.2) *there exists a positive real number* $r > 0$ *such that*

(A_1) $f(t,u) \leq \alpha r$ *for all* $(t,u) \in [0,1] \times [0,r]$,

(A_2) $f(t,u) \geq 0$ *for all* $t \in [0,1]$ *with* $u(t) = 0$.

Then the problem (1.2) *has a positive solution* u *such that* $0 \leq |u| \leq r$.

Example 2.3. Consider the nonlinear fractional differential equation initial value problem

$$D^{3/2}u(t) + D^{1/2}u(t) + u(t) + \sin t = 0, \quad u(0) = 0, \ (0 < t < 1). \tag{2.8}$$

Put $r = 2$ and $\alpha = 3/2$. Since $f(t,u) = u(t) + \sin t \leq u + 1 \leq 3 = \alpha r$ for all $(t,u) \in [0,1] \times [0,2]$ and $f(t,u) = u + \sin t \geq 0$ for all $(t,u) \in [0,1] \times \{0\}$, by using Theorem 2.2 we get that this problem has a positive solution we get that this problem has a positive solution u with $0 \leq \|u\| \leq 2$.

Proof. First, let us to consider the operator $(Tu)(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau$, where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})(0 < \beta < \alpha < 1)$. By using Lemma 2.1, T is completely continuous and note that u is a solution of the problem (1.2) if and only if $u = T(u)$. Let $\Omega_1 = \{u \in P : \|u\| = 0\}$ and $\Omega_2 = \{u \in P : \|u\| u \in \partial\Omega_1\}$ we have $u(t) = 0$ for all $t \in [0,1]$. By using the assumption (A_2), we have

$$(Tu)(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau \geq 0 = \|u\| \tag{2.9}$$

and so $\|Tu\| \geq \|u\|$. Also, for $u \in \partial\Omega_2$ we have $0 \leq u(t) \leq r$ for all $t \in [0,1]$. By using the assumption (A_1) we have

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau \leq \alpha r \int_0^t (t-\tau)^{\alpha-1} d\tau = r t^{\alpha} \leq r = \|u\|. \tag{2.10}$$

This completes the proof. □

Theorem 2.4. *Suppose that in the problem* (2.1) *there exist positive real numbers* $0 < a < b < c$ *such that*

(A_1) $f(t,u) < \alpha a$ *for all* $(t,u) \in [0,1] \times [0,a]$,

(A_2) $f(t,u) > Nb$ *for all* $(t,u) \in [1/2,1] \times [b,c]$, *where*

$$N^{-1} = \inf_{1/2 < t < 1} \left| \int_0^t G(t-s) ds \right|, \tag{2.11}$$

(A_3) $f(t,u) \leq \alpha c$ *for all* $(t,u) \in [0,1] \times [0,c]$.

Then the problem (2.1) *has at least there positive solutions* u_1, u_2, *and* u_3 *such that* $\sup_{0\le t\le 1}|u_1(t)| < a$, $b < \inf_{1/2\le t\le 1}|u_2(t)| < \sup_{1/2\le t\le 1}|u_2(t)| \le c$, $a < \sup_{0\le t\le 1}|u_3(t)| \le c$ *and* $\inf_{1/2\le t\le 1}|u_3(t)| < b$.

Proof. Define $P_c = \{x \in P : \|x\| \le c\}$. Then, $\|u\| \le c$ for all $u \in \overline{P_c}$. Note that, the assumption (A_3) implies that $f(t, u(t)) \le \alpha c$ for all t. Thus,

$$\|Tu\| = \sup_{0\le t\le 1}\left|\int_0^t G(t-\tau)f(\tau, u(\tau))d\tau\right| \le \alpha c\int_0^t (t-\tau)^{\alpha-1}d\tau = \alpha c\frac{t^\alpha}{\alpha} = ct^\alpha \le c. \tag{2.12}$$

Hence, T is a operator on $\overline{P_c}$. Also, note that the assumption (A_1) implies that $f(t, u(t)) < \alpha a$ for all $0 \le t \le 1$. Thus, the condition (c_2) in Lemma 1.2 holds. It is sufficient that we show that the condition (c_1) in Lemma 1.2 holds. Put $u(t) = (b + c)/2$ for all $0 \le t \le 1$. It is easy to see that $u(t) \in P(\theta, b, c)$ and $\theta(u) = \theta((b + c)/2) > b$. Thus, $\{u \in P(\theta, b, c) : \theta(u) > b\} \ne \emptyset$ and so $b \le u(t) \le c$ for all $u \in P(\theta, b, c)$ and $1/2 \le t \le 1$. But, the assumption (A_2) implies that $f(t, u(t)) \ge Nb$ for all $1/2 \le t \le 1$ and so

$$\theta(Tu) = \inf_{1/2\le t\le 1}|(Tu)(t)| = \inf_{1/2\le t\le 1}\left|\int_0^t G(t-\tau)f(\tau, u(\tau))d\tau\right| > NbN^{-1} = b. \tag{2.13}$$

Thus, $\theta(Tu) > b$ for all $u \in P(\theta, b, c)$. This shows that the condition (c_1) in Lemma 1.2 holds. This completes the proof. □

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