Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 821949, 9 pages
doi:10.1155/2012/821949

Research Article

Dirichlet Characters, Gauss Sums, and Inverse Z Transform

Jing Gao[1] and Huaning Liu[2]

[1] *Department of Mathematical Sciences, Xi'an Jiaotong University, Xi'an, Shaanxi, China*
[2] *Department of Mathematics, Northwest University, Xi'an, Shaanxi, China*

Correspondence should be addressed to Jing Gao, jgao@mail.xjtu.edu.cn

Received 26 December 2011; Accepted 9 January 2012

Academic Editor: Karl Joachim Wirths

A generalized Möbius transform is presented. It is based on Dirichlet characters. A general algorithm is developed to compute the inverse Z transform on the unit circle, and an error estimate is given for the truncated series representation.

1. Introduction

We consider a causal, linear, time-invariant system with an infinite impulse response $\{c_j\}_{j=1}^{\infty}$. The system is assumed to be stable and the Z transform $X(z) = \sum_{j=1}^{\infty} c_j z^{-j}$ is convergent for $|z| > r$, where $r < 1$. The frequency response of the system is obtained by evaluating the Z transform on the unit circle.

The arithmetic Fourier transform (AFT) offers a convenient method, based on the construction of weighted averages, to calculate the Fourier coefficients of a periodic function. It was discovered by Bruns [1] at the beginning of the last century. Similar algorithms were studied by Wintner [2] and Sadasiv [3] for the calculation of the Fourier coefficients of even periodic functions. This method was extended in [4] to calculate the Fourier coefficients of both the even and odd components of a periodic function. The Bruns approach was incorporated in [5] resulting in a more computationally balanced algorithm. In [6, 7], Knockaert presented the theory of the generalized Möbius transform and gave a general formulation.

In [8], Schiff et al. applied Wintner's algorithm for the computation of the inverse Z-transform of an infinite causal sequence. Hsu et al. [9] applied two special Möbius inversion formulae to the inverse Z-transform.

The transform pairs play a central part in the arithmetic Fourier transform and inverse Z-transform. In this paper, based on Dirichlet characters, we presented a generalized Möbius transform of which all the transform pairs used in the mentioned papers are the special cases. A general algorithm was developed in Section 2 to compute the inverse Z transform on the unit circle. The algorithm computes each term c_j of the infinite impulse response from sampled values of the Z transform taken at a countable set of points on the unit circle. An error estimate is given in Section 3 for the truncated series representation. A numerical example is given in Section 4. Number theory and Dirichlet characters [10] play an important role in the paper.

2. The Algorithm

According to the Möbius inversion formula for finite series [4], if n is a positive integer and $f(n)$, $g(n)$ are two number-theoretic functions, then

$$g(n) = \sum_{k=1}^{[N/n]} f(kn) \quad \text{iff } f(n) = \sum_{m=1}^{[N/n]} \mu(m)g(mn), \tag{2.1}$$

where $[y]$ denotes the integer part of real number y and $\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ includes } r \text{ distinct prime factors}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

Knockaert [6] extended the Möbius inversion formula and proved the following proposition.

Proposition 2.1. *Let $f_1, f_2, \ldots$ be a sequence of real numbers and $\alpha(n)$, $\beta(n)$ two arithmetical functions. For the transform pair*

$$s_n = \sum_{k=1}^{\infty} \alpha(k) f_{kn}, \qquad f_n = \sum_{k=1}^{\infty} \beta(k) s_{kn} \tag{2.3}$$

to be valid for all sequences f_n, it is necessary and sufficient that

$$\sum_{kl=m} \alpha(k)\beta(l) = \sum_{k|m} \alpha(k)\beta\left(\frac{m}{k}\right) = \delta_{1m} = \begin{cases} 1, & m = 1, \\ 0, & m \neq 1. \end{cases} \tag{2.4}$$

Let G be the group of reduced residue classes modulo q. Corresponding to each character f of G, we define an arithmetical function $\chi = \chi_f$ as follows:

$$\chi(n) = f(\widehat{n}) \quad \text{if } (n, q) = 1, \qquad \chi(n) = 0 \quad \text{if } (n, q) > 1, \tag{2.5}$$

where $\widehat{n} = \{x : x \equiv n (\mathrm{mod}\ q)\}$ and (a, b) denotes the greatest common divisor of a and b.

The function χ is called a Dirichlet character modulo q. The principal character χ_1 is that which has the properties

$$\chi_1(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1. \end{cases} \tag{2.6}$$

If $q \geq 1$, the Euler's totient $\phi(q)$ is defined to be the number of positive integers not exceeding q that are relatively prime to q. There are $\phi(q)$ distinct Dirichlet characters modulo q, each of which is completely multiplicative and periodic with period q. That is, we have

$$\chi(mn) = \chi(m)\chi(n) \quad \forall m, n, \tag{2.7}$$

$$\chi(n+q) = \chi(n) \quad \forall n. \tag{2.8}$$

Conversely, if χ is completely multiplicative and periodic with period q, and if $\chi(n) = 0$ if $(n,q) > 1$, then χ is one of the Dirichlet characters modulo q.

Let $f(n)$ be an arithmetical function. Series of the form $\sum_{n=1}^{\infty} f(n)/n^s$ are called Dirichlet series with coefficients $f(n)$. If $f(n) = \chi(n)$, then the series are called Dirichlet L-functions. For any Dirichlet character χ mod q, the sum

$$G(n,\chi) = \sum_{m=1}^{q} \chi(m)\, e^{2\pi i m n/q} \tag{2.9}$$

is called the Gauss sums associated with χ. If $\chi = \chi_1$, then the Gauss sums reduce to Ramanujan's sum

$$G(n,\chi_1) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e^{2\pi i m n/q} = c_q(n). \tag{2.10}$$

See [10].

Let χ be a Dirichlet character modulo q. We have

$$\sum_{k|m} \chi(k)\mu\left(\frac{m}{k}\right)\chi\left(\frac{m}{k}\right) = \chi(m)\sum_{k|m}\mu\left(\frac{m}{k}\right) = \delta_{1m}. \tag{2.11}$$

In this way, we have defined a generalized Möbius transform pair.

Lemma 2.2. *Let χ be a Dirichlet character modulo q; then transform pair*

$$s_n = \sum_{k=1}^{\infty} \chi(k) f_{kn}, \qquad f_n = \sum_{k=1}^{\infty} \mu(k)\chi(k) s_{kn} \tag{2.12}$$

is valid for all q.

Remarks 1. The transform pairs play a central part in the arithmetic Fourier transform and inverse Z-transform. It is not hard to show that all the transform pairs used in the mentioned papers are the special cases of our generalized Möbius transform. In fact,

(a) let $q = 1$ in Lemma 2.2; we have

$$s_n = \sum_{k=1}^{[N/n]} f_{kn}, \qquad f_n = \sum_{k=1}^{[N/n]} \mu(k) s_{kn}, \tag{2.13}$$

which is Theorem 3 in [4] and Lemma 1 in [8];

(b) let $q = 2^\alpha$ and $\chi = \chi_1$ in Lemma 2.2, where $\alpha \geq 1$ is a positive integer; we have

$$s_n = \sum_{k=1,3,5,\ldots}^{[N/n]} f_{kn}, \qquad f_n = \sum_{k=1,3,5,\ldots}^{[N/n]} \mu(k) s_{kn}, \tag{2.14}$$

which is Case 1 of Lemma 1 in [9];

(c) let $q = 2^\alpha$, $\alpha \geq 2$, and

$$\chi_2(k) = \begin{cases} (-1)^{(k-1)/2} & \text{if } (k,q) = 1, \\ 0 & \text{if } (k,q) > 1 \end{cases} \tag{2.15}$$

in Lemma 2.2, then χ_2 is one of the Dirichlet characters modulo q since $\chi_2(k)$ is completely multiplicative and periodic with period q. We have

$$s_n = \sum_{k=1,3,5,\ldots}^{[N/n]} f_{kn} (-1)^{(k-1)/2}, \qquad f_n = \sum_{k=1,3,5,\ldots}^{[N/n]} \mu(k) s_{kn} (-1)^{(k-1)/2}, \tag{2.16}$$

which is Case 2 of Lemma 1 in [9];

(d) let $\chi = \chi_1$ in Lemma 2.2; we have

$$s_n = \sum_{(k,q)=1} f_{kn}, \qquad f_n = \sum_{(k,q)=1} \mu(k) s_{kn}, \tag{2.17}$$

which is transform pair I of Theorem 4 in [7];

(e) let $q = 4$, p^α or $2p^\alpha$, and $\chi_3(k) = (k/q)$ in Lemma 2.2, where p is an odd prime, $\alpha \geq 1$, and (k/q) is the Legendre's symbol defined as follows:

$$\left(\frac{k}{q}\right) = \begin{cases} 1 & \text{if } (k,q) = 1 \text{ and } n \text{ is a quadratic residue } \bmod q, \\ -1 & \text{if } (k,q) = 1 \text{ and } n \text{ is not a quadratic residue } \bmod q, \\ 0 & \text{if } (k,q) > 1. \end{cases} \tag{2.18}$$

From [10], we know that q admits a primitive root and $(k/q) = (-1)^{\mathrm{ind}(k)}$. We have

$$s_n = \sum_{(k,q)=1} (-1)^{\mathrm{ind}(k)} f_{kn}, \qquad f_n = \sum_{(k,q)=1} \mu(k)(-1)^{\mathrm{ind}(k)} s_{kn}, \tag{2.19}$$

which is transform pair II of Theorem 4 in [7].

From these facts, we claim that Lemma 2.2 is actually an important extension on the Möbius inversion formula. In practice, we can choose the best possible transform pair.

We do not discuss the convergence of the transform pair since in practice it is used only on a truncated series. Next we establish our main theorem.

Theorem 2.3. *Let $X(z) = \sum_{j=1}^{\infty} c_j z^{-j}$ be convergent for $|z| > r$, where $r < 1$. For any fixed $q \geq 1$ and Dirichlet character χ modulo q, the coefficients are given by*

$$c_n = \frac{1}{qn}\sum_{k=1}^{\infty}\frac{\mu(k)\chi(k)}{k}\sum_{r=1}^{q} G(r,\chi)\sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right). \tag{2.20}$$

Proof. On $|z| = 1$, let us write $X(\theta) = X(e^{i\theta}) = \sum_{j=1}^{\infty} c_j e^{-ij\theta}$.

Define

$$s_n = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\left[\frac{1}{n}\sum_{l=1}^{n} X\left(e^{(2\pi i/n)(l+r/q)}\right)\right]. \tag{2.21}$$

Note that for a positive integer k

$$\frac{1}{n}\sum_{m=1}^{n} e^{2\pi i k m/n} = \begin{cases} 1 & \text{if } n \text{ divides } k, \\ 0 & \text{if } n \text{ does not divide } k, \end{cases} \tag{2.22}$$

we have

$$\begin{aligned} s_n &= \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\left[\frac{1}{n}\sum_{l=1}^{n} X\left(e^{(2\pi i/n)(l+r/q)}\right)\right] = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\left[\frac{1}{n}\sum_{l=1}^{n}\sum_{j=1}^{\infty} c_j e^{-2\pi i j(l+r/q)/n}\right] \\ &= \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\left[\frac{1}{n}\sum_{j=1}^{\infty} c_j \sum_{l=1}^{n} e^{-2\pi i j l/n} e^{-2\pi i j r/nq}\right] = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\sum_{l=1}^{\infty} c_{ln} e^{-2\pi i l r/q}. \end{aligned} \tag{2.23}$$

Let $l = qk + s$; then

$$s_n = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\sum_{k=0}^{\infty}\sum_{s=1}^{q} c_{n(qk+s)} e^{-2\pi i s r/q} = \frac{1}{q}\sum_{k=0}^{\infty}\sum_{s=1}^{q} c_{n(qk+s)} \sum_{m=1}^{q} \chi(m) \sum_{r=1}^{q} e^{2\pi i r(m-s)/q}. \tag{2.24}$$

Note that $1 \leq m,\ s \leq q$, so $q \mid (m-s)$ if and only if $m = s$; therefore,

$$s_n = \sum_{k=0}^{\infty} \sum_{s=1}^{q} \chi(s) c_{n(qk+s)} = \sum_{t=1}^{\infty} \chi(t) c_{nt}. \tag{2.25}$$

By Lemma 2.2, we have

$$c_n = \sum_{k=1}^{\infty} \mu(k)\chi(k) s_{nk} = \frac{1}{qn} \sum_{k=1}^{\infty} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right). \tag{2.26}$$

This completes the proof of Theorem 2.3. □

Remarks 2. Let $q = 1$ in Theorem 2.3; we have

$$c_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1)}\right), \tag{2.27}$$

which is the theorem in [8].

Let $q = 2$ in Theorem 2.3 or $q = 4$ and $\chi = \chi_1$ in Theorem 2.3; we easily have

$$c_n = \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu(k)}{2kn} \left[\sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1)}\right) - \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1/2)}\right) \right]. \tag{2.28}$$

Let $q = 4$ and $\chi = \chi_2$ in Theorem 2.3; we have

$$c_n = \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu(k)(-1)^{(k-1)/2} i}{2kn} \left[\sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1/4)}\right) - \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+3/4)}\right) \right]. \tag{2.29}$$

In practice, a large number of coefficients c_n may be calculated. We suppose that a truncation is employed. Next we estimate the error due to the truncation of the series.

3. Error Estimate

In order to estimate the error due to truncation of the series representation of the coefficients c_n, we require the following lemma.

Lemma 3.1. *If f is a function of period 2π, with $f' \in \mathrm{Lip}_1([0, 2\pi])$, then*

$$\left| \int_0^{2\pi} f(\theta) d\theta - \frac{1}{n} \sum_{m=1}^{n} f\left(\theta + \frac{2\pi m}{n}\right) \right| \leq \frac{C}{n^2}, \tag{3.1}$$

uniformly in θ, where C is the Lipschitz constant.

Proof. This is Lemma 3 of [8]. □

Taking $X(z)$ as in Theorem 2.3, we maintain the following theorem.

Theorem 3.2. *The truncation error satisfies*

$$\left| c_n - \frac{1}{qn} \sum_{k=1}^{N} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right) \right| \le \frac{C\phi(q)}{n^2 N}, \tag{3.2}$$

where C is the Lipschitz constant.

Proof. Note that we have

$$0 = c_0 = \frac{1}{2\pi} \int_0^{2\pi} X\left(e^{i\varphi}\right) d\varphi. \tag{3.3}$$

Moreover, $X' \in \mathrm{Lip}_1([0,2\pi])$ by the analyticity of X. By Theorem 2.3 and Lemma 3.1, we have

$$\begin{aligned} &\left| c_n - \frac{1}{qn} \sum_{k=1}^{N} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right) \right| \\ &\quad = \left| \frac{1}{qn} \sum_{k=N+1}^{\infty} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right) \right| \\ &\quad \le \left| \frac{1}{q} \sum_{k=N+1}^{\infty} \mu(k)\chi(k) \sum_{r=1}^{q} G(r,\chi) \frac{C}{n^2 k^2} \right| \le \frac{C\phi(q)}{n^2} \sum_{k=N+1}^{\infty} \frac{1}{k^2} \le \frac{C\phi(q)}{n^2 N}. \end{aligned} \tag{3.4}$$

This completes the proof of Theorem 3.2. □

Table 1: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z - 1/2) - 1,\ q = 1$.

k	c_1	c_2	c_3
1	$3.718281828 + 0.000000720i$	$1.209747301 + 0.000000338i$	$0.453772595 + 0.000000199i$
2	$2.508534526 + 0.000000381i$	$1.034722511 + 0.000000208i$	$0.420637706 + 0.000000128i$
3	$2.054761931 + 0.000000182i$	$1.001587622 + 0.000000138i$	$0.416721106 + 0.000000087i$
4	$2.054761931 + 0.000000182i$	$1.001587622 + 0.000000138i$	$0.416721106 + 0.000000087i$
5	$1.981912218 + 0.000000089i$	$0.999632365 + 0.000000100i$	$0.416660127 + 0.000000062i$
6	$2.015047107 + 0.000000160i$	$1.000120712 + 0.000000131i$	$0.416667696 + 0.000000082i$
7	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
8	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
9	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
10	$2.001055961 + 0.000000140i$	$1.000000538 + 0.000000123i$	$0.416666743 + 0.000000077i$

Table 2: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z - 1/2) - 1,\ q = 2,\ q = 4$, or $\chi = \chi_1$.

k	c_1	c_2	c_3
1	$2.508534526 + 0.000000785i$	$1.034722484 + 0.000000226i$	$0.420637664 + 0.000000130i$
3	$2.087896862 + 0.000000654i$	$1.002075958 + 0.000000213i$	$0.416728633 + 0.000000129i$
5	$2.017002434 + 0.000000624i$	$1.000122546 + 0.000000212i$	$0.416667589 + 0.000000126i$
7	$2.001178059 + 0.000000618i$	$1.000000467 + 0.000000209i$	$0.416666630 + 0.000000121i$
9	$2.001178059 + 0.000000618i$	$1.000000467 + 0.000000209i$	$0.416666630 + 0.000000121i$
11	$2.000201462 + 0.000000617i$	$0.999999985 + 0.000000205i$	$0.416666627 + 0.000000117i$
13	$1.999957311 + 0.000000615i$	$0.999999950 + 0.000000200i$	$0.416666625 + 0.000000113i$
15	$2.000018355 + 0.000000618i$	$0.999999956 + 0.000000204i$	$0.416666626 + 0.000000117i$
17	$2.000003089 + 0.000000614i$	$0.999999953 + 0.000000200i$	$0.416666625 + 0.000000114i$
19	$1.999999268 + 0.000000610i$	$0.999999951 + 0.000000196i$	$0.416666624 + 0.000000111i$

4. An Example

Consider the function

$$X(z) = e^{1/z} + \frac{1}{z - 1/2} - 1, \quad |z| > \frac{1}{2}. \tag{4.1}$$

The few first coefficients are $c_1 = 2$, $c_2 = 1$, and $c_3 = 5/12$. Employing formulae (2.27), (2.28), and (2.29), we obtain the results given in Tables 1, 2, and 3. The results show that formulae (2.28) and (2.29) is quite more accurate than formula (2.27). Choosing carefully the modulo q and the Dirichlet character, we will greatly improve the algorithm.

5. Conclusion

A general algorithm offers a general way to compute the inverse Z transform. It is based on generalized Möbius transform, Dirichlet characters, and Gauss sums. The algorithm computes each term c_j of the infinite impulse response from sampled values of the Z transform taken at a countable set of points on the unit circle. An error estimate and a numerical example are given for the truncated series representation. Choosing carefully the

Table 3: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z - 1/2) - 1$, $q = 4$, and $\chi = \chi_2$.

k	c_1	c_2	c_3
1	1.641470945 + 0.000000164i	0.969199603 + 0.000000195i	0.412817735 + 0.000000128i
3	2.054288680 + 0.000000292i	1.001830856 + 0.000000204i	0.416726726 + 0.000000119i
5	1.983516336 + 0.000000263i	0.999877456 + 0.000000214i	0.416665686 + 0.000000128i
7	1.999338790 + 0.000000262i	0.999999531 + 0.000000205i	0.416666645 + 0.000000122i
9	1.999338790 + 0.000000262i	0.999999531 + 0.000000205i	0.416666645 + 0.000000122i
11	2.000315379 + 0.000000252i	1.000000013 + 0.000000200i	0.416666650 + 0.000000120i
13	2.000071234 + 0.000000261i	0.999999978 + 0.000000204i	0.416666646 + 0.000000122i
15	2.000010194 + 0.000000270i	0.999999971 + 0.000000207i	0.416666642 + 0.000000123i
17	1.999994930 + 0.000000277i	0.999999967 + 0.000000209i	0.416666639 + 0.000000124i
19	1.999998750 + 0.000000271i	0.999999971 + 0.000000208i	0.416666642 + 0.000000123i

modulo q and the Dirichlet character we will greatly improve the algorithm. But this is not exhaustive. Dirichlet characters and Gauss sums play an important role in number theory, and there are so many methods and results associated with them. Any development on the Dirichlet character and Gauss sums may be applied to the inverse Z transform.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China under Grant no. 10901128, the Specialized Research Fund for the Doctoral Program of Higher Education of China under Grant no. 20090201120061, the Natural Science Foundation of the Education Department of Shaanxi Province of China under Grant no. 09JK762, and the Fundamental Research Funds for the Central University.

References

[1] H. Bruns, *Grundlinien des Wissenschaftlichichnen Rechnens*, B. G. Teubner, Leipzig, Germany, 1903.

[2] A. Wintner, *An Arithmetical Approach to Ordinary Fourier Series*, Waverly Press, Baltimore, Md, USA, 1946.

[3] G. Sadasiv, "The arithmetic Fourier transform," *IEEE ASSP Magazine*, vol. 5, no. 1, pp. 13–17, 1988.

[4] I. S. Reed, D. W. Tufts, X. Yu, T. K. Truong, M. T. Shih, and X. Yin, "Fourier analysis and signal processing by use of the Mobius inversion formula," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, no. 3, pp. 458–470, 1990.

[5] I. S. Reed, M. T. Shih, T. K. Truong, E. Hendon, and D. W. Tufts, "A VLSI architecture for simplified arithmetic Fourier transform algorithm," *IEEE Transactions on Signal Processing*, vol. 40, no. 5, pp. 1122–1133, 1992.

[6] L. Knockaert, "Generalized Mobius transform and arithmetic fourier transforms," *IEEE Transactions on Signal Processing*, vol. 42, no. 11, pp. 2967–2971, 1994.

[7] L. Knockaert, "A generalized möbius transform, arithmetic fourier transforms, and primitive roots," *IEEE Transactions on Signal Processing*, vol. 44, no. 5, pp. 1307–1310, 1996.

[8] J. L. Schiff, T. J. Surendonk, and W. J. Walker, "An algorithm for computing the inverse Z transform," *IEEE Transactions on Signal Processing*, vol. 40, no. 9, pp. 2194–2198, 1992.

[9] C. C. Hsu, I. S. Reed, and T. K. Truong, "Inverse Z-transform by Mobius inversion and the error bounds of aliasing in sampling," *IEEE Transactions on Signal Processing*, vol. 42, no. 10, pp. 2823–2830, 1994.

[10] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, NY, USA, 1976.