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Research Article

A Note on the Class of Functions with Bounded Turning

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We consider subclasses of functions with bounded turning for normalized analytic functions in the unit disk. The geometric representation is introduced, some subordination relations are suggested, and the upper bound of the pre-Schwarzian norm for these functions is computed. Moreover, by employing Jack's lemma, we obtain a convex class in the class of functions of bounded turning and relations with other classes are posed.

1. Introduction

Let $U := \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{H}$ denote the space of all analytic functions on U. Here, we suppose that $\mathcal{H}$ is a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots . \tag{1.1}$$

Further, let

$$\mathcal{A} := \left\{ f \in \mathcal{H} : f(0) = f'(0) - 1 = 0 \right\}, \tag{1.2}$$

and let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is called starlike if $f(U)$ is a starlike domain with respect to the origin, and the class of univalent starlike functions is denoted by S^*. It is called convex $\mathcal{C}$ if $f(U)$ is a convex domain. Each univalent starlike function f is characterized by the analytic condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad f(z) \neq 0 \tag{1.3}$$

in U. Also, it is known that $zf'(z)$ is starlike if and only if f is convex, which is characterized by the analytic condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad f'(z) \neq 0 \tag{1.4}$$

in U. Let $f \in \mathcal{H}$, and let g be a univalent function in U, with $f(0) = g(0)$. Then, we say that f is subordinate to g (or g is superordinate to f), denoted by $f(z) \prec g(z)$, if $f(U) \subset g(U)$. For two functions $f, g \in \mathcal{A}$, the Hadamard product is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.5}$$

where a_n and b_n are the coefficients of f and g, respectively.

The pre-Schwarzian derivative T_f of f is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)} \tag{1.6}$$

with the norm

$$\|T_f\| = \sup_{z\in U} |T_f| \left(1 - |z|^2\right). \tag{1.7}$$

It is known that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent. It is also known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$ and that $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. Moreover, it is showed that, when $|T_f| \leq 3.05 f$ is univalent in U and when $|T_f| \leq 2.28329$, f is starlike in U (see [1]). Recently, the sharp norm estimates for well-known integral operators are determined (see [2–4]).

For $0 \leq \nu < 1$, let $B(\nu)$ denote the class of functions f of the form (2.2) so that $\Re\{f'\} > \nu$ in U. The functions in $B(\nu)$ are called functions of bounded turning. By the Nashiro-Warschowski theorem, the functions in $B(\nu)$ are univalent and also close to convex in U. It is well known that $B(\nu) \not\subseteq S^*$ and $S^* \not\subseteq B(\nu)$. In [5], Mocanu obtained a subclass of S^*, which is contained in $B(\nu)$. Recently, Tuneski generalized the class of convex functions with bounded turning (see [6]):

$$\sqrt[k]{f'(z)} \prec \frac{1 + Cz}{1 + Dz}, \quad k \geq 1. \tag{1.8}$$

Different studies of the class of bounded turning functions can be found in [7–10].

In this note we pose the following subclass of bounded turning functions in the the unit disk: for given numbers $\epsilon > 0$ and $\alpha > 0$, let us consider the class $B(p_\epsilon)$:

$$B(p_\epsilon) = \{ f \in \mathcal{A} : \left| (f'(z))^\alpha - 1 \right| < \Re (f'(z))^\alpha - \epsilon, \ z \in U \}. \tag{1.9}$$

It is easy to see that $f \in B(p_\epsilon)$ if and only if

$$f'(z) \prec p_\epsilon := \left(1 - \frac{\epsilon}{k} z\right)^{1/\alpha}, \quad z \in U, \ k \geq 2. \tag{1.10}$$

For the development of the paper we need to define a set Q by all points on the right half-plane as follows:

$$Q = \{ w \in \mathbb{C} : \ |w^\alpha - 1| < \Re(w^\alpha) - \epsilon, \ \alpha > 0 \}. \tag{1.11}$$

Thus it is easy to verify that its boundary satisfies the equation of a parabola

$$v^2 = 2u(1-\epsilon) - (1-\epsilon)^2, \ (w = u + iv). \tag{1.12}$$

2. Main Results

First, our result is in the following form.

Theorem 2.1. *A function $f \in B(p_\epsilon)$, $\epsilon > 0$, if and only if there exists an analytic function p,*

$$p(z) \prec p_\epsilon(z) := \left(1 - \frac{\epsilon}{k} z\right)^{1/\alpha}, \quad k \geq 2, \tag{2.1}$$

such that

$$f(z) = \int_0^z p(t) dt, \quad p(0) = 1. \tag{2.2}$$

Moreover, if function $f_\epsilon \in B(p_\epsilon)$ takes the form

$$f_\epsilon(z) = \frac{(2/\epsilon) - (2/\epsilon)[1 - (\epsilon z/2)]^{(2+\alpha)/\alpha}}{(2+\alpha)/\alpha}, \tag{2.3}$$

then the subordination relation

$$\frac{f(z)}{z} \prec \frac{f_\epsilon(z)}{z}, \quad z \in U \tag{2.4}$$

holds.

Proof. Let $f \in B(p_\epsilon)$, and let

$$p(z) = f'(z) \prec \left(1 - \frac{\epsilon}{k} z\right)^{1/\alpha}. \tag{2.5}$$

Integrating this equation yields (2.2). If f is given in (2.2) with an analytic function $p(z) \prec p_\epsilon(z)$, then by a differentiation of (2.2) we obtain that $f'(z) = p(z)$; therefore,

$$f'(z) \prec \left(1 - \frac{\epsilon}{k} z\right)^{1/\alpha}, \tag{2.6}$$

and consequently $f \in B(p_\epsilon)$.

Now we proceed to prove that $f_\epsilon \in B(p_\epsilon)$. For this purpose we will show that the set

$$Q_\epsilon := \left\{ w \in \mathbb{C} : \left| w^{\alpha/2} - 1 \right| < \sqrt{\Re(w^\alpha) - \epsilon} - 1,\ \Re(w^\alpha) > \epsilon \right\} \subset Q. \tag{2.7}$$

Let $w \in Q_\epsilon$. Then

$$\left| w^{\alpha/2} - 1 \right| < \sqrt{\Re(w^\alpha) - \epsilon} - 1 \Longrightarrow \left| w^{\alpha/2} + 1 \right| < \sqrt{\Re(w^\alpha) - \epsilon} + 1. \tag{2.8}$$

Multiplying these inequalities, we obtain

$$|w^\alpha - 1| < \Re(w^\alpha) - \epsilon - 1 < \Re(w^\alpha) - \epsilon. \tag{2.9}$$

Therefore, $w \in Q$.

Define a function $q_\epsilon(z)$, $z \in U$, as follows:

$$q_\epsilon(z) := \left[1 - \frac{\epsilon z}{2}\right]^{2/\alpha}. \tag{2.10}$$

We suppose that

$$w^{\alpha/2} := \left[q_\epsilon(z)\right]^{\alpha/2} = 1 - \frac{\epsilon z}{2}, \tag{2.11}$$

and thus

$$q_\epsilon(U) = \left\{ w \in \mathbb{C} : \left| w^{\alpha/2} - 1 \right| < \sqrt{\Re(w^\alpha) - \epsilon} - 1,\ \Re(w^\alpha) > \epsilon \right\} \subset Q. \tag{2.12}$$

Hence, $q_\epsilon(z) \prec p_\epsilon(z)$. Putting $q_\epsilon(z)$ in (2.2) implies (2.3). To prove the subordination relation (2.4), first we show that $f_\epsilon(z)/z$ is a convex function. We observe that

$$f_\epsilon(z) = \frac{(2/\epsilon) - (2/\epsilon)[1 - (\epsilon z/2)]^{(2+\alpha)/\alpha}}{(2+\alpha)/\alpha} = z - \frac{\epsilon}{2\alpha} z^2 + \cdots = z + \sum_{n=2}^{\infty} \lambda(\alpha, \epsilon)\ z^n \in \mathcal{A}. \tag{2.13}$$

Let us consider the function

$$F_\epsilon(z) = -\frac{2\alpha}{\epsilon}\left[\frac{f_\epsilon(z)}{z} - 1\right] \in \mathcal{A}. \tag{2.14}$$

Computations give

$$F'_\epsilon(z) = -\frac{2\alpha}{\epsilon}\left[\frac{f'_\epsilon(z)}{z} - \frac{f_\epsilon(z)}{z^2}\right], \tag{2.15}$$

$$F''_\epsilon(z) = -\frac{2\alpha}{\epsilon}\left[\frac{zf''_\epsilon(z) - f'_\epsilon(z)}{z^2} - \frac{z^2 f'_\epsilon(z) - 2zf_\epsilon(z)}{z^4}\right]. \tag{2.16}$$

Also, a calculation implies that

$$f_\epsilon(z) = \frac{(2/\epsilon) - (2/\epsilon)[1-(\epsilon z/2)]^{(2+\alpha)/\alpha}}{(2+\alpha)/\alpha}, \qquad f'_\epsilon(z) = \left[1 - \frac{\epsilon z}{2}\right]^{2/\alpha}, \qquad f''_\epsilon(z) = \frac{-\epsilon}{\alpha}\left[1 - \frac{\epsilon z}{2}\right]^{(2-\alpha)/\alpha}. \tag{2.17}$$

The aim is to show that $1+(zF''_\epsilon(z)/F'_\epsilon(z))$ has positive real part in the unit disk. For $\Re(z) > 0$ and a suitable choice of $0 < \epsilon \le 1$ such that $0 < \Re(1 - \epsilon z/2) < 1$ and by using (2.17), we have

$$\Re\left(1 + \frac{zF''_\epsilon(z)}{F'_\epsilon(z)}\right) = \Re\left(\frac{z^2 f''_\epsilon(z)}{zf'_\epsilon(z) - f_\epsilon(z)} - 1\right) > 0. \tag{2.18}$$

Consequently, we obtain that $F_\epsilon \in \mathcal{K}$, and therefore $f_\epsilon(z)/z$ is a convex function.

Now by using the fact that if $F, G \in \mathcal{K}$, satisfy $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ and $k(z) = z/(1-z)$ is a convex function, immediately we establish (2.4). This completes the proof. □

Next we consider another class of functions of bounded turning. We will estimate the upper bound of these functions by using the pre-Schwarzian norm.

Theorem 2.2. *Consider the class $B(p_\epsilon)$, $\epsilon \in (0,1]$ of functions $f \in \mathcal{A}$ of bounded turning which satisfies the relation*

$$f'(z) \prec p_\epsilon := \left(1 - \frac{\epsilon}{k}z\right)^{1/\alpha}, \quad \alpha > 1,\ k \ge 2. \tag{2.19}$$

Then,

$$\|f\| \le \frac{\epsilon(1+\epsilon/k)}{k\alpha}. \tag{2.20}$$

Proof. Let $f \in B(p_\epsilon)$, and let $P_f := f'(z)$. Then, there exists an analytic function $w : U \to U$ with $w(0) = 0$ and

$$P_f = p_\epsilon \circ w = \left(1 - \frac{\epsilon}{k} w\right)^{1/\alpha}. \tag{2.21}$$

Define a function $F \in \mathcal{A}$ such that $P_F = -p_\epsilon$, that is,

$$F'(z) = -\left(1 - \frac{\epsilon}{k} z\right)^{1/\alpha}, \tag{2.22}$$

and thus

$$F(z) = -\int_0^z p_\epsilon(t)dt = \frac{k\alpha}{\epsilon(\alpha+1)}\left(1 - \frac{\epsilon}{k} z\right)^{(1+\alpha)/\alpha} - \frac{k\alpha}{\epsilon(\alpha+1)}. \tag{2.23}$$

We proceed to determine the quantities $T_F(|z|)$ and $T_f(z)$. Logarithmic differentiation of (2.22) yields that

$$T_F(|z|) = \frac{F''(|z|)}{F'(|z|)} = \frac{\epsilon}{k\alpha(1 - (\epsilon/k)|z|)}, \tag{2.24}$$

and the logarithmic differentiation of (2.21) gives

$$T_f(z) = \frac{f''(z)}{f'(z)} = -\frac{\epsilon}{k\alpha}\left[\frac{w'(z)}{1 - \epsilon/kw(z)}\right], \quad z \neq 0. \tag{2.25}$$

Thus, by triangle inequality and the Schwarz-Pick lemma, we obtain

$$\begin{aligned} \left|T_f(z)\right| &= \frac{\epsilon}{k\alpha}\left|\frac{w'(z)}{1 - (\epsilon/k)w(z)}\right| \leq \frac{\epsilon}{k\alpha}\frac{1 - |w(z)|^2}{\left(1 - |z|^2\right)(|1 - (\epsilon/k)w(z)|)} \\ &\leq \frac{\epsilon}{k\alpha}\frac{1 - |w(z)|}{(1 - |z|)(|1 - (\epsilon/k)w(z)|)} \leq \frac{\epsilon}{k\alpha}\frac{1}{(1 - |z|)} \leq \frac{\epsilon}{k\alpha}\frac{1}{(1 - (\epsilon/k)|z|)} = T_F(|z|). \end{aligned} \tag{2.26}$$

Consequently, we have

$$\left(1 - |z|^2\right)\left|T_f(z)\right| = \left(1 - |z|^2\right)T_F(|z|). \tag{2.27}$$

Therefore, $\|f\| \leq \|F\|$, and this inequality is sharp. Thus, to determine the upper estimate of $f \in B(q_\epsilon)$ it is enough to compute $\|F\|$. Letting $t = |z|$, we have

$$\sup\left(1-t^2\right)T_F(|z|) = \sup\left(1-t^2\right)\frac{\epsilon}{k\alpha(1-(\epsilon/k)t)} \leq \sup\left(1-\frac{\epsilon^2}{k^2}t^2\right)\frac{\epsilon}{k\alpha(1-(\epsilon/k)t)}, \qquad k > \epsilon$$
$$= \sup\frac{\epsilon(1+(\epsilon/k)t)}{k\alpha}. \tag{2.28}$$

Hence, we obtain (2.20). □

By applying Jack's lemma, we pose the sufficient conditions for convex functions f belonging to the subclasses $B(p_\epsilon)$ and $B(q_\epsilon) = (1+\epsilon\omega(z))^{1/\alpha}$.

Lemma 2.3 (see [11]). *Let w be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = m w(z_0), \tag{2.29}$$

where m is a real number and $m \geq 1$.

Theorem 2.4. *Assume that $\epsilon \in [1/2, 1)$ and $\alpha > \epsilon/(1-\epsilon/k)$. If $f \in \mathcal{A}$ is a convex function in U of order $(0 \leq (k\alpha - \epsilon(1+\alpha))/k\alpha(1-\epsilon/k)) < 1$, then $f \in B(p_\epsilon)$.*

Proof. Let $f \in \mathcal{K}(\alpha - \epsilon(1+\alpha)/\alpha(1-\epsilon))$, that is,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{k\alpha-\epsilon(1+\alpha)}{k\alpha(1-\epsilon/k)}, \qquad z \in U. \tag{2.30}$$

From the proof of Theorem 2.2, we have

$$\Re\left\{\frac{zf''(z)}{f'(z)}+1\right\} = \Re\left\{1-\frac{\epsilon}{k\alpha}\left[\frac{zw'(z)}{1-(\epsilon/k)w(z)}\right]\right\} > \frac{k\alpha-\epsilon(1+\alpha)}{k\alpha(1-\epsilon/k)}, \qquad z \in U, \tag{2.31}$$

where w is analytic in U and satisfies $w(0) = 0$ and

$$f'(z) = \left(1-\frac{\epsilon}{k}w(z)\right)^{1/\alpha}. \tag{2.32}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1. \tag{2.33}$$

Then, on using Lemma 2.3 and letting $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = me^{i\theta}$, $m \geq 1$ yields

$$\begin{aligned} \Re\left\{\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right\} &= \Re\left\{1 - \frac{\epsilon}{k\alpha}\frac{z_0 w'(z_0)}{(1-\epsilon/kw(z_0))}\right\} = \Re\left\{1 - \frac{\epsilon}{k\alpha}\frac{me^{i\theta}}{(1-\epsilon/ke^{i\theta})}\right\} \\ &\leq \Re\left\{1 - \frac{\epsilon}{k\alpha}\frac{e^{i\theta}}{(1-\epsilon/ke^{i\theta})}\right\}, \end{aligned} \tag{2.34}$$

which contradicts hypothesis (2.30). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, that is, $f \in B(p_\epsilon)$. □

Theorem 2.5. *Assume that $\epsilon \in (0,1]$ and $\alpha > 1$. If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \frac{\epsilon(\epsilon/k+2)}{k\alpha(\epsilon/k+1)}, \qquad z \in U, \tag{2.35}$$

then $f \in B(q_\epsilon)$.

Proof. Define a function $\omega(z)$ by

$$f'(z) = \left(1 + \frac{\epsilon}{k}\omega(z)\right)^{1/\alpha}. \tag{2.36}$$

Then, ω is analytic in U and satisfies $\omega(0) = 0$. It follows that

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} = \frac{\epsilon}{k\alpha}\Re\left\{\frac{z\omega'(z) + (\epsilon/k)\omega(z) + 1}{1 + (\epsilon/k)\omega(z)}\right\} < \frac{\epsilon(\epsilon/k+2)}{k\alpha(\epsilon/k+1)}. \tag{2.37}$$

In the same manner of Theorem 2.4, we find that

$$\begin{aligned} \Re\left\{\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right\} &= \frac{\epsilon}{\alpha}\Re\left\{\frac{z_0\omega'(z_0) + \epsilon\omega(z_0) + 1}{1 + \epsilon\omega(z_0)}\right\} = \frac{\epsilon}{k\alpha}\Re\left\{\frac{me^{i\theta} + (\epsilon/k)e^{i\theta} + 1}{1 + (\epsilon/k)e^{i\theta}}\right\} \\ &\geq \frac{\epsilon(\epsilon/k+2)}{k\alpha(\epsilon/k+1)}, \end{aligned} \tag{2.38}$$

which contradicts hypothesis (2.35). Therefore, we conclude that $|\omega(z)| < 1$ for all $z \in U$, that is, $f \in B(q_\epsilon)$. □

Corollary 2.6. *Let the assumptions of Theorem 2.5 hold. Then, f is strongly close to convex of order $1/\alpha$.*

Proof. Since $f \in B(q_\epsilon)$, there exists an analytic function $\psi \in U$ such that $\psi(0) = 0$, $|\psi(z)| < 1$ and

$$(f'(z))^{\alpha} = 1 + \frac{\epsilon}{k}\psi(z). \tag{2.39}$$

But

$$\left|\varphi(z)\right| = \left|\frac{(f'(z))^{\alpha} - 1}{\epsilon/k}\right| < 1, \tag{2.40}$$

which implies

$$\left|(f'(z))^{\alpha} - 1\right| < \frac{\epsilon}{k} < 1 \tag{2.41}$$

and thus f is strongly close to convex of order $1/\alpha$. □

3. Conclusion

It is well known that the class of bounded turning functions is not included in the class of the starlike functions and also the starlike functions cannot embed in the class of bounded turning functions. From the above, we conclude that some classes of bounded turning functions can be included in the class of convex functions ($\mathcal{K} \subset B(p_\epsilon)$; Theorem 2.4). Moreover, some classes of bounded turning functions can embed in the class of close-to-convex functions ($B(q_\epsilon) \subset \mathcal{KL}$; Corollary 2.6). Hence, we have the following inclusion relation:

$$\mathcal{K} \subset B \subset \mathcal{KL}. \tag{3.1}$$

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