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*Research Article*

# The First-Integral Method and Abundant Explicit Exact Solutions to the Zakharov Equations

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This paper is concerned with the system of Zakharov equations which involves the interactions between Langmuir and ion-acoustic waves in plasma. Abundant explicit and exact solutions of the system of Zakharov equations are derived uniformly by using the first integral method. These exact solutions are include that of the solitary wave solutions of bell-type for $n$ and $E$, the solitary wave solutions of kink-type for $E$ and bell-type for $n$, the singular traveling wave solutions, periodic wave solutions of triangle functions, Jacobi elliptic function doubly periodic solutions, and Weierstrass elliptic function doubly periodic wave solutions. The results obtained confirm that the first integral method is an efficient technique for analytic treatment of a wide variety of nonlinear systems of partial differential equations.

## 1. Introduction

Zakharov equations

$$\begin{gathered} n_{tt} - c_s^2 n_{xx} = \beta\left(|E|^2\right)_{xx}, \\ iE_t + \alpha E_{xx} = \delta n E \end{gathered} \tag{1.1}$$

have been presented by Zakharov and others in 1972 [1, 2] to model the interactions of laser-plasma. In (1.1), $n$ is the perturbed number density of the ion (in the low frequency response); $E$ is the slow variation amplitude of the electric field intensity; $c_s$ is the thermal transportation velocity of the electron-ion; $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$, $c_s$ are constants. Equations (1.1) are one of the fundamental models governing dynamics of nonlinear waves and describing the interactions between high- and low-frequency waves. In the interaction of laser-plasma the system of Zakharov equations plays an important role (see [3] and references therein).

More recently, some authors considered the exact and explicit solutions of the system of Zakharov equations by different methods in [4–8]. In [3], the author established the traveling wave solutions for (1.1) by analytical method. The extended hyperbolic function method was employed to find some solitary wave solutions for (1.1) in [4, 5]. In [6, 7], the authors obtained elliptic function solutions for (1.1) by the Jacobi elliptic function method and the generalized Jacobi elliptic function expansion method. In [8], some new traveling wave solutions for (1.1) are obtained by using bifunction method and Wu-elimination method. In the above references the authors can obtain only one type of exact solution by one method.

The aim of this paper is to supply a unified method for constructing a series of explicit exact solutions to the system of Zakharov equations (1.1). The first integral method is employed to investigate the system of Zakharov equations (1.1). Through an exhaustive analysis and discussion for different parameters, we uniformly construct a series of explicit exact solutions to (1.1). Compared with most methods used in [4–8] such as the extended hyperbolic function method, Jacobi elliptic function method, and its extension, the first integral method not only gives abundant explicit exact solitary wave solutions, periodic wave solutions of triangle function, but also provides Jacobi elliptic function and Weierstrass elliptic function doubly periodic wave solutions.

The rest of this paper is organized as follows. In Section 2, the outline of the first integral method will be given. Section 3 is the main part of this paper; the method is employed to seek the explicit and exact solutions of the system of Zakharov equations (1.1). In the last section, some conclusion is given.

## 2. The First Integral Method

The first-integral method, which is based on the ring theory of commutative algebra, was first proposed by Prof. Feng Zhaosheng [9] in 2002. The method has been applied by Feng to solve Burgers-KdV equation, the compound Burgers-KdV equation, an approximate Sine-Gordon equation in $(n+1)$-dimensional space, and two-dimensional Burgers-KdV equation [10–14].

Recent years, many authors employed this method to solve different types of nonlinear partial differential equations in physical mathematics. More information about these applications can be found in [15] and references therein. The most advantage is that the first integral method does not have many sophisticated computation in solving nonlinear algebra equations compared to other direct algebra methods. For the sake of completeness, we briefly outline the main steps of this method.

The main steps of this method are summarized as follows.

Given a system of nonlinear partial differential equations, for example, in two independent variables

$$\begin{gathered} P(u_t, u_x, u_{xx}, u_{xt}, \dots) = 0, \\ Q(v_t, v_x, v_{xx}, v_{xt}, \dots) = 0, \end{gathered} \tag{2.1}$$

and using traveling wave transformation $u(x,t) = f(\xi)$, $v(x,t) = g(\xi)$, $\xi = kx + \omega t + \xi_0$ and some other mathematical operations, the systems (2.1) can be reduced to a second order nonlinear ordinary differential equation

$$D(f, f', f'') = 0. \tag{2.2}$$

By introducing new variables $X = f(\xi)$, $Y = f'(\xi)$, or making some other transformations we reduce ordinary differential equation (2.2) to a system of the first order ordinary differential equation

$$\begin{aligned} X' &= Y, \\ Y' &= H(X, Y). \end{aligned} \tag{2.3}$$

Suppose that the first integral of (2.3) has a form as follows:

$$P(X, Y) = \sum_{i=o}^{m} a_i(X) Y^i = 0 \tag{2.4}$$

(in general $m = 1$ or $m = 2$), where $a_i$ $(i = 0, 1, \ldots, m)$ are real polynomials of $X$.

According to the Division theorem there exists polynomials $\alpha(X)$, $\beta(X)$ of variable X in $\Re[X]$ such that

$$\frac{dP}{d\xi} = [\alpha(X) + \beta(X)Y] P(X, Y). \tag{2.5}$$

We determine polynomials $\alpha(X)$, $\beta(X)$, $a_i(X)$ $(i = 0, 1, 2, \ldots)$ from (2.5) and, furthermore, obtain $P(X, Y)$.

Then substituting $X = f(\xi)$, $Y = f'(\xi)$ or other transformations into (2.4), exact solutions to (2.1) is established, through solving the resulting first order integrable differential equation.

## 3. Explicit and Exact Solutions of the System of Zakharov Equations

In this section we will employ the first integral method to construct abundant explicit exact traveling wave solutions to (1.1).

In order to transfer (1.1) into the form of (2.2), we firstly do some transformations for (1.1). Since $E(x, t)$ in (1.1) is a complex function and we are seeking for the traveling wave solutions, we introduce a gauge transformation

$$E(x, t) = e^{i(kx + \omega t + \xi_0)} \varphi(x, t), \tag{3.1}$$

where $\varphi(x, t)$ is real-valued function, $k$, $\omega$ are two real constants to be determined later, and $\xi_0$ is an arbitrary constant. Substituting (3.1) into (1.1), we have

$$n_{tt} - c_s^2 n_{xx} = \beta\left(\varphi^2\right)_{xx}, \tag{3.2}$$

$$\varphi_t + 2\alpha k \varphi_x = 0, \tag{3.3}$$

$$\alpha \varphi_{xx} - \left(\alpha k^2 + \omega\right)\varphi = \delta n \varphi. \tag{3.4}$$

In the view of (3.3), we suppose

$$\varphi(x, t) = \varphi(\xi) = \varphi(x - 2\alpha k t + \xi_1), \tag{3.5}$$

where $\xi_1$ is an arbitrary constant. Substituting (3.5) into (3.4), we infer that

$$n(x,t) = \frac{\alpha\varphi_{xx} - (\omega + \alpha k^2)\varphi}{\delta\varphi} = \frac{\alpha\varphi''(\xi)}{\delta\varphi(\xi)} - \left(\omega + \alpha k^2\right). \tag{3.6}$$

Therefore, we can also assume

$$n(x,t) = \psi(\xi) = \psi(x - 2\alpha k t + \xi_1). \tag{3.7}$$

Substituting (3.7) into (3.2), and integrating the resultant equation twice with respect to $\xi$, we obtain

$$\left(4\alpha^2 k^2 - c_s^2\right)\psi(\xi) = \beta\varphi^2(\xi) + C'\xi + C, \tag{3.8}$$

where $C'$, $C$ are two arbitrary integration constants. For $k = \pm c_s/2\alpha$, (1.1) have one set of solution

$$\begin{aligned} E_0(x,t) &= \pm\left[\frac{C'(x \pm c_s t + \xi_0) + C}{-\beta}\right]^{1/2} e^{[i(\pm(c_s/2\alpha)x + \omega t + \xi_0)]}, \\ n_0(x,t) &= -\frac{\alpha(C')^2}{4\delta[C'(x \pm c_s t + \xi_0) + C]^2} - \left(\omega + \frac{c_s^2}{4\alpha}\right). \end{aligned} \tag{3.9}$$

For $k \neq \pm c_s/2\alpha$, we put $C' = 0$ in (3.8) for the cause of technical. Thus (3.8) becomes

$$\psi(\xi) = \frac{\beta}{4\alpha^2 k^2 - c_s^2}\varphi^2(\xi) + C. \tag{3.10}$$

Substituting (3.5), (3.7), and (3.10) into (3.4), we obtain

$$\varphi''(\xi) - \frac{(\omega + \alpha k^2 + C\delta)}{\alpha}\varphi(\xi) - \frac{\delta\beta}{\alpha\left(4\alpha^2 k^2 - c_s^2\right)}\varphi^3(\xi) = 0. \tag{3.11}$$

Let $l = (\omega + \alpha k^2 + C\delta)/\alpha$, $m = \delta\beta/\alpha(4\alpha^2 k^2 - c_s^2)$; thus (3.11) becomes the Liénard equation

$$\varphi''(\xi) - l\varphi(\xi) - m\varphi^3(\xi) = 0. \tag{3.12}$$

Let $X = \varphi(\xi)$, $Y = X'$, (3.12) can be converted to a system of nonlinear ODEs as follows:

$$\begin{aligned} X' &= Y, \\ Y' &= lX + mX^3. \end{aligned} \tag{3.13}$$

Now the Division theorem is applied to seek the first integral to (3.13). Suppose that $X = X(\xi)$, $Y = Y(\xi)$ are the nontrivial solution to the system (3.13), and its first integral is an irreducible polynomial in $\Re[X, Y]$

$$P(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X) Y^i = 0, \tag{3.14}$$

where $a_i$, $i = 0, 1, 2, \ldots, m$ are polynomials of $X$. According to the Division theorem, there exists a polynomial $T(X, Y) = \alpha(X) + \beta(X)Y$, such that

$$\frac{dP}{d\xi} = \frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \xi} = \left[\alpha(X) + \beta(X)Y\right]\left(\sum_{i=0}^{m} a_i(X) Y^i\right). \tag{3.15}$$

Here we only consider the case of $m = 2$.

Substituting (3.13) and (3.14) into (3.15), one gets

$$\begin{aligned} \frac{dP}{d\xi} &= a_0'(X)Y + a_1'(X)Y^2 + [a_1(X) + 2a_2(X)Y]Y' + a_2'(x)Y^3 \\ &= \left(\alpha(X) + \beta(X)Y\right)\left(a_0(X) + a_1(X)Y + a_2(X)Y^2\right). \end{aligned} \tag{3.16}$$

Collecting all the terms with the same power of $Y$ together and equating each coefficient to zero yields a set of nonlinear algebraic equations as follows:

$$a_2'(X) = \beta(X) a_2(X), \tag{3.17}$$

$$a_1'(X) = \beta(X) a_1(X) + \alpha(X) a_2(X), \tag{3.18}$$

$$a_0'(X) = \beta(X) a_0(X) + \alpha(X) a_1(X) - 2a_2(X)\left(lX + mX^3\right), \tag{3.19}$$

$$a_1(X)\left(lX + mX^3\right) = \alpha(X) a_0(X). \tag{3.20}$$

Because $a_i(X)$ ($i = 0, 1, 2$) are polynomials, from (3.17) we can deduce $\deg[a_2(X)] = 0$, $\beta(X) = 0$; that is, $a_2(X)$ is a constant. For simplicity, we take $\beta(X) = 0$, $a_2(X) = 1$. Then we determine $a_0(X)$, $a_1(X)$, and $\alpha(X)$. From (3.18), we have $\deg[a_1(X)] - 1 = \deg[\alpha(X)]$ or $a_1(X) = 0$, $\alpha(X) = 0$. In what follows we will discuss these two situations.

(a) In the case of $a_1(X) = 0$, $\alpha(X) = 0$.

In this case, (3.18) and (3.20) are satisfied. From (3.19), we can derive $a_0(X) = -(m/2)X^4 - lX^2 - d$, where $d$ is an integral constant. Substituting $a_2(X)$, $a_1(X)$, and $a_0(X)$ into (3.14), one obtain that

$$Y^2 = \frac{m}{2}X^4 + lX^2 + d, \tag{3.21}$$

that is,

$$X' = \pm\sqrt{\frac{m}{2}X^4 + lX^2 + d}. \tag{3.22}$$

Based on the discussion for different parameters, we can obtain the solutions of the nonlinear ordinary differential equation (3.22).

(1) For $d = 0$, (3.22) admits the following five general solutions:

$$
\begin{aligned}
X_1 &= \pm\sqrt{-\frac{2l}{m}}\sec\sqrt{-l}\xi, \quad l < 0,\ m > 0, \\
X_2 &= \pm\sqrt{-\frac{2l}{m}}\csc\sqrt{-l}\xi, \quad l < 0,\ m > 0, \\
X_3 &= \pm\sqrt{\frac{2l}{m}}\operatorname{csch}\sqrt{l}\xi, \quad lm > 0, \\
X_4 &= \pm\sqrt{-\frac{2l}{m}}\operatorname{sech}\sqrt{l}\xi, \quad l > 0,\ m < 0, \\
X_5 &= \pm\frac{1}{\sqrt{m/2}}, \quad l = 0.
\end{aligned} \tag{3.23}
$$

Combining (3.1), (3.7), (3.10), (3.23), and $X = \varphi(\xi)$, one can get the following five sets of explicit exact solution to (1.1):

$$
\begin{gathered}
E_1(x,t) = \pm\sqrt{\frac{2(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}}\sec\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{\alpha}}\xi\right]e^{i\eta}, \\
n_1(x,t) = -\frac{2(\omega + \alpha k^2 + C\delta)}{\delta}\sec^2\left[\sqrt{-\frac{\omega + \alpha k^2 + C\delta}{\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) < 0, \quad \alpha\beta\delta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.24}
$$

$$
\begin{gathered}
E_2(x,t) = \pm\sqrt{\frac{2(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}}\csc\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{\alpha}}\xi\right]e^{i\eta}, \\
n_2(x,t) = -\frac{2(\omega + \alpha k^2 + C\delta)}{\delta}\csc^2\left[\sqrt{-\frac{\omega + \alpha k^2 + C\delta}{\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) < 0, \quad \alpha\beta\delta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.25}
$$

$$
\begin{gathered}
E_3(x,t) = \pm\sqrt{\frac{2(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}}\operatorname{csch}\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{\alpha}}\xi\right]e^{i\eta}, \\
n_3(x,t) = \frac{2(\omega + \alpha k^2 + C\delta)}{\delta}\operatorname{csch}^2\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) > 0, \quad \alpha\beta\delta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.26}
$$

$$E_4(x,t) = \pm\sqrt{\frac{2(\omega+\alpha k^2+C\delta)(4\alpha^2k^2-c_s^2)}{\delta\beta}}\operatorname{sech}\left[\sqrt{\frac{\omega+\alpha k^2+C\delta}{\alpha}}\xi\right]e^{i\eta},$$
$$n_4(x,t) = -\frac{2(\omega+\alpha k^2+C\delta)}{\delta}\operatorname{sech}^2\left[\sqrt{\frac{\omega+\alpha k^2+C\delta}{\alpha}}\xi\right]+C, \tag{3.27}$$
$$\alpha\left(\omega+\alpha k^2+C\delta\right)>0,\quad \alpha\beta\delta\left(4\alpha^2k^2-c_s^2\right)<0,$$

where $\xi = x-2\alpha kt+\xi_1$, $\eta = kx+\omega t+\eta_0$, $k \neq \pm c_s/2\alpha$, $\omega$, $C$ are arbitrary parameters, and $\xi_1, \eta_0$ are two arbitrary constants. One has

$$E_5(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\frac{1}{(x-2\alpha kt+\xi_1)}e^{i\eta},$$
$$n_5(x,t) = \frac{2\alpha}{\delta}\left[\frac{1}{x-2\alpha kt+\xi_1}\right]^2-\frac{\omega+\alpha k^2}{\delta}, \tag{3.28}$$
$$\alpha\delta\beta\left(4\alpha^2k^2-c_s^2\right)>0,$$

where $\eta = kx+\omega t+\eta_0$, $k \neq \pm c_s/2\alpha$, $k$, $\omega$ are arbitrary parameters, and $\xi_1$, $\eta_0$ are two arbitrary constants.

*Remark 3.1.* In the above solutions, solutions $E_1$, $n_1$ (3.24) and $E_2$, $n_2$ (3.25) are explicit exact periodic traveling wave solutions. The solution $E_3$ is an envelope solitary wave solution of bell shape and $n_3$ is a explicit exact solitary wave solutions of bell shape. The $n_3$ be called Langmuir whistler soliton or Langmuir pit soliton according to $n_3$ is positive or negative. The solutions $E_4$, $n_4$ (3.27) are explicit exact singular traveling wave solutions. The singularity will appear as $\xi = 0$ and it represents that the distortions arise from the perturbed number density of the ion $n$ and the electric field intensity $E$ due to instability. While solution $E_5$ (and $n_5$, resp.) (3.28) is an envelope solitary wave solution of bell shape and (a explicit exact solitary wave solutions of bell shape, resp.) in rational function type.

(2) For $d = l^2/2m$, we can obtain following four sets of explicit exact solutions to (3.22)

$$X_6 = \pm\sqrt{\frac{l}{m}}\tan\sqrt{\frac{l}{2}}(\xi+\xi_2),\quad l>0,\ m>0,$$
$$X_7 = \pm\sqrt{\frac{l}{m}}\cot\sqrt{\frac{l}{2}}(\xi+\xi_2),\quad l>0,\ m>0,$$
$$X_8 = \pm\sqrt{-\frac{l}{m}}\tanh\sqrt{-\frac{l}{2}}(\xi+\xi_2),\quad l<0,\ m>0, \tag{3.29}$$
$$X_9 = \pm\sqrt{-\frac{l}{m}}\coth\sqrt{-\frac{l}{2}}(\xi+\xi_2),\quad l<0,\ m>0,$$

where $l = (\omega+\alpha k^2+C\delta)/\alpha$, $m = \delta\beta/\alpha(4\alpha^2k^2-c_s^2)$, and $\xi_2$ is integral constant.

Combining (3.1), (3.7), (3.10), (3.29), and $X = \varphi(\xi)$, we can get the following four explicit exact solution of (1.1):

$$
\begin{gathered}
E_6(x,t) = \pm\sqrt{\frac{(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}} \tan\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{2\alpha}}\xi\right] e^{i\eta}, \\
n_6(x,t) = \frac{\omega + \alpha k^2 + C\delta}{\delta}\tan^2\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{2\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) > 0, \quad \alpha\delta\beta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.30}
$$

$$
\begin{gathered}
E_7(x,t) = \pm\sqrt{\frac{(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}} \cot\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{2\alpha}}\xi\right] e^{i\eta}, \\
n_7(x,t) = \frac{\omega + \alpha k^2 + C\delta}{\delta}\cot^2\left[\sqrt{\frac{\omega + \alpha k^2 + C\delta}{2\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) > 0, \quad \alpha\delta\beta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.31}
$$

$$
\begin{gathered}
E_8(x,t) = \pm\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}} \tanh\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{2\alpha}}\xi\right] e^{i\eta}, \\
n_8(x,t) = -\frac{(\omega + \alpha k^2 + C\delta)}{\delta}\tanh^2\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{2\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) < 0, \quad \alpha\delta\beta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.32}
$$

$$
\begin{gathered}
E_9(x,t) = \pm\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)(4\alpha^2 k^2 - c_s^2)}{\delta\beta}} \coth\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{2\alpha}}\xi\right] e^{i\eta}, \\
n_9(x,t) = -\frac{2(\omega + \alpha k^2 + C\delta)}{\delta}\coth^2\left[\sqrt{\frac{-(\omega + \alpha k^2 + C\delta)}{2\alpha}}\xi\right] + C, \\
\alpha\left(\omega + \alpha k^2 + C\delta\right) < 0, \quad \alpha\delta\beta\left(4\alpha^2 k^2 - c_s^2\right) > 0,
\end{gathered} \tag{3.33}
$$

where $\xi = x - 2\alpha k t + \xi_1$, $\eta = kx + \omega t + \eta_0$, $k \neq \pm c_s/2\alpha$, $\omega$, $C$ are arbitrary parameters, and $\xi_1$, $\eta_0$ are two arbitrary constants.

*Remark 3.2.* The solutions $E_6$, $n_6$ (3.30) and $E_7$, $n_7$ (3.31) are all unbounded periodic traveling wave solutions of triangle function type. The solution $E_8$ is an envelope solitary wave solution of kink type while exact solitary wave solution $n_8$ is dark soliton; that means the density increases as a whole but decreases in part. The exact solutions $E_9$, $n_9$ (3.32) are explicit exact singular traveling wave solutions. The singularity will be appear as $\xi = 0$ and it indicates that the distortions arise from the perturbed number density of the ion $n$ and the electric field intensity $E$ due to instability.

(3) For $d \neq 0$, we obtain elliptic function solutions for (3.22) as follows:

$$X_{10} = \pm\sqrt{-\frac{2(l+1)}{m}}\,\mathrm{sn}\sqrt{\frac{md}{2}}\xi, \qquad d = -\frac{2(l+1)}{m}, \tag{3.34}$$

$$X_{11} = \pm\left[\frac{2d(1+l)}{m(l-1)}\right]^{1/4}\mathrm{cn}\sqrt{\frac{l+1}{2}}\xi, \qquad d = \frac{l^2-1}{2m}, \tag{3.35}$$

$$X_{12} = \pm\sqrt{\frac{2}{m}}\,\mathrm{dn}\sqrt{\frac{l+1}{2}}\xi, \qquad d = \frac{2(l-1)}{m}, \tag{3.36}$$

$$X_{13} = \pm\sqrt{\frac{2}{m}}\,\mathrm{ns}\sqrt{-l-1}\xi, \qquad d = -\frac{2(l+1)}{m}, \tag{3.37}$$

$$X_{14} = \pm\left[-\frac{l-1}{m}\right]^{1/4}\mathrm{nc}\sqrt{\frac{l+1}{2}}\xi, \qquad d = \frac{l^2-1}{2m}, \tag{3.38}$$

$$X_{15} = \pm\sqrt{-\frac{2(l-1)}{m}}\,\mathrm{nd}\sqrt{2-l}\xi, \qquad d = \frac{2(l-1)}{m}, \tag{3.39}$$

$$X_{16} = \pm\sqrt{\frac{2(l-1)}{m}}\,\mathrm{sc}\sqrt{2-l}\xi, \qquad d = \frac{2(l-1)}{m}, \tag{3.40}$$

$$X_{17} = \pm\sqrt{\frac{l^2-1}{2m}}\,\mathrm{sd}\sqrt{\frac{2+l}{2}}\xi, \qquad d = \frac{l^2-1}{2m}, \tag{3.41}$$

$$X_{18} = \pm\sqrt{\frac{2}{m}}\operatorname{cs}\sqrt{2-l}\xi, \qquad d = \frac{2(l-1)}{m}, \tag{3.42}$$

$$X_{19} = \pm\sqrt{-\frac{2(l+1)}{m}}\operatorname{cd}\sqrt{-l-1}\xi, \qquad d = -\frac{2(l+1)}{m}, \tag{3.43}$$

$$X_{20} = \pm\sqrt{\frac{2}{m}}\operatorname{ds}\sqrt{\frac{2+l}{2}}\xi, \qquad d = \frac{l^2-1}{2m}, \tag{3.44}$$

$$X_{21} = \pm\sqrt{\frac{2}{m}}\operatorname{dc}\sqrt{-l-1}\xi, \qquad d = -\frac{2(l+1)}{m}. \tag{3.45}$$

Combining (3.1), (3.7), (3.10), the above results (3.33)–(3.44), and $X = \varphi(\xi)$, we can get the following twelve Jacobi elliptic doubly periodic wave solutions of (1.1)

$$\begin{gathered}
E_{10}(x,t) = \pm\sqrt{-\frac{2(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta}}\operatorname{sn}\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)e^{i\eta},\\
n_{10}(x,t) = -\frac{2(\omega+\alpha k^2+C\delta+\alpha)}{\delta}\operatorname{sn}^2\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)+C,\\
d = -\frac{2(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta},
\end{gathered} \tag{3.46}$$

$$\begin{gathered}
E_{11}(x,t) = \pm\sqrt{\frac{(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta}}\operatorname{cn}\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+\alpha}{2\alpha}}\xi\right)e^{i\eta},\\
n_{11}(x,t) = \frac{\omega+\alpha k^2+C\delta+\alpha}{\delta}\operatorname{cn}^2\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+\alpha}{2\alpha}}\xi\right)+C,\\
d = \frac{\left[(\omega+\alpha k^2+C\delta)^2-\alpha^2\right](4\alpha^2k^2-c_s^2)}{2\alpha\delta\beta},
\end{gathered} \tag{3.47}$$

$$E_{12}(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2 - c_s^2)}{\delta\beta}}\,\mathrm{dn}\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{\alpha}}\,\xi\right)e^{i\eta},$$

$$n_{12}(x,t) = \frac{2\alpha}{\delta}\mathrm{dn}^2\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{\alpha}}\,\xi\right) + C, \tag{3.48}$$

$$d = \frac{2(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta},$$

$$E_{13}(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2 - c_s^2)}{\delta\beta}}\,\mathrm{ns}\left(\sqrt{\frac{-(\omega + \alpha k^2 + C\delta + \alpha)}{\alpha}}\,\xi\right)e^{i\eta},$$

$$n_{13}(x,t) = \frac{2\alpha}{\delta}\mathrm{ns}^2\left(\sqrt{\frac{-(\omega + \alpha k^2 + C\delta + \alpha)}{\alpha}}\,\xi\right) + C, \tag{3.49}$$

$$d = -\frac{2(\omega + \alpha k^2 + C\delta + \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta},$$

$$E_{14}(x,t) = \pm\sqrt{\frac{-(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta}}\,\mathrm{nc}\left(\sqrt{\frac{\omega + \alpha k^2 + C\delta + 2\alpha}{2\alpha}}\,\xi\right)e^{i\eta},$$

$$n_{14}(x,t) = -\frac{\omega + \alpha k^2 + C\delta - \alpha}{\delta}\mathrm{nc}^2\left(\sqrt{\frac{\omega + \alpha k^2 + C\delta + 2\alpha}{2\alpha}}\,\xi\right) + C, \tag{3.50}$$

$$d = \frac{\left[(\omega + \alpha k^2 + C\delta)^2 - \alpha^2\right](4\alpha^2k^2 - c_s^2)}{2\alpha\delta\beta},$$

$$E_{15}(x,t) = \pm\sqrt{\frac{-2(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta}}\,\mathrm{nd}\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{2\alpha}}\,\xi\right)e^{i\eta},$$

$$n_{15}(x,t) = -\frac{2(\omega + \alpha k^2 + C\delta - \alpha)}{\delta}\mathrm{nd}^2\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{2\alpha}}\,\xi\right) + C,$$

$$d = \frac{2(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta}, \tag{3.51}$$

$$E_{16}(x,t) = \pm\sqrt{\frac{2(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta}}\,\mathrm{sc}\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{2\alpha}}\,\xi\right)e^{i\eta},$$

$$n_{16}(x,t) = \frac{2(\omega + \alpha k^2 + C\delta - \alpha)}{\delta}\mathrm{sc}^2\left(\sqrt{\frac{2\alpha - (\omega + \alpha k^2 + C\delta)}{2\alpha}}\,\xi\right) + C,$$

$$d = \frac{2(\omega + \alpha k^2 + C\delta - \alpha)(4\alpha^2k^2 - c_s^2)}{\delta\beta}, \tag{3.52}$$

$$E_{17}(x,t) = \pm\sqrt{\frac{\left[(\omega+\alpha k^2+C\delta)^2-\alpha^2\right](4\alpha^2k^2-c_s^2)}{2\alpha\delta\beta}}\,\mathrm{sd}\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+2\alpha}{2\alpha}}\xi\right)e^{i\eta},$$

$$n_{17}(x,t) = \frac{(\omega+\alpha k^2+C\delta)^2-\alpha^2}{2\alpha\delta}\mathrm{sd}^2\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+2\alpha}{2\alpha}}\xi\right)+C,$$

$$d = \frac{\left[(\omega+\alpha k^2+C\delta)^2-\alpha^2\right](4\alpha^2k^2-c_s^2)}{2\alpha\delta\beta}, \tag{3.53}$$

$$E_{18}(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\mathrm{cs}\left(\sqrt{\frac{2\alpha-(\omega+\alpha k^2+C\delta)}{2\alpha}}\xi\right)e^{i\eta},$$

$$n_{18}(x,t) = \frac{2\alpha}{\delta}\mathrm{cs}^2\left(\sqrt{\frac{2\alpha-(\omega+\alpha k^2+C\delta)}{2\alpha}}\xi\right)+C, \tag{3.54}$$

$$d = \frac{2(\omega+\alpha k^2+C\delta-\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta},$$

$$E_{19}(x,t) = \pm\sqrt{\frac{-2(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta}}\mathrm{cd}\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)e^{i\eta},$$

$$n_{19}(x,t) = -\frac{2(\omega+\alpha k^2+C\delta+\alpha)}{\delta}\mathrm{cd}^2\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)+C,$$

$$d = -\frac{2(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta}, \tag{3.55}$$

$$E_{20}(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\mathrm{ds}\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+2\alpha}{2\alpha}}\xi\right)e^{i\eta},$$

$$n_{20}(x,t) = \frac{2\alpha}{\delta}\mathrm{ds}^2\left(\sqrt{\frac{\omega+\alpha k^2+C\delta+2\alpha}{2\alpha}}\xi\right)+C, \tag{3.56}$$

$$d = \frac{\left[(\omega+\alpha k^2+C\delta)^2-\alpha^2\right](4\alpha^2k^2-c_s^2)}{2\alpha\delta\beta},$$

$$E_{21}(x,t) = \pm\sqrt{\frac{2\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\mathrm{dc}\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)e^{i\eta},$$

$$n_{21}(x,t) = \frac{2\alpha}{\delta}\mathrm{dc}^2\left(\sqrt{\frac{-(\omega+\alpha k^2+C\delta+\alpha)}{\alpha}}\xi\right)+C, \tag{3.57}$$

$$d = -\frac{2(\omega+\alpha k^2+C\delta+\alpha)(4\alpha^2k^2-c_s^2)}{\delta\beta},$$

where $\xi = x - 2\alpha k t + \xi_1$, $\eta = kx + \omega t + \eta_0$, $k \neq \pm c_s/2\alpha$, $\omega$, $C$ are arbitrary parameters, and $\xi_1$, $\eta_0$ are two arbitrary constants.

(4) For $l = 0$, $m \neq 0$, from (3.22) we have

$$X' = \pm\sqrt{\frac{m}{2}X^4 + d}. \tag{3.58}$$

Let $X^2 = Z$; (3.58) becomes

$$Z'(\xi) = \pm\sqrt{2mZ^3 + 4dZ}. \tag{3.59}$$

While $m > 0$, the above equation possesses a Weierstrass elliptic function doubly periodic wave type solution

$$Z = \wp\left(\sqrt{\frac{m}{2}}\xi, \frac{-8d}{m}, 0\right). \tag{3.60}$$

So (3.58) admits a Weierstrass elliptic function doubly periodic wave type solution

$$X_{22} = \pm\sqrt{\wp\left(\sqrt{\frac{m}{2}}\xi, \frac{-8d}{m}, 0\right)}. \tag{3.61}$$

Combining (3.1), (3.7), (3.10), the above result (3.61), and $X = \varphi(\xi)$, we derive that (1.1) admits a Weierstrass elliptic function doubly periodic wave type solution

$$\begin{aligned} E_{22}(x,t) &= \pm\sqrt{\wp\left(\sqrt{\frac{\delta\beta}{2\alpha(4\alpha^2k^2 - c_s^2)}}\xi, \frac{-8d\delta\beta}{\alpha(4\alpha^2k^2 - c_s^2)}, 0\right)}e^{i(kx+\omega t+\xi_0)}, \\ n_{22}(x,t) &= \frac{\beta}{4\alpha^2k^2 - c_s^2}\wp\left(\sqrt{\frac{\delta\beta}{2\alpha(4\alpha^2k^2 - c_s^2)}}\xi, \frac{-8d\delta\beta}{\alpha(4\alpha^2k^2 - c_s^2)}, 0\right) + C, \end{aligned} \tag{3.62}$$

where $\xi = x - 2\alpha k t + \xi_1$.

*Remark 3.3.* The above twelve explicit exact Jacobi elliptic doubly periodic wave solutions $E_{10}$, $n_{10}$ (3.46); $E_{21}$, $n_{21}$ (3.57); and explicit exact Weierstrass elliptic doubly periodic wave solution $E_{22}$, $n_{22}$ (3.62) have not been obtained in the author's previous work [4] or other literature [5–8]. It should be emphasized that explicit exact Weierstrass elliptic doubly periodic wave solution $E_{22}$, $n_{22}$ (3.62) is obtained in this paper firstly.

(b) In the case of $\deg[a_1(X)] - 1 = \deg[\alpha(X)]$.

In this case, we assume that $\deg[\alpha(X)] = k_1$, $\deg[a_0(X)] = k_2$, then we have $\deg[a_1(X)] = k_1 + 1$. Now, by balancing the degrees of both sides of (3.20), we can deduce that $k_2 = 4$. By balancing the degrees of both sides of (3.19), we can also conclude that $k_1 = 1$ or

$k_1 = 0$. If $k_1 = 0$ assuming that $\alpha(X) = A_0$, $a_1(X) = A_1X + A_2$, $a_0(X) = C_4X^4 + C_3X^3 + C_2X^2 + C_1X + C_0$ and substituting them into (3.18)–(3.20); by equating the coefficients of the different powers of $X$ on both sides of (3.18) to (3.20), we can get that $\alpha(X) = a_1(X) = 0$. This contradicts with our assumption. It indicates that $k_1 \neq 0$. While $k_1 = 1$, assuming that $a_0(X) = C_4X^4 + C_3X^3 + C_2X^2 + C_1X + C_0$, $a_1 = A_2X^2 + A_1X + A_0$, $\alpha(X) = B_1X + B_0$ then substituting these representations into (3.18)–(3.20), and by equating the coefficients of the different powers of $X$ on both sides of (3.18) to (3.20), we can obtain an overdetermined system of nonlinear algebraic equations

$$\begin{aligned} 2A_2 &= B_1, \\ A_1 &= B_0, \\ 4C_4 + 2m &= A_2B_1, \\ 3C_3 &= A_1B_1 + A_2B_0, \\ 2C_2 + 2l &= A_1B_0 + A_0B_1, \\ C_1 &= A_2B_0, \\ mA_2 &= B_1C_4, \\ mA_1 &= B_1C_3 + C_4B_0, \\ mA_0 + lA_2 &= B_1C_2 + B_0C_3, \\ lA_1 &= B_1C_1 + B_0C_2, \\ lA_2 &= B_1C_0 + B_0C_1, \\ B_0C_0 &= 0. \end{aligned} \tag{3.63}$$

By analyzing all kinds of possibilities, we have the following.

(1) While $B_0 = C_0 = 0$, it leads to a contradiction.

(2) While $B_0 \neq 0, C_0 = 0$, it also leads to a contradiction.

(3) While $B_0 = 0, C_0 \neq 0$, we can derive that

$$A_0 = \pm\frac{l\sqrt{2m}}{m}, \qquad A_1 = 0, \qquad A_2 = \pm\sqrt{2m}, \qquad C_0 = \frac{l^2}{2m}, \qquad C_1 = C_3 = 0, \qquad C_2 = l, \qquad C_4 = \frac{m}{2}. \tag{3.64}$$

Setting (3.64) in (3.14) yields

$$(X')^2 \pm \left(\sqrt{2m}X^2 + \frac{l}{m}\sqrt{2m}\right)X' + \frac{m}{2}X^4 + lX^2 + \frac{l^2}{2m} = 0, \tag{3.65}$$

that is,

$$\frac{dX}{X^2 + l/m} = \pm\sqrt{\frac{m}{2}}(\xi + \xi_0). \tag{3.66}$$

Solving (3.66) we can obtain solutions $X_6$, $X_7$, $X_8$, and $X_9$ again. Consequently, we obtain explicit exact solutions $E_6$, $n_6$; $E_7$, $n_7$; $E_8$, $n_8$; $E_9$, $n_9$ to (1.1). Here we will not list them one by one.

## 4. Summary and Conclusions

In summary, we employ the first integral method to uniformly construct a series of explicit exact solutions for a system of Zakharov equations. Abundant explicit exact solutions to Zakharov equations are obtained through an exhaustive analysis and discussion of different parameters. The exact solutions obtained in this paper include that of the solitary wave solutions of bell type for $n$ and $E$, the solitary wave solutions of kink-type for $E$ and bell type for $n$, the two kinds of singular traveling wave solutions, four kinds of periodic wave solutions of triangle functions, twelve kinds of Jacobi elliptic function doubly periodic solutions, and one kind of Weierstrass elliptic function doubly periodic wave solutions. Among these are entirely new solutions that first reported in this paper. Some known results of previous references are enriched greatly. The results indicate that the first integral method is a very effective method to solve nonlinear differential equation. The method also is readily applicable to a large variety of other nonlinear evolution equations in physical mathematics. Of course, the first integral method also has its limitations. These solutions are not general and by no means exhaust all possibilities. Such solitary wave solutions of a compound of the bell shape and the kink-shape for $n$ and $E$ established in [4] cannot be obtained by using the first integral method. Some Jacobi elliptic function solutions obtained in [16] also cannot be established in here.

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