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Research Article

Existence and Global Exponential Stability of Periodic Solution to Cohen-Grossberg BAM Neural Networks with Time-Varying Delays

Kaiyu Liu, Zhengqiu Zhang, and Liping Wang

College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

Correspondence should be addressed to Kaiyu Liu, liukyhnu@yahoo.com.cn

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We investigate first the existence of periodic solution in general Cohen-Grossberg BAM neural networks with multiple time-varying delays by means of using degree theory. Then using the existence result of periodic solution and constructing a Lyapunov functional, we discuss global exponential stability of periodic solution for the above neural networks. Our result on global exponential stability of periodic solution is different from the existing results. In our result, the hypothesis for monotonicity ineqiality conditions in the works of Xia (2010) Chen and Cao (2007) on the behaved functions is removed and the assumption for boundedness in the works of Zhang et al. (2011) and Li et al. (2009) is also removed. We just require that the behaved functions satisfy sign conditions and activation functions are globally Lipschitz continuous.

1. Introduction

In 1983, Cohen and Grossberg [1] constructed a kind of simplified neural networks that are now called Cohen-Grossberg neural networks (CGNNs); they have received increasing interesting due to their promising potential applications in many fields such as pattern recognition, parallel computing, associative memory, and combinatorial optimization. Such applications heavily depend on the dynamical behaviors. Thus, the qualitative analysis of the dynamical behaviors is a necessary step for the practical design and application of neural networks (or neural system [2–4]). The stability of Cohen-Grossberg neural network with or without delays has been widely studied by many researchers, and various interesting results have been reported [5–14].

On the other hand, since the pioneering work of Kosko [15, 16], a series of neural networks related to bidirectional associative memory models have been proposed. These models generalized the single-layer autoassociative Hebbian correlator to a class of two-layer pattern-matched heteroassociative circuits. Bidirectional associative memory neural networks have also been used in many fields such as pattern recognition and automatic control and image and signal processing. During the last years, many authors have discussed the existence and global stability of BAM neural networks [17–20]. In recent years, a few authors [17, 21–26] discussed global stability of Cohen-Grossberg BAM neural networks.

As is well known, the studies on neural dynamical system not only involve a discussion of stability properties but also involve other dynamic behavior, such as periodic oscillatory behavior, chaos, and bifurcation. In many applications, periodic oscillatory behavior is of great interest; it has been found in applications in learning theory. Hence, it is of prime importance to study periodic oscillatory solutions of neural networks.

This motivates us to consider periodic solutions of Cohen-Grossberg BAM neural networks. Recently, a few authors discussed the existence and stability of periodic solution to Cohen-Grossberg BAM neural networks with delays [27–31].

In [27], the authors proposed a class of bidirectional Cohen-Grossberg neural networks with distributed delays as follows:

$$\begin{aligned}
&\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\\
&= -a_i(x_i(t))\left[b_i(t,x_i(t)) - \sum_{j=1}^{m} p_{ij}(t)\int_0^{\infty} K_{ji}(u)\times f_j\big(t,\lambda_j y_j(t-u)\big)\mathrm{d}u - I_i(t)\right],\quad i=1,2,\dots,n,\\
&\frac{\mathrm{d}y_j(t)}{\mathrm{d}t}\\
&= -c_j(y_j(t))\left[d_j\big(t,y_j(t)\big) - \sum_{i=1}^{n} q_{ji}(t)\int_0^{\infty} L_{ij}(u)\times g_i\big(t,\mu_i x_i(t-u)\big)\mathrm{d}u - J_j(t)\right],\quad j=1,2,\dots,m.
\end{aligned} \tag{1.1}$$

By using the Lyapunov functional method and some analytical techniques, some sufficient conditions were obtained for global exponential stability of periodic solutions to these networks.

In [28], the authors discussed the following Cohen-Grossberg-type BAM neural networks with time-varying delays:

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} p_{ij}(t) f_j\big(\lambda_j y_j(t-\tau_{ij}(t))\big) - I_i(t)\right],\quad i=1,2,\dots,n,\\
\frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -c_j(y_j(t))\left[d_j(y_j(t)) - \sum_{i=1}^{n} q_{ji}(t) g_i\big(\mu_i x_i(t-\sigma_{ji}(t))\big) - J_j(t)\right],\quad j=1,2,\dots,m,
\end{aligned} \tag{1.2}$$

where $n, m \geq 2$ are the number of neurons in the networks with initial value conditions:

$$x_i(\theta) = \phi_i(\theta),\quad \theta \in [-r_1, 0],\quad y_j(\theta) = \phi_j(\theta),\quad \theta \in [-r_2, 0], \tag{1.3}$$

where $r_1 = \max_{1\leq i\leq n,\, 1\leq j\leq m,\, 0\leq t\leq \omega}\{\sigma_{ji}(t)\}$, $r_2 = \max_{1\leq i\leq n, 1\leq j\leq m, 0\leq t\leq\omega}\{\tau_{ij}(t)\}$, $a_i(x_i(t))$, $b_i(x_i(t))$, $c_j(y_j(t))$, $d_j(y_j(t))$ are continuous functions, $f_j(\lambda_j y_j(t-\tau_{ji}(t)))$, $g_i(\mu_i x_i(t-\delta_{ij}(t)))$ are continuous functions, λ_j, μ_i are parameters, $I_i(t)$ and $J_j(t)$ are continuous functions, x_i and y_j denote the state variables of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V at time t, respectively, $a_i(x_i(t)) > 0$, $c_j(y_j(t)) > 0$ represent amplification functions of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, $b_i(x_i(t)), d_j(y_j(t))$ are appropriately behaved functions of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, f_j, g_i are the activation functions of the jth neurons from the neural field F_V and the ith neurons from the neural field F_U, respectively, I_i, J_j are the exogenous inputs of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, p_{ij} and q_{ji} are the connection weights, which denote the strengths of connectivity between the neuron j from the neural field F_V and the neuron i from the neural field F_U, and $\tau_{ij}(t)$, $\sigma_{ij}(t)$ correspond to the transmission time delays.

By using the analysis method and inequality technique, some sufficient conditions were obtained to ensure the existence, uniqueness, global attractivity, and exponential stability of the periodic solution to this neural networks.

In [29, 30], the authors discussed, respectively, two Cohen-Grossberg BAM neural networks on time scales. When time scale T becomes R, the existence and global exponential stability of periodic solution are obtained in [29, 30] under the assumptions that activation functions satisfy global Lipschitz conditions and boundedness conditions and behaved functions satisfy some inequality conditions.

In [31], the authors discussed the following Cohen-Grossberg BAM neural networks of neutral type with delays:

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} a_{ij}(t) f_j\left(y_j\left(t-\tau_{ij}(t)\right)\right)\right. \\
&\qquad\left. - \sum_{j=1}^{m} b_{ij}(t) f_j\left(y_j\left(t-\sigma_{ij}(t)\right)\right) - I_i(t)\right], \quad i = 1, 2, \ldots, n, \\
\frac{dy_j(t)}{dt} &= -c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right) - \sum_{i=1}^{n} c_{ji}(t) g_i\left(x_i\left(t-p_{ji}(t)\right)\right)\right. \\
&\qquad\left. - \sum_{i=1}^{n} d_{ji}(t) g_i\left(x_i\left(t-q_{ji}(t)\right)\right) - J_j(t)\right], \quad j = 1, 2, \ldots, m.
\end{aligned} \tag{1.4}$$

Under the assumptions that activation functions satisfy global Lipschitz conditions and behaved functions satisfy some inequality conditions, global exponential stability of periodic solution is obtained for system (1.4).

In this paper, our purpose is to obtain a new sufficient condition for the existence and global exponential stability of periodic solution of system (1.2). The paper is organized as follows. In Section 2, we discuss the existence of periodic solution of system (1.2) by using coincidence degree theory and inequality technique. In Section 3, we study the global exponential stability of periodic solution of system (1.2) by using the existence result of periodic solution and constituting Lyapunov functional. Our result on global exponential stability of periodic solution is different from the existing results. In our result, the hypotheses for monotonicity inequalities in [27, 28] on behaved functions are replaced with sign conditions and the assumption for boundedness in [29, 30] on activation functions is removed.

2. Existence of Periodic Solution

In this section, we first establish the existence of at least a periodic solution by applying the coincidence degree theory. To establish the existence of at least a periodic solution by applying the coincidence degree theory, we recall some basic tools in the frame work of Mawhin's coincidence degree [32] that will be used to investigate the existence of periodic solutions.

Let X, Z be Banach spaces, L: $\operatorname{Dom} L \subset X \to Z$ a linear mapping, and $N : X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to \operatorname{Ker} L$ and $Q : Z \to Z/\operatorname{Im} L$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L/_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of the map $L/_{\operatorname{Dom} L \cap \operatorname{Ker} P}$ by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $(QN)(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof of our existence theorem, we will use the continuation theorem of Gaines and Mawhin [32].

Lemma 2.1 (continuation theorem). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(a) $Lx \neq \lambda N(x)$, *for all* $\lambda \in (0,1)$, $x \in \partial\Omega$,

(b) $QN(x) \neq 0$, *for all* $x \in \operatorname{Ker} L \cap \partial\Omega$,

(c) $\deg(JQNx, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then, $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

For the sake of convenience, we introduce some notations.
$|\cdot|$ denotes the norm in R, $\overline{f} = \max_{0 \le t \le \omega}|f(t)|$, $\underline{f} = \min_{0 \le t \le \omega}|f(t)|$, where $f(t)$ is a continuously periodic function with common period ω. Our main result on the existence of at least a periodic solution for system (1.2) is stated in the following theorem.

Theorem 2.2. *One assume that the following conditions holds:*

(i) $p_{ij}(t)$, $q_{ji}(t)$, $I_i(t)$, $J_j(t)$ *are continuously periodic functions on* $t \in [0, +\infty)$ *with common period* $\omega > 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(ii) $a_i(\cdot)$ *and* $c_j(\cdot)$ *are continuously bounded, that is, there exist positive constants* l_i, l_i^*, k_j, k_j^* $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ *such that*

$$\begin{aligned} l_i &\le a_i \le l_i^*, \\ k_j &\le c_j \le k_j^*; \end{aligned} \tag{2.1}$$

(iii) $b_i(x_i(t))$ *and* $d_j(y_j(t))$ *are continuous and there exist positive constants* M_i, N_j $(i = 1,\dots,n, j = 1,\dots,m)$ *such that for all* $x, y \neq x \in R$*,*

$$\begin{aligned} \operatorname{sign}(x-y)\left[b_i(x)-b_i(y)\right] &\geq M_i|x-y|, \\ \operatorname{sign}(x-y)\left[d_j(x)-d_j(y)\right] &\geq N_j|x-y|; \end{aligned} \tag{2.2}$$

(iv) *there exist positive constants* A_j, B_i $(i = 1,\dots,n,\ j = 1,2,\dots,m)$ *such that for all* $x, y \in R$*,*

$$\begin{aligned} \left|f_j(x)-f_j(y)\right| &\leq A_j|x-y|, \\ \left|g_i(x)-g_i(y)\right| &\leq B_i|x-y|; \end{aligned} \tag{2.3}$$

(v) *there exist two positive constants* $r_i > 1,\ i = 1,2$ *with* $\tau'_{ij} < \min\{1, 1-r_1^{-1}\} < 1$ *and* $\sigma'_{ji} < \min\{1, 1-r_2^{-1}\} < 1$ *such that for* $i = 1,\dots,n;\ j = 1,\dots,m$*,*

$$\begin{aligned} l_i M_i &> \sum_{j=1}^{m} l_i^* \overline{p_{ij}} A_j \lambda_j \sqrt{r_1}, \\ k_j N_j &> \sum_{i=1}^{n} k_j^* \overline{q_{ji}} B_i \mu_i \sqrt{r_2}. \end{aligned} \tag{2.4}$$

Then, system (1.2) *has at least one* ω*-periodic solution.*

Proof. In order to apply Lemma 2.1 to system (1.2), let

$$\begin{gathered} X = \left\{u = \left(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\right)^T \in C\left(R, R^{m+n}\right) : u(t+\omega) = u(t)\right\}, \\ Z = \left\{z \in C\left(R, R^{m+n}\right) : z(t+\omega) = z(t)\right\}. \end{gathered} \tag{2.5}$$

Define

$$\|u\| = \max_{t\in[0,\omega]} \sum_{i=1}^{n} |x_i(t)| + \max_{t\in[0,\omega]} \sum_{j=1}^{m} \left|y_j(t)\right|, \quad u \in X \text{ or } Z. \tag{2.6}$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces.

Let for $u \in X$

$$\begin{gathered} Nu = \begin{pmatrix} H_i(t) \\ K_j(t) \end{pmatrix} = \begin{pmatrix} -a_i(x_i(t)) \left[b_i(x_i(t)) - \displaystyle\sum_{j=1}^{m} p_{ij}(t) f_j\left(\lambda_j y_j\left(t-\tau_{ij}(t)\right)\right) - I_i(t)\right] \\ -c_j\left(y_j(t)\right) \left[d_j\left(y_j(t)\right) - \displaystyle\sum_{i=1}^{n} q_{ji}(t) g_i\left(\mu_i x_i\left(t-\sigma_{ji}(t)\right)\right) - J_j(t)\right] \end{pmatrix}, \\ Lu = u' = \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \quad Pu = \frac{1}{\omega}\int_0^{\omega} u(t)\mathrm{d}t, \quad u \in X, \quad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\mathrm{d}t, \quad z \in Z. \end{gathered} \tag{2.7}$$

Then, it follows that $\operatorname{Ker} L = R^{(m+n)}$, $\operatorname{Im} L = \{z \in Z : \int_0^{\omega} z(t)dt = 0\}$ is closed in Z, $\dim \operatorname{Ker} L = m + n = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{2.8}$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_{p:} \operatorname{Im} L \rightarrow \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{s} z(t)\,\mathrm{d}t\,\mathrm{d}s. \tag{2.9}$$

Then,

$$QNu = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega} H_1(s)\mathrm{d}s \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} H_2(s)\mathrm{d}s \\ \vdots \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} H_n(s)\mathrm{d}s \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} K_1(s)\mathrm{d}s \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} K_2(s)\mathrm{d}s \\ \vdots \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} K_m(s)\mathrm{d}s \end{pmatrix},$$

$$K_p(I-Q)Nu = \begin{pmatrix} \int_0^t H_1(s)\mathrm{d}s - \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{t} H_1(s)\mathrm{d}s\mathrm{d}t + \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega} H_1(s)\mathrm{d}s \\ \int_0^t H_2(s)\mathrm{d}s - \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{t} H_2(s)\mathrm{d}s\mathrm{d}t + \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega} H_2(s)\mathrm{d}s \\ \vdots \\ \int_0^t H_n(s)\mathrm{d}s - \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{t} H_n(s)\mathrm{d}s\mathrm{d}t + \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega} H_n(s)\mathrm{d}s \\ \int_0^t K_1(s)\mathrm{d}s - \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{t} K_1(s)\mathrm{d}s\mathrm{d}t + \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega} K_1(s)\mathrm{d}s \\ \vdots \\ \int_0^t K_m(s)\mathrm{d}s - \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{t} K_m(s)\mathrm{d}s\mathrm{d}t + \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega} K_m(s)\mathrm{d}s \end{pmatrix}. \tag{2.10}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_p(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Condition (iii) in Theorem 2.2 implies that for all $x \in R$

$$\begin{aligned} \operatorname{sign} x b_i(x) &\geq M_i|x| + \operatorname{sign} x b_i(0), \\ \operatorname{sign} x d_j(x) &\geq N_j|x| + \operatorname{sign} x d_j(0). \end{aligned} \tag{2.11}$$

Condition (iv) in Theorem 2.2 implies that for all $x \in R$

$$\begin{aligned} \left|f_j(x)\right| &\leq A_j|x| + \left|f_j(0)\right|, \\ \left|g_i(x)\right| &\leq B_i|x| + \left|g_j(0)\right|. \end{aligned} \tag{2.12}$$

Corresponding to the operator equation $Lx = \lambda Nx,\ \lambda \in (0,1)$, we have for $i = 1,2,\ldots,n,\ j = 1,\ldots,m$

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= \lambda H_i(t), \\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= \lambda K_j(t). \end{aligned} \tag{2.13}$$

Assume that $u \in X$ is a solution of system (2.13) for some $\lambda \in (0,1)$. Multiplying the first equation of system (2.13) by $x_i(t)$ and integrating over $[0,\omega]$, we have

$$\begin{aligned} &\int_0^\omega x_i(t) \operatorname{sign} x_i(t) \operatorname{sign} x_i(t) \\ &\quad \times \left\{ a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^m p_{ij}(t)\ f_j\left(\lambda_j y_j\left(t - \tau_{ij}(t)\right)\right) - I_i(t) \right] \right\} \mathrm{d}t = 0. \end{aligned} \tag{2.14}$$

Multiplying the second equation of system (2.13) by $y_j(t)$ and integrating over $[0,\omega]$, we have

$$\begin{aligned} &\int_0^\omega y_j(t) \operatorname{sign} y_j(t) \operatorname{sign} y_j(t) \\ &\quad \times \left\{ c_j\left(y_j(t)\right) \left[d_j\left(y_j(t)\right) - \sum_{i=1}^n q_{ji}(t) g_i\left(\mu_i x_i\left(t - \sigma_{ji}(t)\right)\right) - J_j(t) \right] \right\} \mathrm{d}t = 0. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$
\begin{aligned}
&l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \\
&\quad \le l_i^* \int_0^{\omega} |x_i(t)| \left\{ -a_i \operatorname{sign} x_i(t) b_i(0) + \sum_{j=1}^{m} \overline{p_{ij}} \left(A_j \lambda_j \left| y_j(t-\tau_{ij}(t)) \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right\} \mathrm{d}t,
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
&k_i N_j \int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \\
&\quad \le k_j^* \int_0^{\omega} \left| y_j(t) \right| \left\{ -c_j \operatorname{sign} y_j(t) d_j(0) + \sum_{i=1}^{n} \overline{q_{ji}} \left(B_i \mu_i \left| x_i(t-\sigma_{ji}(t)) \right| + \left| g_i(0) \right| \right) + \overline{J_j} \right\} \mathrm{d}t.
\end{aligned} \tag{2.17}
$$

Hence,

$$
\begin{aligned}
&l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \\
&\quad \le l_i^* \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \\
&\times \left\{ \sum_{j=1}^{m} \overline{p_{ij}} \left[A_j \lambda_j \left(\int_0^{\omega} \left| y_j(t-\tau_{ij}(t)) \right|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left| f_j(0) \right| \right] + l_i^* |b_i(0)| + \sqrt{\omega}\, \overline{I_i} \right\},
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
&k_j N_j \int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \\
&\quad \le k_j^* \left(\int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \right)^{1/2} \\
&\times \left\{ \sum_{i=1}^{n} \overline{q_{ji}} \left[B_i \mu_i \left(\int_0^{\omega} \left| x_i(t-\sigma_{ji}(t)) \right|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left| g_i(0) \right| \right] + k_j^* \left| d_j(0) \right| + \sqrt{\omega}\, \overline{J_j} \right\}
\end{aligned} \tag{2.19}
$$

Denoting $s = t - \tau_{ij}(t) = g(t),\ \sigma = t - \sigma_{ji}(t) = h(t)$, then

$$
\left(\int_0^{\omega} \left| y_j(t-\tau_{ij}(t)) \right|^2 \mathrm{d}t \right)^{1/2} = \left(\int_0^{\omega} \frac{\left| y_j(s) \right|^2}{1-\tau_{ij}'(g^{-1}(s))} \mathrm{d}s \right)^{1/2}, \tag{2.20}
$$

$$
\left(\int_0^{\omega} \left| x_i(t-\sigma_{ji}(t)) \right|^2 \mathrm{d}t \right)^{1/2} = \left(\int_0^{\omega} \frac{|x_i(\sigma)|^2}{1-\sigma_{ji}'(h^{-1}(\sigma))} \mathrm{d}\sigma \right)^{1/2}. \tag{2.21}
$$

Substituting (2.20) into (2.18) and substituting (2.21) into (2.19) give for $i = 1,\ldots,n$, $j = 1,\ldots,m$

$$
\begin{aligned}
& l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \\
& \quad \le l_i^* \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \\
& \times \left\{ \sum_{j=1}^{m} \overline{p_{ij}} A_j \lambda_j \sqrt{r_1} \left(\int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left(\sum_{j=1}^{m} \overline{p_{ij}} \left| f_j(0) \right| + |b_i(0)| + \overline{I_i} \right) \right\},
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
& k_j N_j \int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \\
& \quad \le k_j^* \left(\int_0^{\omega} \left| y_j(t) \right|^2 \right)^{1/2} \\
& \times \quad \left\{ \sum_{i=1}^{n} \overline{q_{ji}} B_i \mu_i \sqrt{r_2} \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left(\sum_{i=1}^{n} \overline{q_{ji}} \left| g_i(0) \right| + \left| d_j(0) \right| + \overline{J_j} \right) \right\}.
\end{aligned}
\tag{2.23}
$$

Denoting for the sake of convenience

$$
\begin{aligned}
\max_{1 \le i \le n} \left\{ \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \right\} &= \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t \right)^{1/2}, \\
\max_{1 \le j \le m} \left\{ \left(\int_0^{\omega} \left| y_j(t) \right|^2 \mathrm{d}t \right)^{1/2} \right\} &= \left(\int_0^{\omega} \left| y_{j_0}(t) \right|^2 \mathrm{d}t \right)^{1/2},
\end{aligned}
\tag{2.24}
$$

where, $i_0 \in \{1,2,\ldots,n\}$, $j_0 \in \{1,2,\ldots,m\}$, and from (2.22) and (2.23), we obtain

$$
\begin{aligned}
l_{i_0} M_{i_0} \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t \right)^{1/2} \le{}& l_{i_0}^* \sum_{j=1}^{m} \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1} \left(\int_0^{\omega} \left| y_{j_0}(t) \right|^2 \mathrm{d}t \right)^{1/2} \\
& + l_{i_0}^* \sqrt{\omega} \left(\sum_{j=1}^{m} \overline{p_{i_0 j}} \left| f_j(0) \right| + |b_{i_0}(0)| + \overline{I_{i_0}} \right),
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
k_{j_0} N_{j_0} \left(\int_0^{\omega} \left| y_{j_0}(t) \right|^2 \mathrm{d}t \right)^{1/2} \le{}& k_{j_0}^* \sum_{i=1}^{n} \overline{q_{j_0 i}} B_i \mu_i \sqrt{r_2} \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t \right)^{1/2} \\
& + k_{j_0}^* \sqrt{\omega} \left(\sum_{i=1}^{n} \overline{q_{j_0 i}} \left| g_i(0) \right| + \left| d_{j_0}(0) \right| + \overline{J_{j_0}} \right).
\end{aligned}
\tag{2.26}
$$

Now we consider two possible cases for (2.26) and (2.25):

$$\begin{aligned}
&\text{(i)}\ \left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \le \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2},\\
&\text{(ii)}\ \left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} > \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{2.27}$$

When $(\int_0^\omega |y_{j_0}(t)|^2 \mathrm{d}t)^{1/2} \le (\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t)^{1/2}$, from (2.25), we have

$$\left(l_{i_0} M_{i_0} - l_{i_0}^* \sum_{j=1}^m \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1}\right)\left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} \le l_{i_0}^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{i_0 j}} \left|f_j(0)\right| + |b_{i_0}(0)| + \overline{I_{i_0}}\right). \tag{2.28}$$

Thus,

$$\begin{aligned}
\left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} &\le \frac{l_{i_0}^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{i_0 j}} \left|f_j(0)\right| + |b_{i_0}(0)| + \overline{I_{i_0}}\right)}{l_{i_0} M_{i_0} - l_{i_0}^* \sum_{j=1}^m \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1}}\\
&\le \max_{1 \le i \le n}\left\{\frac{l_i^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{ij}} \left|f_j(0)\right| + |b_i(0)| + \overline{I_i}\right)}{l_i M_i - l_i^* \sum_{j=1}^m \overline{p_{ij}} A_j \lambda_j \sqrt{r_1}}\right\}\\
&\overset{\text{def}}{=} d_1.
\end{aligned} \tag{2.29}$$

Therefore,

$$\begin{aligned}
\left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} &\le \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2}\\
&\le d_1.
\end{aligned} \tag{2.30}$$

(ii) When $(\int_0^\omega |y_{j_0}(t)|^2 \mathrm{d}t)^{1/2} > (\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t)^{1/2}$, from (2.26), we have

$$\left(k_{j_0} N_{j_0} - k_{j_0}^* \sum_{i=1}^n \overline{q_{j_0 i}} B_i \mu_i \sqrt{r_2}\right)\left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \le k_{j_0}^* \sqrt{\omega}\left(\sum_{i=1}^n \overline{q_{j_0 i}} \left|g_i(0)\right| + \left|d_{j_0}(0)\right| + \overline{J_{j_0}}\right). \tag{2.31}$$

Thus,

$$\begin{aligned}\left(\int_0^{\omega}\left|y_{j_0}(t)\right|^2\mathrm{d}t\right)^{1/2} &\le \frac{k_{j_0}^*\sqrt{\omega}\left(\sum_{i=1}^{n}\overline{q_{j_0 i}}\left|g_i(0)\right|+\left|d_{j_0}(0)\right|+\overline{J_{j_0}}\right)}{k_{j_0}N_{j_0}-k_{j_0}^*\sum_{i=1}^{n}\overline{q_{j_0 i}}B_i\mu_i\sqrt{r_2}} \\ &\le \max_{1\le j\le m}\left\{\frac{k_j^*\sqrt{\omega}\left(\sum_{i=1}^{n}\overline{q_{ji}}\left|g_i(0)\right|+\left|d_j(0)\right|+\overline{J_j}\right)}{k_jN_j-k_j^*\sum_{i=1}^{n}\overline{q_{ji}}B_i\mu_i\sqrt{r_2}}\right\} \\ &\overset{\text{def}}{=} d_2.\end{aligned} \tag{2.32}$$

Therefore,

$$\begin{aligned}\left(\int_0^{\omega}\left|x_{i_0}(t)\right|^2\mathrm{d}t\right)^{1/2} &\le \left(\int_0^{\omega}\left|y_{j_0}(t)\right|^2\mathrm{d}t\right)^{1/2} \\ &\le d_2.\end{aligned} \tag{2.33}$$

Hence, from (2.30) and (2.33), we have for $i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ t\in[0,\omega]$

$$\left(\int_0^{\omega}\left|x_i(t)\right|^2\mathrm{d}t\right)^{1/2} < \max\{d_1,d_2\}\overset{\text{def}}{=} d, \tag{2.34}$$

$$\left(\int_0^{\omega}\left|y_j(t)\right|^2\mathrm{d}t\right)^{1/2} < \max\{d_1,d_2\} = d. \tag{2.35}$$

Multiplying the first equation of system (2.13) by $x_i'(t)$ and integrating over $[0,\omega]$, from (2.20) and (2.35) and the fact that

$$\int_0^{\omega} a_i(x_i(t))b_i(x_i(t))x_i'(t)\,\mathrm{d}t = 0, \tag{2.36}$$

it follows that

$$\begin{aligned}\left(\int_0^{\omega}\left|x_i'(t)\right|^2\mathrm{d}t\right)^{1/2} &\le l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\left(\int_0^{\omega}\left|y_j\left(t-\tau_{ij}(t)\right)\right|\mathrm{d}t\right)^{1/2}+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right) \\ &\le l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\sqrt{r_1}\left(\int_0^{\omega}\left|y_j(t)\right|^2\mathrm{d}t\right)^{1/2}+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right) \\ &< \max_{1\le i\le n}\left\{l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\sqrt{r_1}d+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right)\right\}\overset{\text{def}}{=} c_1.\end{aligned} \tag{2.37}$$

Similarly, multiplying the second equation of system (2.13) by $y_j(t)$ and integrating over $[0,\omega]$, from (2.21) and (2.34) and the fact that

$$\int_0^{\omega} c_j(y_j(t))d_j(y_j(t))y_j'(t)\mathrm{d}t = 0, \tag{2.38}$$

it follows that there exists a positive constant c_2 such that

$$\left(\int_0^{\omega}\left|y_j'(t)\right|^2\mathrm{d}t\right)^{1/2} < c_2. \tag{2.39}$$

From (2.34) and (2.35), it follows that there exist points t_i and $\overline{t_j}$ such that

$$|x_i(t_i)| < \frac{d}{\sqrt{\omega}}, \tag{2.40}$$

$$\left|y_j\left(\overline{t_j}\right)\right| < \frac{d}{\sqrt{\omega}}. \tag{2.41}$$

Since for all $t \in [0,\omega]$,

$$\begin{aligned}|x_i(t)| &\le |x_i(t_i)| + \int_0^{\omega}\left|x_i'(t)\right|\mathrm{d}t \\ &\le |x_i(t_i)| + \sqrt{\omega}\left(\int_0^{\omega}\left|x_i'(t)\right|^2\right)^{1/2},\end{aligned} \tag{2.42}$$

$$\begin{aligned}\left|y_j(t)\right| &\le \left|y_j(t_i)\right| + \int_0^{\omega}\left|y_j'(t)\right|\mathrm{d}t \\ &\le \left|y_j(t)\right| + \sqrt{\omega}\left(\int_0^{\omega}\left|y_j'(t)\right|^2\right)^{1/2},\end{aligned} \tag{2.43}$$

then from (2.40)–(2.43), we have for $t \in [0,\omega]$, $i = 1,\dots,n$, $j = 1,\dots,m$

$$\begin{aligned}|x_i(t)| &\le \frac{d}{\sqrt{\omega}} + \sqrt{\omega}c_1, \\ \left|y_j(t)\right| &\le \frac{d}{\sqrt{\omega}} + \sqrt{\omega}c_2.\end{aligned} \tag{2.44}$$

Obviously, $d/\sqrt{\omega}$, $\sqrt{\omega}c_1$, and $\sqrt{\omega}c_2$ are all independent of λ. Now let

$$\begin{aligned}\Omega = \Bigg\{ &u = (x_1,x_2,\dots,x_n;y_1,y_2,\dots,y_m)^T \in X : \\ &\|u\| < n\left(\frac{d}{\sqrt{\omega}} + r_1 + \sqrt{\omega}c_1\right) + m\left(\frac{d}{\sqrt{\omega}} + r_2 + \sqrt{\omega}c_2\right)\Bigg\},\end{aligned} \tag{2.45}$$

where $r_1,\ r_2$ are two chosen positive constants such that the bound of Ω is larger. Then, Ω is bounded open subset of X. Hence, Ω satisfies requirement (a) in Lemma 2.1. We prove that (b) in Lemma 2.1 holds. If it is not true, then when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$ we have

$$\begin{aligned} &QNu \\ &\quad = \left(\frac{1}{\omega}\int_0^\omega H_1(t)\mathrm{d}t, \frac{1}{\omega}\int_0^\omega H_2(t)\mathrm{d}t, \dots, \frac{1}{\omega}\int_0^\omega H_n(t)\mathrm{d}t; \frac{1}{\omega}\int_0^\omega K_1(t)\mathrm{d}t, \dots, \frac{1}{\omega}\int_0^\omega K_m(t)\mathrm{d}t\right)^T \\ &\quad = (0, \dots, 0)^T. \end{aligned} \tag{2.46}$$

Therefore, there exist points ξ_i $(i = 1, 2, \dots, n)$ and η_j $(j = 1, 2, \dots, m)$ such that

$$\begin{aligned} H_i(\xi_i) &= 0, \\ K_j(\eta_j) &= 0. \end{aligned} \tag{2.47}$$

From this and following the arguments of (2.40) and (2.41), we have for forall $i = 1, 2, \dots, n, j = 1, 2, \dots, m, t \in [0, \omega]$

$$\begin{aligned} |x_i(t)| &< \frac{d}{\sqrt{\omega}}, \\ |y_j(t)| &< \frac{d}{\sqrt{\omega}}. \end{aligned} \tag{2.48}$$

Hence,

$$\|u\| < n\frac{d}{\sqrt{\omega}} + m\frac{d}{\sqrt{\omega}}. \tag{2.49}$$

Thus, $u \in \Omega \cap R^{(m+n)}$. This contradicts the fact that $u \in \partial\Omega \cap R^{(m+n)}$. Hence, this proves that (b) in Lemma 2.1 holds. Finally, we show that (c) in Lemma 2.1 holds. We only need to prove that $\deg\{-JQNu, \Omega \cap \operatorname{Ker} L, (0,0)^T\} \neq (0, 0, \dots, 0)^T$. Now, we show that

$$\begin{aligned} &\deg\left\{-JQNu, \Omega \cap \operatorname{Ker} L, (0, 0, \dots, 0)^T\right\} \\ &\quad = \deg\left\{(l_1M_1x_1, l_2M_2x_2, \dots, l_nM_nx_n; k_1N_1y_1, \dots, k_mN_my_m)^T,\ \Omega \cap \operatorname{Ker} L, (0, \dots, 0)^T\right\}. \end{aligned} \tag{2.50}$$

To this end, we define a mapping $\phi : \operatorname{Dom} L \times [0, 1] \to X$ by

$$\begin{aligned} &\phi(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_m, \mu) \\ &\quad = -\frac{\mu}{\omega}\left(\int_0^\omega H_1(t)\mathrm{d}t, \int_0^\omega H_2(t)\mathrm{d}t, \dots, \int_0^\omega H_n(t)\mathrm{d}t, \int_0^\omega K_1(t)\mathrm{d}t, \dots, \int_0^\omega K_m(t)\mathrm{d}t\right) \\ &\qquad + (1-\mu)(l_1M_1x_1, l_2M_2x_2, \dots, l_nM_nx_n;\ k_1N_1y_1, \dots, k_mN_my_m), \end{aligned} \tag{2.51}$$

where $\mu \in [0,1]$ is a parameter. We show that when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$, $\phi(x_1, x_2, \ldots, x_n;\ y_1, \ldots, y_m, \mu) \neq (0,0,\ldots,0)^T$. If it is not true, then when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$, $\phi(x_1, x_2, \ldots, x_n; y_1, \ldots, y_m, \mu) = (0,0,\ldots,0)^T$. Thus, constant vector u with $u \in \partial\Omega$ satisfies for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$,

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\left\{a_i(x_i)\left[b_i(x_i) - a_i(x_i)\sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t + (1-\mu) l_i M_i x_i = 0,\\
&\frac{\mu}{\omega}\int_0^{\omega}\left\{c_j(y_j)\left[d_j(y_j) - c_j(y_j)\sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i u_i) - J_j(t)\right]\right\}\mathrm{d}t + (1-\mu) k_j N_j y_j = 0.
\end{aligned} \tag{2.52}$$

That is,

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} x_i\left\{a_i(x_i)(b_i(x_i) - b_i(0)) + a_i(x_i) b_i(0) - a_i(x_i)\left[\sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t\\
&\qquad + (1-\mu) l_i M_i |x_i| = 0,
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} y_j\left\{c_j(y_j)(d_j(y_j) - d_j(0)) + c_j(y_j) d_j(0) - c_j(y_j)\left[\sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i u_i) - J_j(t)\right]\right\}\mathrm{d}t\\
&\qquad + (1-\mu) k_j N_j |y_j| = 0.
\end{aligned} \tag{2.54}$$

Denote $|y_{j_0}| = \max_{1\le j\le m}\{|y_j|\}, |x_{i_0}| = \max_{1\le i\le n}\{|x_i|\}$. □

Claim 1. We claim that $|x_{i_0}| < (d/\sqrt{\omega}) + \sqrt{\omega}c_1 + r_1$, otherwise, $|x_{i_0}| \ge (d/\sqrt{\omega}) + \sqrt{\omega}c_1 + r_1$. We consider two possible cases: (i) $|y_{j_0}| \le |x_{i_0}|$ and (ii) $|y_{j_0}| > |x_{i_0}|$.

(i) When $|y_{j_0}| \le |x_{i_0}|$, we have

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} x_{i_0}\left\{a_i(x_{i_0})(b_i(x_{i_0}) - b_i(0)) + a_i(x_{i_0})\left[b_i(0) - \sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t\\
&\quad + (1-\mu) l_i M_i |x_{i_0}|\\
&\qquad \ge \mu l_i M_i |x_{i_0}| - l_i^{*}\left[|b_i(0)| + \sum_{j=1}^{m}\overline{p_{ij}}(\lambda_j A_j |y_j| + |f_j(0)|) + \overline{I_i}\right] + (1-\mu) l_i M_i |x_{i_0}|
\end{aligned}$$

$$\begin{aligned}
&\geq l_i M_i |x_{i_0}| - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&\geq \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) |x_{i_0}| - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&\geq \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) \left(\frac{d_1}{\sqrt{\omega}} + \sqrt{\omega} c_1 + r_1 \right) - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&> \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) r_1 \\
&> 0,
\end{aligned} \tag{2.55}$$

which contradicts (2.53).

(ii) When $|y_{j_0}| > |x_{i_0}|$, we have

$$\begin{aligned}
&\frac{\mu}{\omega} \int_0^{\omega} \operatorname{sign} y_{j_0} \left\{ c_j(y_{j_0}) \left(d_j(y_{j_0}) - d_j(0) \right) + c_j(y_{j_0}) \left[d_j(0) - \sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i x_i) - J_j(t) \right] \right\} \mathrm{dt} \\
&\quad + (1-\mu) k_j N_j \left| y_{j_0} \right| \\
&\quad \geq \mu k_j N_j \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_i| + \left| g_i(0) \right| \right) + \overline{J_j} \right] + (1-\mu) k_j N_j \left| y_{j_0} \right| \\
&\quad \geq k_j N_j \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad \geq \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad \geq \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) \left(\frac{d_2}{\sqrt{\omega}} + \sqrt{\omega} c_1 + r_1 \right) - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad > \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) r_1 \\
&\quad > 0,
\end{aligned} \tag{2.56}$$

which contradicts (2.54). From the discussion of (i) and (ii), Claim 1 holds.

Claim 2. We claim that $|y_{j_0}| < (d/\sqrt{\omega}) + \sqrt{\omega} c_2 + r_2$, otherwise, $|y_{j_0}| \geq (d/\sqrt{\omega}) + \sqrt{\omega} c_2 + r_2$. We consider two possible cases: (i) $|x_{i_0}| \leq |y_{j_0}|$ and (ii) $|x_{i_0}| > |y_{j_0}|$.

The proofs of (i) and (ii) are similar to those of (ii) and (1) in Claim 1, respectively, therefore Claim 2 holds.

Thus, $|x_i| < (d_1/\sqrt{\omega}) + c_1\sqrt{\omega} + r_1$ and $|y_j| < (d_2/\sqrt{\omega}) + \sqrt{\omega}c_2 + r_2$. Thus, $u \in \Omega \cap R^{(m+n)}$. This contradicts the fact that $u \in \partial\Omega \cap R^{(m+n)}$. According to the topological degree theory and by taking $J = I$ since $\operatorname{Ker} L = \operatorname{Im} Q$, we obtain

$$\begin{aligned}
&\deg\left\{-JQNu, \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\phi(u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_m, 1), \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\phi(u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_m, 0), \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\left(l_1 M_1 x_1, l_2 M_2 x_2, \ldots, l_n M_n x_n; k_1 N_1 y_1, \ldots, k_m N_m y_m\right)^T, \Omega \cap \operatorname{Ker} L, (0,\ldots,0)^T\right\} \\
&\quad \neq 0.
\end{aligned} \tag{2.57}$$

So far, we have proved that Ω satisfies all the assumptions in Lemma 2.1. Therefore, system (1.2) has at least one ω-periodic solution.

3. Global Exponential Stability of Periodic Solution

In this section, by constructing a Lyapunov functional, we derive new sufficient conditions for global exponential stability of a periodic solution of system (1.2).

Theorem 3.1. *In addition to all conditions in Theorem 2.2, one assumes further that the following conditions hold:*

(H_1) *there exists two positive constants $r_i \geq 1$ $(i = 1,2)$ with $M_i > \sum_{j=1}^{m} \overline{q_{ji}}\mu_i B_i r_2$ and $N_j > \sum_{i=1}^{n} \overline{p_{ij}}\lambda_j A_j r_1$ such that $\tau'_{ij} < \min\{1, 1 - r_1^{-1}\} < 1$ and $\sigma'_{ji} < \min\{1, 1 - r_2^{-1}\} < 1$;*

(H_2) *there exist constants τ_{ij} and σ_{ji}, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, such that*

$$0 < \tau_{ij}(t) < \tau_{ij}, \quad 0 < \sigma_{ji}(t) < \sigma_{ji}. \tag{3.1}$$

Then, the ω periodic solution of system (1.2) *is globally exponentially stable.*

Proof. By Theorem 2.2, system (1.2) has at least one ω periodic solution, say, $u^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t); y_1^*(t), \ldots, y_m^*(t))^T$. Suppose that $u(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^T$ is an arbitrary ω periodic solution of system (1.2). From (H_1), we can choose a suitable θ such that

$$\begin{aligned}
M_i &> \frac{\theta}{l_i} + \sum_{j=1}^{m} \overline{q_{ji}}\mu_i B_i r_2 \exp(\theta\tau_{ij}), \\
N_j &> \frac{\theta}{k_j} + \sum_{i=1}^{n} \overline{p_{ij}}\lambda_j A_j r_i \exp(\theta\sigma_{ji}).
\end{aligned} \tag{3.2}$$

We define a Lyapunov functional as follows for $t > 0, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$:

$$
\begin{aligned}
V(t) = \exp(\theta t)&\left\{ \sum_{i=1}^{n} \left| \int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s \right| + \sum_{j=1}^{m} \left| \int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \right| \mathrm{d}s \right\} \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j \int_{t-\tau_{ij}(t)}^{t} \exp\left[\theta\left(\sigma + \tau_{ij}\left(g^{-1}(\sigma) \right) \right) \right] \frac{\left| y_j(\sigma) - y_j^*(\sigma) \right|}{1 - \tau_{ij}'(g^{-1}(\sigma))} \mathrm{d}\sigma \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{q_{ji}} \mu_i B_i \int_{t-\sigma_{ji}(t)}^{t} \exp\left[\theta\left(\sigma + \sigma_{ji}\left(h^{-1}(\sigma) \right) \right) \right] \frac{\left| x_i(\sigma) - x_i^*(\sigma) \right|}{1 - \sigma_{ji}'(h^{-1}(\sigma))} \mathrm{d}\sigma,
\end{aligned}
\tag{3.3}
$$

where $g(t) = t - \tau_{ij}(t),\ h(t) = t - \sigma_{ji}(t),\ i = 1, 2, \ldots, n,\ j = 1, \ldots, m$. Calculating the upper right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (1.2), we obtain

$$
\begin{aligned}
D^+V(t) \le \exp(\theta t) \sum_{i=1}^{n} &\left\{ \theta \left| \int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s \right| - M_i \left| x_i(t) - x_i^*(t) \right| \right. \\
&\left. + \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j \left| y_j(t - \tau_{ij}(t)) - y_j^*(t - \tau_{ij}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{j=1}^{m} &\left\{ \theta \left| \int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s \right| - N_j \left| y_j(t) - y_j^*(t) \right| \right. \\
&\left. + \sum_{i=1}^{n} \overline{q_{ji}} \mu_i B_i \left| x_i(t - \sigma_{ji}(t)) - x_i^*(t - \tau_{ji}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j &\left\{ \frac{\left| y_j(t) - y_j^*(t) \right| \exp\left[\theta \tau_{ij}(s^{-1}(t)) \right]}{1 - \tau_{ij}'(s^{-1}(t))} \right. \\
&\left. - \left| y_j(t - \tau_{ij}(t)) - y_j^*(t - \tau_{ij}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{q_{ji}} \mu_i B_i &\left\{ \frac{\left| x_i(t) - x_i^*(t) \right| \exp\left[\theta \sigma_{ji}(h^{-1}(t)) \right]}{1 - \sigma_{ji}'(h^{-1}(t))} \right. \\
&\left. - \left| x_i(t - \sigma_{ji}(t)) - x_i^*(t - \sigma_{ji}(t)) \right| \right\}.
\end{aligned}
\tag{3.4}
$$

Since there exist points ξ_i, η_j such that

$$\left|\int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s\right| = \frac{1}{a_i(\xi_i)} \left|x_i(t) - x_i^*(t)\right|,$$
$$\left|\int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s\right| = \frac{1}{c_j(\eta_j)} \left|y_j(t) - y_j^*(t)\right|, \tag{3.5}$$

from (3.4), we have

$$\begin{aligned} D^+V(t) \le &- \exp(\theta t) \sum_{j=1}^{m} \left\{ N_j - \frac{\theta}{k_j} - \sum_{i=1}^{n} \overline{p_{ij}} \lambda_j A_j r_1 \exp(\theta \sigma_{ji}) \right\} \left| y_j(t) - y_j^*(t) \right| \\ &- \exp(\theta t) \sum_{i=1}^{n} \left\{ M_i - \frac{\theta}{l_i} - \sum_{j=1}^{m} \overline{q_{ji}} \mu_i B_i r_2 \exp(\theta \tau_{ij}) \right\} \left| x_i(t) - x_i^*(t) \right|. \end{aligned} \tag{3.6}$$

In view of (3.2), it follows that $V(t) < V(0)$. Therefore,

$$\exp(\theta t) \left\{ \sum_{i=1}^{n} \left| \int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s \right| + \sum_{j=1}^{m} \left| \int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s \right| \right\} < V(t) < V(0). \tag{3.7}$$

Equation (3.3) implies that

$$\begin{aligned} V(0) < &\sum_{i=1}^{n} \left\{ \frac{1}{l_i} + \sum_{j=1}^{m} \overline{w_{ji}} \mu_i B_i r_2 \int_{-\sigma_{ji}(0)}^{0} \exp\left[\theta\left(\sigma + \sigma_{ji}\left(h^{-1}(\sigma)\right)\right)\right] \mathrm{d}\sigma \right\} \sup_{0 \le \le \omega} \left| x_i(s) - x_i^*(s) \right| \\ &+ \sum_{j=1}^{m} \left\{ \frac{1}{k_j} + \sum_{i=1}^{n} \overline{h_{ij}} \lambda_j A_j r_1 \int_{-\tau_{ij}}^{0} \exp\left[\theta\left(\sigma + \tau_{ij}\left(g^{-1}(\sigma)\right)\right)\right] \mathrm{d}\sigma \right\} \sup_{0 \le s \le \omega} \left| y_j(s) - y_j^*(s) \right|. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) gives

$$\begin{aligned} &\sum_{i=1}^{n} \left| x_i(t) - x_i^*(t) \right| + \sum_{j=1}^{m} \left| y_j(t) - y_j^*(t) \right| \\ &\quad < \frac{M}{N} \exp(-\theta t) \left\{ \sum_{i=1}^{n} \sup_{0 \le s \le \omega} \left| x_i(s) - x_i^*(s) \right| + \sum_{j=1}^{m} \sup_{0 \le s \le \omega} \left| y_j(s) - y_j^*(s) \right| \right\}, \end{aligned} \tag{3.9}$$

where

$$M = \max_{1\le i\le n, 1\le j\le m}\left\{\frac{1}{l_i}+\sum_{i=1}^{n}\overline{h_{ij}}\lambda_j A_j r_1\int_{-\tau_{ij}(0)}^{0}\exp\left[\theta+\tau_{ij}\left(g^{-1}(\sigma)\right)\right]\mathrm{d}\sigma,\right.$$
$$\left.\frac{1}{k_j}+\sum_{j=1}^{m}\overline{w_{ji}}\mu_i B_i r_2\int_{-\sigma_{ji}(0)}^{0}\exp\left[\theta+\sigma_{ji}\left(h^{-1}(\sigma)\right)\right]\mathrm{d}\sigma\right\}, \tag{3.10}$$
$$N = \min\left\{\frac{1}{l_i^*},\frac{1}{k_j^*}\right\}.$$

The proof of Theorem 3.1 is complete. □

4. An Example

Example 4.1. Consider the following Cohen-Grossberg BAM neural networks with time-varying delays:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(2+\sin x_1)\left\{200x_1(t)+100\sin x_1(t)-(2+\sin t)\left|y_1\left[t-\left(1+\frac{\sin t}{2}\right)\right]\right|-\sin t\right\},$$
$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -(3+\cos y_1)\left\{200y_1(t)+100\sin y_1(t)-(2+\cos t)\left|x_1\left[t-\left(1+\frac{\sin t}{3}\right)\right]\right|-\cos t\right\}. \tag{4.1}$$

In Theorem 3.1,

$$A_1=1,\quad B_1=1,\quad l_1=1,\quad l_1^*=3,\quad k_1=2,\quad k_1^*=4,\quad M_1=100,$$
$$N_1=100,\quad \overline{p_{11}}=3,\quad \overline{q_{11}}=3,\quad \lambda_1=\mu_1=1,$$
$$\tau_{11}'=\frac{\cos t}{2},\quad \sigma_{11}'=\frac{\cos t}{3}. \tag{4.2}$$

Since

$$1-\frac{\cos t}{2}\ge 1-\frac{|\cos t|}{2}\ge\frac{1}{2},\qquad 1-\frac{\cos t}{3}\ge 1-\frac{|\cos t|}{3}\ge\frac{2}{3}, \tag{4.3}$$

then $r_1=2$, $r_2=3/2$.

Since

$$M_1=100>\overline{q_{11}}\mu_1 B_1 r_2=\frac{9}{2},\qquad N_1=100>\overline{p_{11}}\lambda_1 A_1 r_1=6,$$
$$l_1M_1=100>l_1^*\overline{p_{11}}A_1\sqrt{r_1}=9\sqrt{2},\qquad k_1N_1=200>\overline{q_{11}}B_1\mu_1\sqrt{r_2}=12\sqrt{\frac{3}{2}}, \tag{4.4}$$

then conditions (H_1), (H_2), and (v) are satisfied. It is easy to prove that the rest of the conditions in Theorem 3.1 are satisfied. By Theorem (3.2) , system (4.1) has a unique ω periodic solution that is globally exponentially stable.

5. Conclusion

We investigate first the existence of the periodic solution in general Cohen-Grossberg BAM neural networks with multiple time-varying delays by means of using degree theory. Then, using the existence result of periodic solution and constructing a Lyapunov functional, we discuss global exponential stability of periodic solution for the above neural networks. In our result, the hypotheses for monotonicity in [27, 28] on the behaved functions are replaced with sign conditions and the assumption for boundedness on activation functions is removed. We just require that the behaved functions satisfy sign conditions and activation functions are globally Lipschitz continuous.

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