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Research Article

Application of Rational Second Kind Chebyshev Functions for System of Integrodifferential Equations on Semi-Infinite Intervals

M. Tavassoli Kajani,[1] S. Vahdati,[2] Zulkifly Abbas,[3] and Mohammad Maleki[4]

[1] *Department of Mathematics, Islamic Azad University, Khorasgan Branch, 81515-158 Isfahan, Iran*
[2] *Department of Mathematics, Khansar Faculty of Computer and Mathematics, University of Isfahan, Isfahan 81746-73441, Iran*
[3] *Department of Physics, Faculty of Science, University Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[4] *Department of Mathematics, Islamic Azad University, Mobarakeh Branch, 84819-14411 Isfahan, Iran*

Correspondence should be addressed to Zulkifly Abbas, za@science.upm.edu.my

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Rational Chebyshev bases and Galerkin method are used to obtain the approximate solution of a system of high-order integro-differential equations on the interval $[0,\infty)$. This method is based on replacement of the unknown functions by their truncated series of rational Chebyshev expansion. Test examples are considered to show the high accuracy, simplicity, and efficiency of this method.

1. Introduction

In recent years, there has been a growing interest in the system of integrodifferential equations (IDE), which arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, and so forth [1–8]. The systems of integrodifferential equations are generally difficult to solve analytically, thus finding efficient computational algorithms for obtaining numerical solution is required.

There are various techniques for solving systems of IDE, for example, operational Tau method [9, 10], Adomian decomposition method [11], Galerkin method [12], rationalized Haar functions method [13], He's homotopy perturbation method (HPM) [14, 15], and Ghebyshev polynomial [16].

A number of problems arising in science and engineering are set in semi-infinite domains. One can apply different spectral methods that are used to solve problems in semi-infinite domains. The first approach is using Laguerre polynomials [17–20]. The second approach is replacing semi-infinite domain with $[0, L]$ interval by choosing L, sufficiently large. This method is named domain truncation [21]. The third approach is reformulating original problem in semi-infinite domain to singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [22]. The fourth approach of spectral method is based on rational orthogonal functions.

Boyd [23] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo et al. [24] introduced a new set of rational Legendre functions which are mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd et al. [25] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre, and mapped Fourier sine.

The authors of [26–29] applied spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on a rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev and Legendre functions and then they applied these matrices together with the Tau method to reduce the solution of these problems to the solution of system of algebraic equations.

Zarebnia and Ali Abadi [30] used Sinc-Collocation method for solving system of nonlinear second-order integrodifferential equations of the Fredholm type. Rational second (third) kind Chebyshev (RSC) functions, for the first time, were proposed by Tavassoli Kajani and Ghasemi Tabatabaei [31] to find the numerical solution of Lane-Emden equation.

This paper outlines the application of rational second kind Chebyshev functions and Galerkin method to the following system of linear high-order integrodifferential equations on the interval $[0, \infty)$. Two problems of such equations are solved to make clear the application of the proposed method. One has

$$\sum_{i=1}^{l}\left(\sum_{j=0}^{m}\nu_{pij}(x)y_i^{(j)}(x)+\lambda_{ip}\int_a^b\left(k_{ip}(x,t)y_i(t)\right)dt\right)=f_p(x),\quad p=1,2,\ldots,l,$$
$$y_i^{(j)}(0)=y_{ij},\quad i=1,2,\ldots,l,\ j=0,1,\ldots,m-1,$$
$$x\in[0,\infty), \tag{1.1}$$

where $0 \le a < b < \infty$.

2. Properties of RSC Functions

In this section, we present some properties of RSC functions.

2.1. RSC Functions

The second kind Chebyshev polynomials $U_n(x)$, $n \ge 0$, are orthogonal in the interval $[-1, 1]$ with respect to the weight function $\sqrt{1-x^2}$ and we find that $U_n(x)$ satisfies the recurrence relation [32]

$$U_0(x)=1,\qquad U_1(x)=2x,$$
$$U_n(x)=2xU_{n-1}(x)-U_{n-2}(x),\quad n\ge 2. \tag{2.1}$$

The RSC functions are defined by [31, 33]

$$R_n(x) = U_n\left(\frac{x-1}{x+1}\right), \tag{2.2}$$

thus RSC functions satisfy

$$\begin{gathered} R_0(x) = 1, \qquad R_1(x) = 2\left(\frac{x-1}{x+1}\right), \\ R_n(x) = 2\left(\frac{x-1}{x+1}\right)R_{n-1}(x) - R_{n-2}(x), \quad n \geq 2. \end{gathered} \tag{2.3}$$

2.2. Function Approximation

Let $w(x) = 4\sqrt{x}/(x+1)^3$ denotes a nonnegative, integrable, real-valued function over the interval $I = [0, +\infty)$. We define

$$L_w^2(I) = \left\{y : I \longrightarrow \mathbb{R} \mid y \text{ is measurable and } \|y\|_w < \infty\right\}, \tag{2.4}$$

where

$$\|y\|_w = \left(\int_0^\infty |y(x)|^2 w(x)dx\right)^{1/2} \tag{2.5}$$

is the norm induced by the scalar product

$$\langle y, z\rangle_w = \int_0^\infty y(x)z(x)w(x)dx. \tag{2.6}$$

Thus $\{R_n(x)\}_{n\geq 0}$ denote a system which is mutually orthogonal under (2.6), that is,

$$\int_0^\infty R_n(x)R_m(x)w(x)dx = \frac{\pi}{2}\delta_{nm}, \tag{2.7}$$

where δ_{nm} is the Kronecker delta function. This system is complete in $L_w^2(I)$; as a result, any function $y \in L_w^2(I)$ can be expanded as follows:

$$y(x) = \sum_{k=0}^{\infty} y_k R_k(x), \tag{2.8}$$

with

$$y_k = \frac{2}{\pi}\langle y, R_k\rangle_w. \tag{2.9}$$

The y_k's are the expansion coefficients associated with the family $\{R_k(x)\}$. If the infinite series in (2.8) is truncated, then it can be written as

$$y(x) \simeq y_N(x) = \sum_{k=0}^{N} y_k R_k(x) = Y^T R(x), \tag{2.10}$$

where $Y = [y_0, y_1, \dots, y_N]^T$ and $R(x) = [R_0(x), R_1(x), \dots, R_N(x)]^T$.

We can also approximate the function $k(x,t)$ in $L_w^2(I \times I)$ as follows:

$$k(x,t) \simeq k_N(x,t) = R^T(x) K R(t), \tag{2.11}$$

where K is an $(N+1) \times (N+1)$ matrix that

$$K_{ij} = \frac{2}{\pi^2} \langle R_i(x), \langle k(x,t), R_j(t) \rangle \rangle, \quad i,j = 0,1,\dots,N. \tag{2.12}$$

Moreover, from recurrence relation in (2.3) we have

$$R(0) = \left[1, -2, 3, -4, \dots, (-1)^N (N+1)\right]^T = \mathbf{e}_1. \tag{2.13}$$

2.3. Product Integration of the RSC Functions

We also use the matrix P_{ab} as follows:

$$P_{ab} = \int_a^b R(t) R^T(t)\, dt. \tag{2.14}$$

To illustrate the calculation P_{ab} we choose $a = 0$ and $b = 1$, then we obtain

$$P_{01} = \begin{bmatrix} 1 & 2-4\ln 2 & 11-16\ln 2 & 28-40\ln 2 & \frac{167}{3}-80\ln 2 & \cdots \\ 2-4\ln 2 & 12-16\ln 2 & 30-44\ln 2 & \frac{200}{3}-96\ln 2 & \frac{374}{3}-180\ln 2 & \cdots \\ 11-16\ln 2 & 30-44\ln 2 & \frac{203}{3}-96\ln 2 & \frac{380}{3}-184\ln 2 & \frac{3329}{15}-320\ln 2 & \cdots \\ 28-40\ln 2 & \frac{200}{3}-96\ln 2 & \frac{380}{3}-184\ln 2 & \frac{3344}{15}-320\ln 2 & \frac{5396}{15}-520\ln 2 & \cdots \\ \frac{167}{3}-80\ln 2 & \frac{374}{3}-180\ln 2 & \frac{3329}{15}-320\ln 2 & \frac{5396}{15}-520\ln 2 & \frac{19451}{35}-800\ln 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.15}$$

2.4. Operational Matrix of Derivative

The derivative of the vector $R(x)$ defined in (2.10) can be approximated by

$$R'(x) \simeq DR(x), \tag{2.16}$$

where D is named the $n \times n$ operational matrix of derivative. Differentiating (2.3) we get

$$\begin{gathered} R_0'(x) = 0, \qquad R_1'(x) = \frac{5}{4}R_0(x) - R_1(x) + \frac{1}{4}R_2(x), \\ R_n'(x) = (R_1(x)R_{n-1}(x))' - R_{n-2}'(x), \quad n \geq 2. \end{gathered} \tag{2.17}$$

By using (2.17) the matrix D can be calculated. The matrix D is a lower Hessenberg matrix and can be expressed as $D = D_1 + D_2$, where D_1 is a tridiagonal matrix which is obtained from

$$D_1 = \operatorname{diag}\left(\frac{-2+7i}{4}, -i, \frac{i}{4}\right), \quad i = 0, \ldots, n-1, \tag{2.18}$$

and the d_{ij} elements of matrix D_2 are obtained from

$$d_{ij} = \begin{cases} 0, & i \leq j+1, \\ (-1)^{i+j+1}(2j), & i > j+1. \end{cases} \tag{2.19}$$

2.5. The Product Operational Matrix

The following property of the product of two rational Chebyshev vectors will also be used:

$$R(x)R^T(x)Y = \tilde{Y}R(x), \tag{2.20}$$

where $\tilde{Y}$ is called $(N+1) \times (N+1)$ product operational matrix for the vector Y. Using (2.20) and the orthogonal property, the elements $\tilde{Y}_{ij}$, $i = 0, \ldots, N$, $j = 0, \ldots, N$ of the matrix $\tilde{Y}$ can be calculated from

$$\tilde{Y}_{ij} = \frac{2}{\pi}\sum_{k=0}^{N} c_k g_{ijk}, \tag{2.21}$$

where g_{ijk} is given by

$$g_{ijk} = \int_0^\infty R_i(x)R_j(x)R_k(x)w(x)dx. \tag{2.22}$$

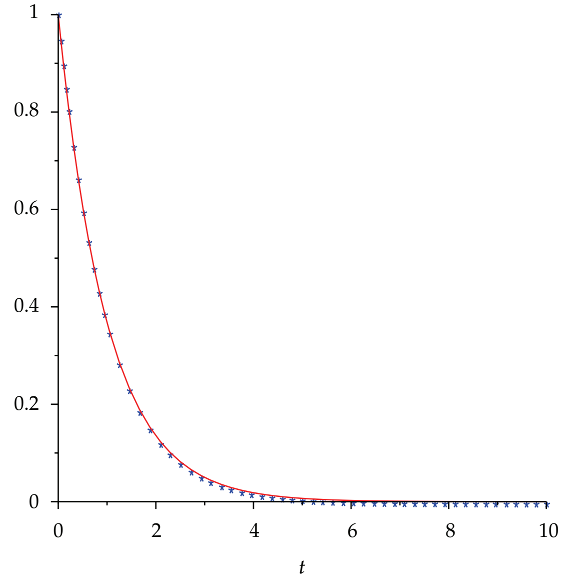

Figure 1: Graph of the exact and numerical solutions of $y_1(x)$ for $N = 13$; symbols correspond to the numerical solution.

3. Solving System of Integrodifferential Equations over Semi-Infinite Interval

Consider the following system of integrodifferential equations:

$$\sum_{i=1}^{l}\left(\sum_{j=0}^{m}\nu_{pij}(x)y_i^{(j)}(x)+\lambda_{ip}\int_a^b\left(k_{ip}(x,t)y_i(t)\right)dt\right)=f_p(x),\quad p=1,2,\ldots,l,$$
$$y_i^{(j)}(0)=y_{ij},\quad i=1,2,\ldots,l,\quad j=0,1,\ldots,m-1,$$
$$x\in[0,\infty). \tag{3.1}$$

Using (2.10) and (2.11) to approximate y_i, f_p, k_{ip}, and ν_{pij} when $i,p = 1,2,\ldots,l$ and $j = 0,\ldots,m$, we have

$$y_i(x)\simeq Y_i^TR(x),\qquad f_p(x)\simeq F_p^TR(x),\qquad k_{ip}(x,t)\simeq R^T(t)K_{ip}R(x),\qquad \nu_{pij}(x)\simeq V_{pij}^TR(x). \tag{3.2}$$

According to the operational matrix of derivative we can approximate $y_i^{(j)}$ as

$$y_i^{(j)}(x)\simeq Y_i^TR^{(j)}(x)\simeq Y_i^TD^jR(x),$$
$$y_i^{(j)}(0)\simeq Y_i^TD^jR(0)=Y_i^TD^j\mathbf{e}_1. \tag{3.3}$$

With substituting these approximations in (3.1) we have

$$\sum_{i=1}^{l}\left(\sum_{j=0}^{m} Y_i^T D^j R(x) R^T(x) V_{pij} + \lambda_{ip}\int_a^b Y_i^T R(t) R^T(t) K_{ip} R(x)\,dt\right) = F_p^T R(x),$$
$$Y_i^T D^j \mathbf{e}_1 = y_{ij}, \quad i = 1,2,\ldots,l,\ j = 0,1,\ldots,m-1,\ p = 1,2,\ldots,l. \tag{3.4}$$

Then using (2.20) we obtain

$$\sum_{i=1}^{l}\left(\sum_{j=0}^{m} Y_i^T D^j \widetilde{V}_{pij} R(x) + \lambda_{ip} Y_i^T\left(\int_a^b R(t) R^T(t)\,dt\right) K_{ip} R(x)\right) = F_p^T R(x),$$
$$Y_i^T D^j \mathbf{e}_1 = y_{ij}, \quad i = 1,2,\ldots,l,\ j = 0,1,\ldots,m-1,\ p = 1,2,\ldots,l, \tag{3.5}$$

which can be simplified using (2.14)

$$\sum_{i=1}^{l}\left(\sum_{j=0}^{m} Y_i^T D^j \widetilde{V}_{pij} + \lambda_{ip} Y_i^T P_{ab} K_{ip}\right) = F_p^T, \quad p = 1,2,\ldots,l,$$
$$Y_i^T D^j \mathbf{e}_1 = y_{ij}, \quad i = 1,2,\ldots,l,\ j = 0,1,\ldots,n-1. \tag{3.6}$$

By solving this linear system of algebraic equations we can find vectors Y_i, $i = 1,2,\ldots,n$, and then approximate the solutions $y_i(x)$ as

$$y_i(x) \simeq Y_i^T R(x). \tag{3.7}$$

4. Numerical Examples

Example 4.1. Consider the following system of linear integrodifferential equations:

$$\frac{1}{x+1} y_1''(x) + y_1'(x) + y_2''(x) + \int_0^1 \frac{y_1(t) + y_2(t)}{(x+1)^2 (t+1)^2}\,dt = \frac{x-15}{4(x+1)^3},$$
$$y_1'(x) - 2y_2'(x) + y_2''(x) + 24\int_0^1\left(\frac{y_1(t)}{(x+1)^2(t+1)^2} + \frac{y_2(t)}{(x+1)^3(t+1)^3}\right)dt = \frac{8x-1}{(x+1)^3}, \tag{4.1}$$
$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = -1, \quad y_2'(0) = 2.$$

The exact solution of this example is $y_1(x) = 1$ and $y_2(x) = (x-1)/(x+1)$.

Table 1: Numerical results of Example 4.2.

x	$y_1(x)$			$y_2(x)$		
	$N = 11$	$N = 13$	Exact	$N = 11$	$N = 13$	Exact
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	0.82435	0.81643	0.81873	0.67203	0.66964	0.67032
0.4	0.67955	0.66654	0.67032	0.45203	0.44825	0.44933
0.6	0.56024	0.54411	0.54881	0.30436	0.29994	0.30119
0.8	0.46199	0.44410	0.44933	0.20516	0.20062	0.20190
1.0	0.38119	0.36238	0.36788	0.13844	0.13413	0.13534
1.2	0.31473	0.29559	0.30119	0.09351	0.08965	0.09072
1.4	0.26007	0.24100	0.24660	0.06321	0.05991	0.06081
1.6	0.21512	0.19637	0.20190	0.04274	0.04006	0.04076
1.8	0.17819	0.15987	0.16530	0.02886	0.02681	0.02732
2.0	0.14787	0.13003	0.13534	0.01942	0.01798	0.01832
2.2	0.12300	0.10562	0.11080	0.01297	0.01211	0.01228
2.4	0.10261	0.08565	0.09072	0.00855	0.00822	0.00823
2.6	0.08589	0.06932	0.07427	0.00550	0.00565	0.00552
2.8	0.07218	0.05596	0.06081	0.00339	0.00395	0.00370
3.0	0.06094	0.04503	0.04970	0.00193	0.00285	0.00248

We solved Example 4.1 using the present method with $N = 3$, and we obtained $Y_1^T = [1, 0, 0, 0]$ and $Y_2^T = [0, 0.5, 0, 0]$, which imply

$$y_1(x) = Y_1^T R(x) = 1, \qquad y_2(x) = Y_2^T R(x) = \frac{x-1}{x+1} \tag{4.2}$$

that are the exact solutions.

Example 4.2. Next, consider the following system of integrodifferential equations with the exact solution $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$:

$$\begin{gathered} y_1''(x) + y_1'(x) + y_2''(x) + 2y_2'(x) + \int_0^1 6e^{-t-x}(y_1(t) + y_2(t))dt = \left(5 - 3e^{-2} - 2e^{-3}\right)e^{-x}, \\ 4e^{-x}y_1''(x) - y_2''(x) + \int_0^1 12e^{-2(t+x)}(y_1(t) + y_2(t))dt = \left(5 - 4e^{-3} - 3e^{-4}\right)e^{-2x}, \\ y_1(0) = 1, \quad y_1'(0) = -1, \quad y_2(0) = 1, \quad y_2'(0) = -2. \end{gathered} \tag{4.3}$$

We solved this example by using the method described in Section 3 for $N = 11$ and $N = 13$. Results are shown in Table 1 and Figures 1 and 2. The errors for large values of x are shown in Table 2. It is seen that the proposed method provides accurate results even for large values of x.

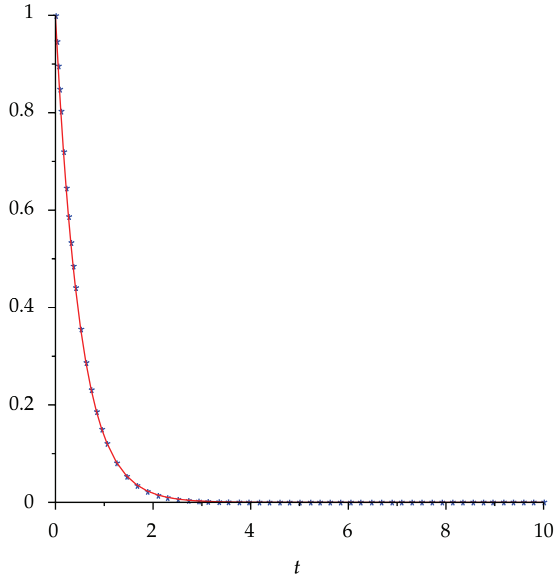

Figure 2: Graph of the exact and numerical solutions of $y_2(x)$ for $N = 13$; symbols correspond to the numerical solution.

Table 2: Absolute error for large values of x.

	$N = 11$			
	$x = 50$	$x = 100$	$x = 200$	$x = 400$
$e(y_1(x) - Y_1^T R(x))$	3.1×10^{-4}	7.9×10^{-5}	1.1×10^{-6}	5.3×10^{-7}
$e(y_2(x) - Y_2^T R(x))$	4.5×10^{-3}	2.4×10^{-3}	1.2×10^{-4}	6.3×10^{-5}

5. Conclusion

The fundamental goal of this paper has been to construct an approximation to the solution of the integrodifferential equations system in a semi-infinite interval. In the above discussion, the Galerkin method with RSC functions, which have the property of orthogonality, is employed to achieve this goal. Advantages of this method is that we do not reform the problem to a finite domain, and with a small value of N accurate results are obtained. There is a good agreement between obtained results and exact values that demonstrates the validity of the present method for this type of problems and gives the method a wider applicability.

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