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Research Article

Regularity for Variational Evolution Integrodifferential Inequalities

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We deal with the regularity for solutions of nonlinear functional integrodifferential equations governed by the variational inequality in a Hilbert space. Moreover, by using the simplest definition of interpolation spaces and the known regularity result, we also prove that the solution mapping from the set of initial and forcing data to the state space of solutions is continuous, which very often arises in application. Finally, an example is also given to illustrate our main result.

1. Introduction

In this paper, we deal with the regularity for solutions of nonlinear functional integrodifferential equations governed by the variational inequality in a Hilbert space H:

$$\begin{aligned}&\left(x'(t)+Ax(t),x(t)-z\right)+\phi(x(t))-\phi(z)\\&\quad\le\left(\int_0^t k(t-s)g(s,x(s))ds+h(t),x(t)-z\right),\quad \text{a.e., } 0<t\le T,\ z\in H,\\&\qquad x(0)=x_0,\end{aligned}\tag{VIP}$$

where A is a unbounded linear operator associated with a sesquilinear form satisfying Gårding's inequality and $\phi : H \to (-\infty,+\infty]$ is a lower semicontinuous, proper convex function. The nonlinear mapping g is a Lipschitz continuous from $\mathbb{R}\times V$ into H in the second coordinate, where V is a dense subspace of H.

The background of these problems has emerged vigorously in such applied fields as automatic control theory, network theory, and the dynamic systems.

By using the subdifferential operator $\partial\phi$, the control system (VIP) is represented by the following nonlinear functional differential equation on H:

$$x'(t) + Ax(t) + \partial\phi(x(t)) \ni \int_0^t k(t-s)g(s,x(s))ds + h(t), \quad 0 < t \leq T,$$
$$x(0) = x_0. \tag{NDE}$$

In Section 4.3.2 of Barbu [1] (also see Section 4.3.1 in [2]) is widely developed the existence of solutions for the case $g \equiv 0$. Recently, the regular problem for solutions of the nonlinear functional differential equations with a nonlinear hemicontinuous and coercive operator A was studied in [3]. Some results for solutions of a class of semilinear equations with the nonlinear terms have been dealt with in [3–7]. As for nontrivial physical examples from the field of visco-elastic materials modeled by integrodifferential equations on Banach spaces, we refer to [8].

In this paper, we will define $\phi_\epsilon : H \to H(\epsilon > 0)$ such that the function ϕ_ϵ is Fréchet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is a single valued and Lipschitz continuous on H with Lipschitz constant ϵ^{-1}, where $\partial\phi_\epsilon = \epsilon^{-1}(I-(I+\epsilon\partial\phi)^{-1})$ as is seen in Corollary 2.2 in [1, Chapter II]. It is also well-known results that $\lim_{\epsilon\to 0}\phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$, where $(\partial\phi)^0$ is the minimal segment of $\partial\phi$. Now, we introduce the smoothing system corresponding to (NDE) as follows:

$$x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = \int_0^t k(t-s)g(s,x(s))ds + h(t), \quad 0 < t \leq T,$$
$$x(0) = x_0. \tag{SDE 1}$$

First we recall some regularity results and a variation of constant formula for solutions of the semilinear functional differential equation (in the case $g \equiv 0$ in (SDE 1)):

$$x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = h(t) \tag{1.1}$$

in a Hilbert space H.

Next, based on the regularity results for (1.1), we intend to establish the regularity for solutions of (NDE). Here, our approach is that results of a class of semilinear equations as (1.1) on L^2-regularity remain valid under the above formulation perturbed of nonlinear terms. Here, we note that sine A is not bounded operator H into itself, the Lipschitz continuity of nonlinear terms must be defined on some adjusted spaces (see Section 3). Moreover, using the simplest definition of interpolation spaces and known regularity, we have that the solution mapping from the set of initial and forcing data to the state space of solutions is continuous, which very often arises in application. Finally, an example is also given to illustrate our main result.

2. Preliminaries

Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H. The norms of V, H and V^* are denoted by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively. The inner product in H is defined by $(\cdot,\cdot)$. The embeddings

$$V \hookrightarrow H \hookrightarrow V^* \tag{2.1}$$

are continuous. Then the following inequality easily follows:

$$\|u\|_* \leq |u| \leq \|u\|, \quad \forall u \in V. \tag{2.2}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re}\ \ a(u,u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2, \quad \omega_1 > 0,\ \omega_2 \geq 0. \tag{2.3}$$

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u,v), \quad u, v \in V. \tag{2.4}$$

Then A is a bounded linear operator from V to V^* and $-A$ generates an analytic semigroup in both of H and V^* as is seen in [9, Theorem 6.1]. The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V;\ Au \in H\} \tag{2.5}$$

is also denoted by A. From the following inequalities:

$$\omega_1 \|u\|^2 \leq \text{Re}\ \ a(u,u) + \omega_2 |u|^2 \leq C|Au|\ |u| + \omega_2 |u|^2 \leq \max\{C, \omega_2\} \|u\|_{D(A)} |u|, \tag{2.6}$$

where

$$\|u\|_{D(A)} = \left(|Au|^2 + |u|^2\right)^{1/2} \tag{2.7}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_1 > 0$ such that

$$\|u\| \leq C_1 \|u\|_{D(A)}^{1/2} |u|^{1/2}. \tag{2.8}$$

Thus, we have the following sequence:

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.9}$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations* (2.8), (2.9), *one has*

$$(D(A), H)_{1/2,2} = V, \tag{2.10}$$

where $(D(A), H)_{1/2,2}$ denotes the real interpolation space between $D(A)$ and H (Section 2.4 of [10] or [11]).

The following abstract linear parabolic equation:

$$\begin{aligned} x'(t) + Ax(t) &= h(t), \quad 0 < t \leq T, \\ x(0) &= x_0, \end{aligned} \tag{LE}$$

has a unique solution $x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T, H)$ for each $T > 0$ if $x_0 \in V \equiv (D(A), H)_{1/2,2}$ and $h \in L^2(0, T; H)$. Moreover, one has

$$\|x\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T,H)} \leq C_2\Big(\|x_0\|_{(D(A),H)_{1/2,2}} + \|h\|_{L^2(0,T;H)}\Big), \tag{2.11}$$

where C_2 depends on T and M (see [12, Theorem 2.3], [13]).

In order to substitute H for the intermediate space V considering A as an operator in $B(V, V^*)$ instead of $B(D(A), H)$ one proves the following result.

Lemma 2.2. *Let $T > 0$. Then*

$$H = \left\{ x \in V^* : \int_0^T \left\| Ae^{tA}x \right\|_*^2 dt < \infty \right\}. \tag{2.12}$$

Hence, it implies that $H = (V, V^*)_{1/2,2}$ in the sense of intermediate spaces generated by an analytic semigroup.

Proof. Put $u(t) = e^{tA}x$ for $x \in H$. From the result of Theorem 2.3 in [12] it follows

$$u \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*), \tag{2.13}$$

hence

$$\int_0^T \left\| Ae^{tA}x \right\|_*^2 dt = \int_0^T \|u'(t)\|_*^2 dt < \infty. \tag{2.14}$$

Conversely, suppose that $x \in V^*$ and $\int_0^T \|Ae^{tA}x\|_*^2 dt < \infty$. Put $u(t) = e^{tA}x$. Then since A is an isomorphism from V to V^* there exists a constant $c > 0$ such that

$$\int_0^T \|u(t)\|^2 dt \leq c\int_0^T \|Au(t)\|_*^2 dt = c\int_0^T \left\| Ae^{tA}x \right\|_*^2 dt. \tag{2.15}$$

Thus, we have $u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$. By using the definition of real interpolation spaces by trace method, it is known that the embedding $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H)$ is continuous. Hence, it follows $x = u(0) \in H$. □

In view of Lemma 2.2 we can apply (2.11) to (LE) in the space V^* as follows.

Proposition 2.3. *Let $x_0 \in H$ and $h \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of* (LE) *belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H) \tag{2.16}$$

and satisfying

$$\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2\Big(|x_0| + \|h\|_{L^2(0,T;V^*)}\Big), \tag{2.17}$$

where C_2 is a constant depending on T.

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y) + (x^*, x - y),\ y \in V\}. \tag{2.18}$$

First, let us concern with the following perturbation of subdifferential operator:

$$\begin{gathered} x'(t) + Ax(t) + \partial\phi(x(t)) \ni h(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{gathered} \tag{VE}$$

Using the regularity for the variational inequality of parabolic type in case where $\phi : V \to (-\infty, +\infty]$ is a lower semicontinuous, proper convex function as is seen in [1, Section 4.3] one has the following result on (VE).

Proposition 2.4. *(1) Let $h \in L^2(0,T;V^*)$ and $x_0 \in V$ satisfying that $\phi(x_0) < \infty$. Then* (VE) *has a unique solution:*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H), \tag{2.19}$$

which satisfies

$$\begin{gathered} x'(t) = \big(h(t) - Ax(t) - \partial\phi(x(t))\big)^0, \\ \|x\|_{L^2\cap W^{1,2}\cap C} \le C_3\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big), \end{gathered} \tag{2.20}$$

where C_3 is a constant and $L^2 \cap W^{1,2} \cap C = L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H)$.

(2) Let A be symmetric and let us assume that there exist $g \in H$ such that for every $\epsilon > 0$ and any $y \in D(\phi)$

$$J_\epsilon(y + \epsilon g) \in D(\phi), \qquad \phi(J_\epsilon(y + \epsilon g)) \le \phi(y). \tag{2.21}$$

Then for $h \in L^2(0,T;H)$ and $x_0 \in \overline{D(\phi)} \cap V$, (VE) *has a unique solution:*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \hookrightarrow C([0,T];H), \tag{2.22}$$

which satisfies

$$\|x\|_{L^2 \cap W^{1,2} \cap C} \le C_3\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;H)}\Big). \tag{2.23}$$

Remark 2.5. When the principal operator A is bounded from H to itself, we assume that $\phi : H \to (-\infty, +\infty]$ is a lower semicontinuous, proper convex function and $g : [0,T] \times H \to H$ be a nonlinear mapping satisfying the following:

$$\big|g(t,x_1) - g(t,x_2)\big| \le L|x_1 - x_2|, \quad \forall x_1, x_2 \in H. \tag{2.24}$$

Then it is easily seen that the result of (2) of Proposition 2.4. is immediately obtained.

Remark 2.6. Here, we remark that if V is compactly embedded in H and $x \in L^2(0,T;V))$ (or the semigroup operator $S(t)$ is compact), the following embedding:

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow L^2(0,T;H) \tag{2.25}$$

is compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $(x_0, f) \mapsto x$ is compact from $V \times L^2(0,T;V^*)$ to $L^2(0,T;H)$, which is also applicable to optimal control problem.

3. Regularity for Solutions

We start with the following assumption.

Assumption (F). Let $g : [0,T] \times V \to H$ be a nonlinear mapping satisfying the following:

$$\big|g(t,x) - g(t,y)\big| \ \le L\|x - y\|, \quad g(t,0) = 0 \ \forall x, y \in V \tag{3.1}$$

for a positive constant L.

For $x \in L^2(0,T;V)$ we set

$$f(t,x) = \int_0^t k(t-s) g(s, x(s)) ds, \tag{3.2}$$

where k belongs to $L^2(0,T)$.

Lemma 3.1. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $f(\cdot,x) \in L^2(0,T;H)$. And*

$$\|f(\cdot,x)\|_{L^2(0,T;H)} \le L\|k\|_{L^2}\sqrt{T}\|x\|_{L^2(0,T;V)}. \tag{3.3}$$

Moreover, if $x_1, x_2 \in L^2(0,T;H)$, then

$$\|f(\cdot,x_1) - f(\cdot,x_2)\|_{L^2(0,T;H)} \le L\|k\|\sqrt{T}\|x_1 - x_2\|_{L^2(0,T;V)}. \tag{3.4}$$

The proof is immediately obtained from Assumption (F).

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\left\{\frac{\|x - J_\epsilon x\|_*^2}{2\epsilon} + \phi(J_\epsilon x) : x \in H\right\}, \tag{3.5}$$

where $J_\epsilon = (I + \epsilon\partial\phi)^{-1}$. Then the function ϕ_ϵ is Frećhet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $\partial\phi_\epsilon = \epsilon^{-1}(I - (I + \epsilon\partial\phi)^{-1})$ as is seen in Corollary 2.2 in [1, Chapter II]. It is also well-known results that $\lim_{\epsilon\to 0}\phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$, where $(\partial\phi)^0$ is the minimal segment of $\partial\phi$.

Now, one introduces the smoothing system corresponding to (NDE) as follows:

$$\begin{aligned} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) &= f(t,x) + h(t), \quad 0 < t \le T, \\ x(0) &= x_0. \end{aligned} \tag{SDE 2}$$

Since $-A$ generates a semigroup $S(t)$ on H, the mild solution of (SDE 2) can be represented by

$$x_\epsilon(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s,x_\epsilon) + h(s) - \partial\phi_\epsilon(x_\epsilon(s))\}ds. \tag{3.6}$$

One will use a fixed point theorem and a step and step method to get the global solution for (NDE). Then one needs the following hypothesis.

Assumption (A). $(\partial\phi)^0$ is uniformly bounded, that is,

$$\left|(\partial\phi)^0 x\right| \le M_1, \quad x \in V. \tag{3.7}$$

Lemma 3.2. *For given ϵ, $\lambda > 0$, let x_ϵ and x_λ be the solutions of* (SDE 2) *corresponding to ϵ and λ, respectively. Then there exists a constant C independent of ϵ and λ such that*

$$\|x_\epsilon - x_\lambda\|_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T. \tag{3.8}$$

Proof. From (SDE 2) we have

$$x_\epsilon'(t) - x_\lambda'(t) + A(x_\epsilon(t) - x_\lambda(t)) + \partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)) = f(t, x_\epsilon) - f(t, x_\lambda), \tag{3.9}$$

and hence, from (2.3) and multiplying by $x_\epsilon(t) - x_\lambda(t)$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_\epsilon(t) - x_\lambda(t)|^2 + \omega_1\|x_\epsilon(t) - x_\lambda(t)\|^2 + \big(\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)\big)\\ &\quad\le \big(f(t, x_\epsilon) - f(t, x_\lambda), x_\epsilon(t) - x_\lambda(t)\big) + \omega_2|x_\epsilon(t) - x_\lambda(t)|^2.\end{aligned} \tag{3.10}$$

Here, we note

$$\begin{aligned}&\big|f(t, x_\epsilon) - f(t, x_\lambda)\big| \le L\|k\|_{L^2}\|x_\epsilon(\cdot) - x_\lambda(\cdot)\|_{L^2(0,t;V)}\\ &\int_0^T \|x_\epsilon(\cdot) - x_\lambda(\cdot)\|^2_{L^2(0,t;V)}dt = T\int_0^T \|x_\epsilon(t) - x_\lambda(t)\|^2 dt.\end{aligned} \tag{3.11}$$

Thus, we have

$$\begin{aligned}&\big(f(t, x_\epsilon) - f(t, x_\lambda), x_\epsilon(t) - x_\lambda(t)\big)\\ &\quad\le \big|f(t, x_\epsilon) - f(t, x_\lambda)\big| \cdot |x_\epsilon(t) - x_\lambda(t)|\\ &\quad\le \frac{\omega_1}{2T(L\|k\|_{L^2})^2}\big|f(t, x_\epsilon) - f(t, x_\lambda)\big|^2 + \frac{T(L\|k\|_{L^2})^2}{2\omega_1}|x_\epsilon(t) - x_\lambda(t)|^2\\ &\quad\le \frac{\omega_1}{2T}\|x_\epsilon(\cdot) - x_\lambda(\cdot)\|^2_{L^2(0,t;H)} + \frac{T(L\|k\|_{L^2})^2}{2\omega_1}|x_\epsilon(t) - x_\lambda(t)|^2.\end{aligned} \tag{3.12}$$

Therefore, by using the monotonicity of $\partial\phi$ and integrating (3.10) over $[0, T]$ it holds

$$\begin{aligned}&\frac{1}{2}|x_\epsilon(t) - x_\lambda(t)|^2 + \frac{\omega_1}{2}\int_0^T \|x_\epsilon(t) - x_\lambda(t)\|^2 dt\\ &\quad\le \int_0^T \big(\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), \lambda\partial\phi_\lambda(x_\lambda(t)) - \epsilon\partial\phi_\epsilon(x_\epsilon(t))\big)dt\\ &\qquad+ \left\{\frac{T(L\|k\|_{L^2})^2}{2\omega_1} + \omega_2\right\}\int_0^T |x_\epsilon(t) - x_\lambda(t)|^2 dt.\end{aligned} \tag{3.13}$$

Here, we used that

$$\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}\Big(x_\epsilon(t) - \big(I + \epsilon\partial\phi\big)^{-1}x_\epsilon(t)\Big). \tag{3.14}$$

Since $|\partial\phi_\epsilon(x)| \le |(\partial\phi)^0 x|$ for every $x \in D(\partial\phi)$ it follows from Assumption (A) and using Gronwall's inequality that

$$\|x_\epsilon - x_\lambda\|_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T. \tag{3.15}$$

□

Let $x \in L^1(0,T;V)$. Then it is well known that

$$\lim_{h\to 0} h^{-1}\int_0^h \|x(t+s) - x(t)\| ds = 0 \tag{3.16}$$

for almost all point of $t \in (0,T)$.

Definition 3.3. The point t which permits (3.16) to hold is called the Lebesgue point of x.

We establish the following results on the solvability of (NDE).

Theorem 3.4. *Let Assumptions (F) and (A) be satisfied. Then for every $(x_0, h) \in V \times L^2(0,T;V^*)$,* (NDE) *has a unique solution:*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H), \tag{3.17}$$

and there exists a constant C_4 depending on T such that

$$\|x\|_{L^2\cap W^{1,2}\cap C} \le C_4\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big). \tag{3.18}$$

Proof. Let us fix $T_0 > 0$ such that

$$C_1C_2\Big(\epsilon^{-1} + \sqrt{T_0}\ L\|k\|_{L^2}\Big)\left(\frac{T_0}{\sqrt{2}}\right)^{1/2} < 1. \tag{3.19}$$

Let $y \in L^2(0,T_0;V)$. Then $f(\cdot, y(\cdot)) \in L^2(0,T_0;H)$ from Assumption (F). Set

$$(Fx)(t) = f(t, x(t)) - \partial\phi_\epsilon(x(t)), \quad 0 \le t \le T_0. \tag{3.20}$$

Then from Lemma 3.1 it follows that

$$|(Fx_1)(t) - (Fx_2)(t)| \le \Big(\epsilon^{-1} + \sqrt{T_0}L\|k\|_{L^2}\Big)\|x_1(t) - x_2(t)\|. \tag{3.21}$$

For $i = 1, 2$, we consider the following equation:

$$\begin{gathered} x_i'(t) + Ax_i(t) = \big(Fy_i\big)(t) + h(t), \quad 0 < t \le T_0, \\ x_i(0) = x_0. \end{gathered} \tag{3.22}$$

Then

$$\begin{gathered}\frac{d}{dt}(x_1(t)-x_2(t))+A(x_1(t)-x_2(t))=(Fy_1)(t)-(Fy_2)(t), \quad t>0,\\ x_1(0)-x_2(0)=0.\end{gathered} \tag{3.23}$$

From (2.11) it follows that

$$\|x_1-x_2\|_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \le C_2\|Fy_1-Fy_2\|_{L^2(0,T_0;H)}. \tag{3.24}$$

Using the Hölder inequality we also obtain that

$$\begin{aligned}\|x_1-x_2\|_{L^2(0,T_0;H)} &= \left\{\int_0^{T_0}|x_1(t)-x_2(t)|^2dt\right\}^{1/2}\\ &= \left\{\int_0^{T_0}\left|\int_0^t(\dot{x}_1(\tau)-\dot{x}_2(\tau))d\tau\right|^2dt\right\}^{1/2}\\ &\le \left\{\int_0^{T_0}t\int_0^t|\dot{x}_1(\tau)-\dot{x}_2(\tau)|^2d\tau dt\right\}^{1/2}\\ &\le \frac{\sqrt{T_0}}{2}\|x_1-x_2\|_{W^{1,2}(0,T_0;H)}.\end{aligned} \tag{3.25}$$

Therefore, in terms of (2.8) and (3.25) we have

$$\begin{aligned}&\|x_1-x_2\|_{L^2(0,T_0;V)} \le C_1\|x_1-x_2\|^{1/2}_{L^2(0,T_0;D(A_0))}\|x_1-x_2\|^{1/2}_{L^2(0,T_0;H)}\\ &\le C_1\|x_1-x_2\|^{1/2}_{L^2(0,T_0;D(A_0))}\left(\frac{T_0}{\sqrt{2}}\right)^{1/2}\|x_1-x_2\|^{1/2}_{W^{1,2}(0,T_0;H)}\\ &\le C_1\left(\frac{T_0}{\sqrt{2}}\right)^{1/2}\|x_1-x_2\|_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)}\\ &\le C_1C_2\left(\frac{T_0}{\sqrt{2}}\right)^{1/2}\|Fy_1-Fy_2\|_{L^2(0,T_0:H)}\\ &\le C_1C_2\left(\epsilon^{-1}+\sqrt{T_0}L\|k\|_{L^2}\right)\left(\frac{T_0}{\sqrt{2}}\right)^{1/2}\|y_1-y_2\|_{L^2(0,T_0;V)}.\end{aligned} \tag{3.26}$$

So by virtue of the condition (3.19) the contraction principle gives that (SDE 2) has a unique solution in $[0,T_0]$. Thus, letting $\lambda \to 0$ in Lemma 3.1 we can see that there exists a constant C independent of ϵ such that

$$\|x_\epsilon-x\|_{C([0,T_0];H)\cap L^2(0,T_0;V)} \le C\epsilon, \quad 0<T_0, \tag{3.27}$$

and hence, $\lim_{\epsilon\to 0} x_\epsilon(t) = x(t)$ exists in H. From Assumption (F) and (3.27) it follows that

$$\begin{aligned} f(\cdot, x_\epsilon) &\longrightarrow f(\cdot, x), \quad \text{strongly in } L^2(0, T_0; H), \\ Ax_n &\longrightarrow Ax, \quad \text{strongly in } L^2(0, T_0; V^*). \end{aligned} \tag{3.28}$$

Since $\partial\phi_\epsilon(x_\epsilon)$ is uniformly bounded by Assumption (A), from (3.27), (3.28) we have that

$$\frac{d}{dt}x_\epsilon \longrightarrow \frac{d}{dt}x, \quad \text{weakly in } L^2(0, T_0; V^*), \tag{3.29}$$

therefore

$$\partial\phi_\epsilon(x_\epsilon) \longrightarrow f(\cdot, x) + h - x' - Ax, \quad \text{weakly in } L^2(0, T_0; V^*). \tag{3.30}$$

Since $(I + \epsilon\partial\phi)^{-1}x_\epsilon \to x$ strongly and $\partial\phi$ is demiclosed, we have that

$$f(\cdot, x) + h - x' - Ax \in \partial\phi(x) \quad \text{in } L^2(0, T_0; V^*). \tag{3.31}$$

Thus we have proved that $x(t)$ satisfies a.e. on $(0, T_0)$ the equation (NDE).

Let y be the solution of

$$\begin{aligned} y'(t) + Ay(t) + \partial\phi(y(t)) &\ni 0, \quad 0 < t \le T_0, \\ y(0) &= x_0, \end{aligned} \tag{3.32}$$

then, it implies

$$\frac{d}{dt}(x(t) - y(t)) + A(x(t) - y(t)) + \partial\phi(x(t)) - \partial\phi(y(t)) \ni f(t, x) + h(t). \tag{3.33}$$

Noting that $\|\cdot\| \le |\cdot| \le \|\cdot\|$, by multiplying by $x(t) - y(t)$ and using the monotonicity of $\partial\phi$ and (2.3), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + \omega_1\|x(t) - y(t)\|^2 \\ &\quad \le \omega_2|x(t) - y(t)|^2 + |f(t, x) + h(t)| \cdot \|x(t) - y(t)\|. \end{aligned} \tag{3.34}$$

Since

$$|f(t, x) + h(t)| \cdot \|x(t) - y(t)\| \le \frac{1}{2\omega_1}|f(t, x) + h(t))|^2 + \frac{\omega_1}{2}\|x(t) - y(t)\|^2 \tag{3.35}$$

for every $c > 0$ and by integrating on (3.34) over $(0,t)$ we have

$$\begin{aligned}&|x(t)-y(t)|^2+\omega_1\int_0^t\|x(s)-y(s)\|^2ds\\&\quad\le\frac{1}{\omega_1}\|f(\cdot,x)+h\|_{L^2(0,T_0;V^*)}+2\omega_2\int_0^t|x(s)-y(s)|^2ds\end{aligned}\tag{3.36}$$

and by Gronwall's inequality:

$$|x(t)-y(t)|^2+\omega_1\int_0^t\|x(s)-y(s)\|^2ds\le\omega_1^{-1}e^{2\omega_2T_0}\|f(\cdot,x)+h\|^2_{L^2(0,T_0;V^*)}.\tag{3.37}$$

Let us fix $T_0 > T_1 > 0$ so that T_1 is a Lebesgue point of x, $\phi(x(T_1)) < \infty$, and

$$\omega_1^{-1}e^{2\omega_2T_1}\sqrt{T_1}L\|k\|_{L^2}<\omega_1.\tag{3.38}$$

Put

$$N=\sqrt{\omega_1^{-2}e^{\omega_2T_1}},\tag{3.39}$$

then from Assumption (F) it follows

$$\begin{aligned}\|x-y\|_{L^2(0,T_1;V)}&\le N\|f(\cdot,x)+h\|_{L^2(0,T_1;V^*)}\\&\le N\sqrt{T_1}\,L\|k\|_{L^2}\|x\|_{L^2(0,T_1;V)}+N\|h\|_{L^2(0,T_1:V^*)}\end{aligned}\tag{3.40}$$

and hence, from (2.17) in Proposition 2.3, we have that

$$\begin{aligned}&\|x\|_{L^2(0,T_1;V)}\\&\quad\le\frac{1}{1-N\sqrt{T_1}L\|k\|_{L^2}}\left(\|y\|_{L^2(0,T_1;V)}+N\|h\|_{L^2(0,T_1:V^*)}\right)\\&\quad\le\frac{1}{1-N\sqrt{T_1}L\|k\|_{L^2}}\left\{C_2(1+\|x_0\|)+N\|h\|_{L^2(0,T_1:V^*)}\right\}\\&\quad\le C_4\left(1+\|x_0\|+\|h\|_{L^2(0,T_1:V^*)}\right)\end{aligned}\tag{3.41}$$

for some positive constant C_4. Since the condition (3.38) is independent of initial values, noting the Assumption (A), the solution of (NDE) can be extended to the internal $[0, nT_1]$ for natural number n, that is, for the initial $x(nT_1)$ in the interval $[nT_1, (n+1)T_1]$, as analogous estimate (3.41) holds for the solution in $[0,(n+1)T_1]$. The norm estimate of x in $W^{1,2}(0,T;H)$ can be obtained by acting on both side of (NDE) by $x'(t)$ and by using

$$\frac{d}{dt}\phi(x(t)) = \left(g(t), \frac{d}{dt}x(t)\right), \quad \text{a.e., } 0 < t, \tag{3.42}$$

for all $g(t) \in \partial\phi(x(t))$. Furthermore, the estimate (3.18) is immediately obtained from (3.41). □

Theorem 3.5. *Let Assumptions (F) and (A) be satisfied and $(x_0, h) \in V \times L^2(0,T;V^*)$, then the solution x of* (NDE) *belongs to $x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and the mapping:*

$$V \times L^2(0,T;V^*) \ni (x_0, h) \longmapsto x \in L^2(0,T;V) \cap C([0,T];H) \tag{3.43}$$

is continuous.

Proof. If $(x_0, h) \in V \times L^2(0,T;V^*)$ then x belongs to $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ form Theorem 3.4. Let $(x_{0i}, h_i) \in V \times L^2(0,T;V^*)$ and x_i be the solution of (NDE) with (x_{0i}, h_i) in place of (x_0, h) for $i = 1, 2$. Multiplying on (NDE) by $x_1(t) - x_2(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1\|x_1(t) - x_2(t)\|^2 \\ &\quad \le \omega_2|x_1(t) - x_2(t)|^2 + \left|f(t,x_1) - f(t,x_2)\right|\|x_1(t) - x_2(t)\| \\ &\qquad + \|h_1(t) - h_2(t)\|_*\|x_1(t) - x_2(t)\|. \end{aligned} \tag{3.44}$$

Let us fix $T_1 > T_2 > 0$ so that T_2 is a Lebesgue point of x, $\phi(x(T_2) < \infty$, and

$$\omega_1 - \omega_1^{-1}e^{2\omega_2 T_2}\sqrt{T_2}\, L\|K\|_{L^2} > 0. \tag{3.45}$$

Since

$$\|h_1(t) - h_2(t))\|_*\|x_1(t) - x_2(t)\| \le \frac{1}{\omega_1}\|h_1(t) - h_2(t)\|_*^2 + \frac{\omega_1}{4}\|x_1(t) - x_2(t)\|^2, \tag{3.46}$$

by integrating on (3.44) over $[0,T_2]$ where $T_2 < T$ and as is seen in (3.37), it follows

$$\begin{aligned} &\|x_1 - x_2\|^2_{C([0,T_2];H)} + \frac{\omega_1}{2}\|x_1 - x_2\|^2_{L^2(0,T_2;V)} \\ &\quad \le \|x_{01} - x_{02}\|^2 + \frac{1}{\omega_1}\left\|f(t,x_1) - f(t,x_2)\right\|^2_{L^2(0,T_2;H)} + \frac{2}{\omega_1}\|h_1 - h_2\|_{L^2(0,T_2;V^*)} \\ &\quad \le \|x_{01} - x_{02}\|^2 + \omega_1^{-1}\sqrt{T_2}\, L\|K\|_{L^2}\|x_1 - x_2\|^2_{L^2(0,T_2;V)} + \frac{2}{\omega_1}\|h_1 - h_2\|_{L^2(0,T_2;V^*)}. \end{aligned} \tag{3.47}$$

Putting that

$$N_1 \equiv \min \left[1, \left\{\frac{\omega_1}{2} - \omega_1^{-1}\sqrt{T_2}L\|K\|_{L^2}\right\}\right]^{1/2}, \qquad N_2 \equiv \max\left\{1, \frac{2}{\omega_1}\right\}, \tag{3.48}$$

we have

$$\|x_1 - x_2\|_{L^2 \cap C} \leq \frac{2N_2}{\sqrt{1 - N_1}}(\|x_{01} - x_{02}\| + \|h_1 - h_2\|). \tag{3.49}$$

Suppose $(x_{0n}, h_n) \rightarrow (x_0, h)$ in $V \times L^2(0, T; V^*)$, and let x_n and x be the solutions (SDE 2) with (x_{0n}, h_n) and (x_0, h), respectively. Then, by virtue of (3.44) and (3.49), we see that $x_n \rightarrow x$ in $L^2(0, T_2, V) \cap W^{1,2}(0, T_2, V^*) \hookrightarrow C([0, T_2]; H)$. This implies that $x_n(T_2) \rightarrow x(T_2)$ in H. Therefore the same argument shows that $x_n \rightarrow x$ in

$$L^2(T_2, \min\{2T_2, T\}; V) \cap C([T_2, \min\{2T_2, T\}]; H). \tag{3.50}$$

Repeating this process, we conclude that $x_n \rightarrow x$ in $L^2(0, T; V) \cap W^{1,2}(0, T_2, V^*) \hookrightarrow C([0, T_2]; H)$. □

4. Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We define the following spaces:

$$\begin{aligned} H^1(\Omega) &= \left\{u : u, \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, \ldots, n\right\}, \\ H^2(\Omega) &= \left\{u : u, \frac{\partial x}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega),\ i, j = 1, 2, \ldots, n\right\}, \\ H_0^1(\Omega) &= \left\{u : u \in H^1(\Omega), u|_{\partial\Omega} = 0\right\} = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1(\Omega), \end{aligned} \tag{4.1}$$

where $\partial/\partial x_i u$ and $\partial^2/\partial x_i \partial x_j u$ are the derivative of u in the distribution sense. The norm of $H_0^1(\Omega)$ is defined by

$$\|u\| = \left\{\int_\Omega \sum_{i=1}^n \left(\frac{\partial u(x)}{\partial x_i}\right)^2 dx\right\}^{1/2}. \tag{4.2}$$

Hence $H_0^1(\Omega)$ is a Hilbert space. Let $H^{-1}(\Omega) = H_0^1(\Omega)^*$ be a dual space of $H_0^1(\Omega)$. For any $l \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$, the notation (l, v) denotes the value l at v. In what follows, we consider the regularity for given equations in the spaces:

$$V = H_0^1(\Omega) = \left\{u \in H^1(\Omega);\ u = 0 \text{ on } \partial\Omega\right\}, \quad H = L^2(\Omega), V^* = H^{-1}(\Omega) \tag{4.3}$$

as introduced in Section 2. We deal with the Dirichlet condition's case as follows.

Assume that $a_{ij} = a_{ji}$ are continuous and bounded on $\overline{\Omega}$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, that is, there exists a positive number δ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2, \quad \forall \xi \in \overline{\Omega}. \tag{4.4}$$

Let

$$b_i \in L^{\infty}(\Omega), \qquad c \in L^{\infty}(\Omega), \qquad \beta_i = \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} + b_i. \tag{4.5}$$

For each $u, v \in H_0^1(\Omega)$, let us consider the following sesquilinear form:

$$a(u,v) = \int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\overline{\partial v}}{\partial x_j} + \sum_{j=1}^{n} \beta_i \frac{\partial u}{\partial x_i} \overline{v} + cu\overline{v} \right\} dx. \tag{4.6}$$

Since $\{a_{ij}\}$ is real symmetric, by (4.4) the inequality:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi}_j \geq \delta|\xi|^2 \tag{4.7}$$

holds for all complex vectors $\xi = (\xi_1, \ldots, \xi_n)$. By hypothesis, there exists a constant K such that $|\beta_i(x)| \leq K$ and $c(x) \leq K$ hold a.e., hence

$$\begin{aligned} \text{Re }\ a(u,u) &\geq \int_{\Omega} \delta \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 dx - K \int_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right| |u| dx - K \int_{\Omega} |u|^2 dx \\ &\geq \delta \int_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 dx - K \int_{\Omega} \sum_{i=1}^{n} \left(\frac{\epsilon}{2} \left| \frac{\partial u}{\partial x_i} \right|^2 + \frac{1}{2\epsilon} |u|^2 \right) dx - K \int_{\Omega} |u|^2 dx \\ &= \left(\delta - \frac{\epsilon}{2} K \right) \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx - \left(\frac{nK}{2\epsilon} + K \right) \int_{\Omega} |u|^2 dx. \end{aligned} \tag{4.8}$$

By choosing $\epsilon = \delta K^{-1}$, we have

$$\begin{aligned} \text{Re }\ a(u,u) &\geq \frac{\delta}{2} \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx - \left(\frac{nK^2}{2\delta} + K \right) \int_{\Omega} |u|^2 dx \\ &= \frac{\delta}{2} \|u\|_1^2 - \left(\frac{nK^2}{2\delta} + K + \frac{\delta}{2} \right) \|u\|^2. \end{aligned} \tag{4.9}$$

By virtue of Lax-Milgram theorem, we know that for any $v \in V$ there exists $f \in V^*$ such that

$$a(u,v) = (f,v). \tag{4.10}$$

Therefore, we know that the associated operator $A : V \to V^*$ defined by

$$(Au,v) = -a(u,v), \quad u,v \in V \tag{4.11}$$

is bounded and satisfies conditions (2.3) in Section 2.

Let $g : [0,T] \times V \to H$ be a nonlinear mapping defined by

$$g(t,u(t,x)) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s,\nabla u(s,x))ds. \tag{4.12}$$

We assume the following.

Assumption (F1). The partial derivatives $\sigma_i(s,\xi)$, $\partial/\partial t\, \sigma_i(s,\xi)$ and $\partial/\partial \xi_j \sigma_i(s,\xi)$, exist and continuous for $i = 1,2$, $j = 1,2,\ldots,n$, and $\sigma_i(s,\xi)$ satisfies an uniform Lipschitz condition with respect to ξ, that is, there exists a constant $L > 0$ such that

$$\left|\sigma_i(s,\xi) - \sigma_i\left(s,\widehat{\xi}\right)\right| \le L\left|\xi - \widehat{\xi}\right|, \tag{4.13}$$

where $|\cdot|$ denotes the norm of $L^2(\Omega)$.

Lemma 4.1. *If Assumption (F1) is satisfied, then the mapping $t \mapsto g(t,\cdot)$ is continuously differentiable on $[0,T]$ and $u \mapsto g(\cdot,u)$ is Lipschitz continuous on V.*

Proof. Put

$$g_1(s,u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s,\nabla u), \tag{4.14}$$

then we have $g_1(s,u) \in H^{-1}(\Omega)$. For each $w \in H_0^1(\Omega)$, we satisfy the following that

$$(g_1(s,u),w) = -\sum_{i=1}^n \left(\sigma_i(s,\nabla u), \frac{\partial}{\partial x_i}w\right). \tag{4.15}$$

The nonlinear term is given by

$$g(t,u) = \int_0^t g_1(s,u)ds. \tag{4.16}$$

For any $w \in H_0^1(\Omega)$, if u and $\hat{u}$ belong to $H_0^1(\Omega)$, by Assumption (F1) we obtain

$$\left|\left(g(t,u) - g(t,\hat{u})\right), w\right| \le LT\|u - \hat{u}\|\|w\|. \tag{4.17}$$

□

We set

$$f(t,u) = \int_0^t k(t-s) \int_0^s \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(\tau, \nabla u(\tau,x)) d\tau ds, \tag{4.18}$$

where k belongs to $L^2(0,T)$. Let $\phi : H_0^1(\Omega) \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Now in virtue of Lemma 4.1, we can apply the results of Theorem 3.4 as follows.

Theorem 4.2. *Let Assumption (F1) be satisfied. Then for any $u_0 \in H_0^1(\Omega)$ and $h \in L^2(0,T;H^{-1}(\Omega))$, the following nonlinear problem:*

$$\begin{gathered}
\left(u'(t) + Au(t), u(t) - z\right) + \phi(u(t)) - \phi(z) \\
\le \left(f(t,u) + h(t), u(t) - z\right), \quad a.e., \ 0 < t \le T, \ z \in L^2(\Omega), \\
u(0) = u_0
\end{gathered} \tag{4.19}$$

has a unique solution:

$$u \in L^2\left(0,T; H_0^1(\Omega)\right) \cap W^{1,2}\left(0,T; H^{-1}(\Omega)\right) \hookrightarrow C\left([0,T]; L^2(\Omega)\right). \tag{4.20}$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$\|u\|_{L^2 \cap W^{1,2}} \le C_T\left(1 + \|u_0\| + \|h\|_{L^2(0,T;H^{-1}(\Omega))}\right). \tag{4.21}$$

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