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Research Article

Strong Convergence Theorems for a Finite Family of λ_i-Strict Pseudocontractions in 2-Uniformly Smooth Banach Spaces by Metric Projections

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A new hybrid projection algorithm is considered for a finite family of λ_i-strict pseudocontractions. Using the metric projection, some strong convergence theorems of common elements are obtained in a uniformly convex and 2-uniformly smooth Banach space. The results presented in this paper improve and extend the corresponding results of Matsushita and Takahshi, 2008, Kang and Wang, 2011, and many others.

1. Introduction

Let E be a real Banach space and let E^* be the dual spaces of E. Assume that J is the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing between E and E^*.

Let C be a closed convex subset of a real Banach space E. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \tag{1.2}$$

for all $x, y \in C$. Also a mapping $T : C \to C$ is called a λ-strict pseudocontraction if there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in C$ and for some $j(x-y) \in J(x-y)$, the following holds:

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2. \tag{1.3}$$

From (1.3) we can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant $L = (1+\lambda)/\lambda$.

It is well-known that the classes of nonexpansive mappings and pseudocontractions are two kinds important nonlinear mappings, which have been studied extensively by many authors (see [1–8]).

In [9] Reich considered the Mann iterative scheme $\{x_n\}$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad x_1 \in C \tag{1.4}$$

for nonexpansive mappings, where $\{\alpha_n\}$ is a sequence in $(0,1)$. Under suitable conditions, the author proved that $\{x_n\}$ converges weakly to a fixed point of T. In 2005, Kim and Xu [10] proved a strong convergence theorem for nonexpansive mappings by modified Mann iteration. In 2008, Zhou [11] extended and improved the main results of Kim and Xu to the more broad 2-uniformly smooth Banach spaces for λ-strict pseudocontractive mappings.

On the other hand, by using metric projection, Nakajo and Takahashi [12] introduced the following iterative algorithms for the nonexpansive mapping T in the framework of Hilbert spaces:

$$\begin{gathered} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{1.5}$$

where $\{\alpha_n\} \subset [0,\alpha]$, $\alpha \in [0,1)$, and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T.

In 2006, Xu [13] extended Nakajo and Takahashi's theorem to Banach spaces by using the generalized projection.

In 2008, Matsushita and Takahashi [14] presented the following iterative algorithms for the nonexpansive mapping T in the framework of Banach spaces:

$$\begin{gathered} x_0 = x \in C, \\ C_n = \overline{\mathrm{co}}\{z \in C : \|z - Tz\| \leq t_n\|x_n - T x_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{1.6}$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, J is normalized duality mapping, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, they proved that $\{x_n\}$ generated by (1.6) converges strongly to a fixed point of nonexpansive mapping T.

Recently, Kang and Wang [15] introduced the following hybrid projection algorithm for a pair of nonexpansive mapping T in the framework of Banach spaces:

$$\begin{gathered}
x_0 = x \in C, \\
y_n = \alpha_n T_1 x_n + (1-\alpha_n) T_2 x_n, \\
C_n = \overline{\text{co}}\{z \in C : \|z - T_1 z\| + \|z - T_2 z\| \le t_n \|x_n - y_n\|\}, \\
x_{n+1} = P_{C_n} x, \quad n = 0, 1, 2, \ldots,
\end{gathered} \tag{1.7}$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and P_{C_n} is the metric projection from E onto C_n. Then, they proved that $\{x_n\}$ generated by (1.7) converges strongly to a fixed point of two nonexpansive mappings T_1 and T_2.

In this paper, motivated by the research work going on in this direction, we introduce the following iterative for finding fixed points of a finite family of λ_i-strict pseudocontractions in a uniformly convex and 2-uniformly smooth Banach space:

$$\begin{gathered}
x_0 = x \in C, \\
y_n = \sum_{i=1}^{N} \alpha_{n,i} T_i x_n, \\
C_n = \overline{\text{co}}\left\{z \in C : \sum_{i=1}^{N} \|z - T_i z\| \le t_n \|x_n - y_n\|\right\}, \\
x_{n+1} = P_{C_n} x, \quad n = 1, 2, \ldots,
\end{gathered} \tag{1.8}$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, $\{\alpha_{n,i}\}$ is N sequences in [0,1] and $\sum_{i=1}^{N} \alpha_{n,i} = 1$ for each $n \ge 0$, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and P_{C_n} is the metric projection from E onto C_n. we prove defined by (1.8) converges strongly to a common fixed point of a finite family of λ_i-strictly pseudocontractions, the main results of Kang and Wang is extended and improved to strictly pseudocontractions.

2. Preliminaries

In this section, we recall the well-known concepts and results which will be needed to prove our main results. Throughout this paper, we assume that E is a real Banach space and C is a nonempty subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. We also assume that E^* is the dual space of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping. Some properties of duality mapping have been given in [16].

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}. \tag{2.1}$$

E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. The modulus of smoothness of E is defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}. \tag{2.3}$$

A Banach space E is said to be *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. A Banach space E is said to be *q-uniformly smooth*, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$.

If E is a reflexive, strictly convex, and smooth Banach space, then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y \in C} \|y - x\|. \tag{2.4}$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. Let $x \in E$ and $u \in C$. Then it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u)\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

For the details on the metric projection, refer to [17–20].

In the sequel, we make use the following lemmas for our main results.

Lemma 2.1 (see [21]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2 \tag{2.6}$$

for any $x, y \in E$.

Lemma 2.2 (see [11]). *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with the best smooth constant $K > 0$ and let $T : C \to C$ be a λ-strict pseudocontraction. For $\alpha \in (0, 1) \cap (0, \lambda/K^2]$, we define $T_\alpha x = (1 - \alpha)x + \alpha Tx$. Then $T_\alpha : C \to E$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.3 (demiclosed principle, see [22]). *Let E be a real uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 (see [23]). *Let C be a closed convex subset of a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\| T\left(\sum_{j=0}^{m} \mu_j z_j\right) - \sum_{j=0}^{m} \mu_j T z_j \right\|\right) \leq \max_{0 \leq j < k \leq m} \left(\|z_j - z_k\| - \|Tz_j - Tz_k\|\right), \tag{2.7}$$

for all $m \geq 1$, $\{\mu_j\}_{j=0}^{m} \in \Delta^m$, $\{z_j\}_{j=0}^{m} \subset C \cap B_r$, and $T \in Lip(C, 1)$, where $\Delta^m = \{\{\mu_0, \mu_1, \ldots, \mu_m\} : 0 \leq \mu_j\ (0 \leq j \leq m)$ and $\sum_{j=0}^{m} \mu_j = 1\}$, $B_r = \{x \in E : \|x\| \leq r\}$, and $Lip(C, 1)$ is the set of all nonexpansive mappings from C into E.

3. Main Results

Now we are ready to give our main results in this paper.

Lemma 3.1. *Let C be a closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K > 0$, and $T : C \to C$ be a λ-strict pseudocontraction. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\alpha\left\| T\left(\sum_{j=0}^{m} \mu_j z_j\right) - \sum_{j=0}^{m} \mu_j T z_j \right\|\right) \leq \alpha \max_{0 \leq j < k \leq m} \left(\|z_j - Tz_j\| + \|z_k - Tz_k\|\right), \tag{3.1}$$

for all $m \geq 1$, $\{\mu_j\}_{j=0}^{m} \in \Delta^m$, $\{z_j\}_{j=0}^{m} \subset C \cap B_r$, where $\alpha \in (0, 1) \cap (0, \lambda/K^2]$, $\Delta^m = \{\{\mu_0, \mu_1, \ldots, \mu_m\} : 0 \leq \mu_j\ (0 \leq j \leq m)$ and $\sum_{j=0}^{m} \mu_j = 1\}$, $B_r = \{x \in E : \|x\| \leq r\}$.

Proof. Define the mapping $T_\alpha : C \to C$ as $T_\alpha x = (1 - \alpha)x + \alpha T x$, for all $x \in C$. Then T_α is nonexpansive. From Lemma 2.4, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ and such that

$$\gamma\left(\left\| T_\alpha\left(\sum_{j=0}^{m} \mu_j z_j\right) - \sum_{j=0}^{m} \mu_j T_\alpha z_j \right\|\right) \leq \max_{0 \leq j < k \leq m} \left(\|z_j - z_k\| - \|T_\alpha z_j - T_\alpha z_k\|\right). \tag{3.2}$$

Hence

$$
\begin{aligned}
\gamma\left(\alpha\left\|T\left(\sum_{j=0}^{m}\mu_j z_j\right)-\sum_{j=0}^{m}\mu_j T z_j\right\|\right) &= \gamma\left(\left\|T_\alpha\left(\sum_{j=0}^{m}\mu_j z_j\right)-\sum_{j=0}^{m}\mu_j T_\alpha z_j\right\|\right)\\
&\le \max_{0\le j<k\le m}\left(\|z_j-z_k\|-\|T_\alpha z_j-T_\alpha z_k\|\right)\\
&\le \max_{0\le j<k\le m}\left(\|z_j-T_\alpha z_j\|+\|z_k-T_\alpha z_k\|\right)\\
&= \alpha\max_{0\le j<k\le m}\left(\|z_j-Tz_j\|+\|z_k-Tz_k\|\right).
\end{aligned}
\tag{3.3}
$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K>0$, assume that for each i $(i=1,2,\ldots,N)$, $T_i : C \to C$ is a λ_i-strict pseudocontraction for some $0<\lambda_i<1$ such that $\mathcal{F}=\cap_{i=1}^{N}\mathcal{F}(T_i)\neq\emptyset$. Let $\{\alpha_{n,i}\}$ be N sequences in [0,1] with $\sum_{i=1}^{N}\alpha_{n,i}=1$ for each $n\ge 0$ and $\{t_n\}$ be a sequence in (0,1) with $t_n \to 0$. Let $\{x_n\}$ be a sequence generated by* (1.8)*, where $\overline{\mathrm{co}}\{z\in C : \sum_{i=1}^{N}\|z-T_i z\|\le t_n\|x_n-y_n\|\}$ denotes the convex closure of the set $\{z\in C : \sum_{i=1}^{N}\|z-T_i z\|\le t_n\|x_n-y_n\|\}$ and P_{C_n} is the metric projection from E onto C_n. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x$.*

Proof. (I) First we prove that $\{x_n\}$ is well defined and bounded.

It is easy to check that C_n is closed and convex and $\mathcal{F}\subset C_n$ for all $n\ge 0$. Therefore $\{x_n\}$ is well defined.

Put $p=P_{\mathcal{F}}x$. Since $\mathcal{F}\subset C_n$ and $x_{n+1}=P_{C_n}x$, we have that

$$
\|x_{n+1}-x\|\le\|p-x\|
\tag{3.4}
$$

for all $n\ge 0$. Hence $\{x_n\}$ is bounded.

(II) Now we prove that $\|x_n-T_i x_n\|\to 0$ as $n\to\infty$ for all $i\in\{1,2,\ldots,N\}$.

Since $x_{n+1}\in C_n$, there exist some positive integer $m\in\mathbb{N}$ ($\mathbb{N}$ denotes the set of all positive integers), $\{\mu_i\}\in\Delta^m$ and $\{z_i\}_{i=0}^{m}\subset C$ such that

$$
\left\|x_{n+1}-\sum_{j=0}^{m}\mu_j z_j\right\|<t_n,
\tag{3.5}
$$

$$
\sum_{i=1}^{N}\|z_j-T_i z_j\|\le t_n\|x_n-y_n\|
\tag{3.6}
$$

for all $j \in \{0,1,\dots,m\}$. Put $r_0 = \sup_{n\geq 1}\|x_n - p\|$ and $\lambda = \min_{1\leq i\leq N}\{\lambda_i\}$. Take $\alpha \in (0,1) \cap (0,\lambda/K^2]$. It follows from Lemma 2.2 and (3.5) that

$$\|x_n - T_i x_n\| = \frac{1}{\alpha}\left\|(T_{i\alpha}x_n - p) + (p - x_n)\right\| \leq \frac{2r_0}{\alpha}, \tag{3.7}$$

$$\begin{aligned}
\left\|T_i\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_i x_{n+1}\right\| &\leq \frac{1}{\alpha}\left(\left\|T_{i\alpha}\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_{i\alpha}x_{n+1}\right\| + (1-\alpha)\left\|\sum_{j=0}^{m}\mu_j z_j - x_{n+1}\right\|\right)\\
&\leq \left(\frac{2}{\alpha} - 1\right)\left\|\sum_{j=0}^{m}\mu_j z_j - x_{n+1}\right\|\\
&\leq \left(\frac{2}{\alpha} - 1\right)t_n
\end{aligned} \tag{3.8}$$

for all $i \in \{1,2,\dots,N\}$. Moreover, (3.7) implies

$$\|x_n - y_n\| \leq \frac{2r_0}{\alpha}. \tag{3.9}$$

It follows from Lemma 3.1, (3.5)–(3.9) that

$$\begin{aligned}
\sum_{i=1}^{N}\|x_{n+1} - T_i x_{n+1}\| &\leq \sum_{i=1}^{N}\left(\left\|x_{n+1} - \sum_{j=0}^{m}\mu_j z_j\right\| + \left\|\sum_{j=0}^{m}\mu_j (z_j - T_i z_j)\right\|\right.\\
&\qquad \left. + \left\|\sum_{j=0}^{m}\mu_j T_i z_j - T_i\left(\sum_{j=0}^{m}\mu_j z_j\right)\right\| + \left\|T_i\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_i x_{n+1}\right\|\right)\\
&\leq \frac{2N}{\alpha}\left\|x_{n+1} - \sum_{j=0}^{m}\mu_j z_j\right\| + \sum_{j=0}^{m}\mu_j\left(\sum_{i=1}^{N}\|z_j - T_i z_j\|\right)\\
&\quad + \sum_{i=1}^{N}\left\|\sum_{j=0}^{m}\mu_j T_i z_j - T_i\left(\sum_{j=0}^{m}\mu_j z_j\right)\right\|\\
&\leq \frac{2N}{\alpha}t_n + t_n\|y_n - x_n\| + \sum_{i=1}^{N}\frac{1}{\alpha}\gamma^{-1}\left(\alpha \max_{0\leq k<j\leq m}\left(\|z_k - T_i z_k\| + \|z_j - T_i z_j\|\right)\right)\\
&\leq \frac{2N + 2r_0}{\alpha}t_n + \frac{N}{\alpha}\gamma^{-1}(4r_0 t_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.10}$$

This shows that

$$\|x_n - T_i x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.11}$$

for all $i \in \{1,2,\dots,N\}$.

(III) Finally, we prove that $x_n \to p = P_{\mathcal{F}}x$.

It follows from the boundedness of $\{x_n\}$ that there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup v$ as $i \to \infty$. Since for each $i \in \{0, 1, \dots, N\}$, T_i is a λ_i-strict pseudocontraction, then T_i is demiclosed. one has $v \in \mathcal{F}$.

From the weakly lower semicontinuity of the norm and (3.4), we have

$$\begin{aligned} \|p - x\| &\leq \|v - x\| \leq \liminf_{i\to\infty} \|x_{n_i}\| - x \\ &\leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|p - x\|. \end{aligned} \tag{3.12}$$

This shows $p = v$ and hence $x_{n_i} \rightharpoonup p$ as $i \to \infty$. Therefore, we obtain $x_n \rightharpoonup p$. Further, we have that

$$\lim_{n\to\infty} \|x_n - x\| = \|p - x\|. \tag{3.13}$$

Since E is uniformly convex, we have $x_n - x \to p - x$. This shows that $x_n \to p$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K > 0$, assume that $T : C \to C$ is a λ-strict pseudocontraction for some $0 < \lambda < 1$ such that $\mathcal{F}(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= x \in C, \\ C_n &= \overline{\mathrm{co}}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}, \\ x_{n+1} &= P_{C_n} x, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{3.14}$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$. $\overline{\mathrm{co}}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}$ denotes the convex closure of the set $\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}$ and P_{C_n} is the metric projection from E onto C_n. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}(T)} x$.

Proof. Set $T_1 = T, T_k = I$ for all $2 \leq k \leq N$, and $\alpha_{n,1} = 1$, $\alpha_{n,k} = 0$ for all $2 \leq k \leq N$ in Theorem 3.2. The desired result can be obtained directly from Theorem 3.2. □

Remark 3.4. At the end of the paper, we would like to point out that concerning the convergence problem of iterative sequences for strictly pseudocontractive mappings has been considered and studied by many authors. It can be consulted the references [24–37].

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