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*Research Article*

# Multiplicity of Solutions for a Class of Fourth-Order Elliptic Problems with Asymptotically Linear Term

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We study the following fourth-order elliptic equations: $\Delta^2 u + a\Delta u = f(x,u)$, $x \in \Omega$, $u = \Delta u = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $f(x,u)$ is asymptotically linear with respect to $u$ at infinity. Using an equivalent version of Cerami's condition and the symmetric mountain pass lemma, we obtain the existence of multiple solutions for the equations.

## 1. Introduction and Main Results

In this paper, we will investigate the existence of multiple solutions to the following fourth-order elliptic boundary value problem:

$$\begin{aligned} \Delta^2 u + a\Delta u = f(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta^2$ is the biharmonic operator, $a < \lambda_1$ ($\lambda_1$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$) is a parameter. We assume that $f(x,u)$ satisfies the following hypotheses.

($f_1$) $f(x,u) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

($f_2$) $\lim_{|u| \to 0} f(x,u)/u = 0$ uniformly for $x \in \Omega$.

$(f_3)$ $\lim_{|u|\to\infty} f(x,u)/u = \ell$ uniformly for $x \in \Omega$, where $\ell \in (0,+\infty)$ is a constant, or $\ell = +\infty$, and there exists $C > 0$, $q \in [2, 2^*)$ such that

$$\left|f(x,u)\right| \leq C\left(1 + |u|^{q-1}\right), \tag{1.2}$$

where $2^* = 2N/(N-4)$.

$(f_4)$ $f(x,u)$ is odd in $u$.

$(f_5)$ $\lim_{|u|\to\infty}(f(x,u)u - 2F(x,u)) = +\infty$ uniformly for $x \in \Omega$, where $F(x,u) = \int_0^u f(x,t)dt$.

$(f_6)$ $f(x,u)/u$ is nondecreasing with respect to $u \geq 0$, for a.e. $x \in \Omega$.

Problem (1.1) is usually used to describe some phenomena appeared in different physical, engineering and other sciences. In recent years, there are many results for the fourth-order elliptic equations. In [1], Lazer and McKenna considered the fourth-order problem:

$$\begin{gathered} \Delta^2 u + a\Delta u = d\left((u+1)^+ - 1\right), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where $u^+ = \max\{u, 0\}$ and $d \in \mathbb{R}$. They pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. They also presented a mathematical model for the bridge that takes account of the fact that the coupling provided by the stays connecting the suspension cable to the deck of the road bed is fundamentally nonlinear (see [1–3]). Since then, more general nonlinear fourth-order elliptic boundary value problems have been studied. Problem (1.1) and (1.3) have been studied extensively in recent years, we refer the reader to [4–14].

For problem (1.3), Lazer and McKenna [2] proved the existence of $2k-1$ solutions when $N = 1$ and $d > \lambda_k(\lambda_k - c)$ ($\lambda_k$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) by the global bifurcation method. In [4], Tarantello found a negative solution when $d > \lambda_1(\lambda_1 - c)$ by a degree argument. For Problem (1.1), when $f(x,u) = bg(x,u)$, the existence of two or three nontrivial solutions has been obtained in [5, 6] for $g(x,u)$ under certain conditions by using variational methods. In [7], positive solutions of problem (1.1) were got when $f$ satisfies the local superlinearity and sublinearity. When $f$ is asymptotically linear at infinity, the existence of three nontrivial solutions has been obtained in [8] by using variational method, and the existence of a nontrivial solution has been obtained in [9] by using the mountain pass theorem. For more similar problems, we refer to [10–20] and the references therein.

In this paper, we prove a new existence result about a multiple solutions of problem (1.1) under the assumption that $f(x,u)$ is asymptotically linear with respect to $u$ at infinity. In this case, the Ambrosetti-Rabinowitz condition ((AR) condition for short) does not hold, hence it is difficult to verify the classical $(\mathrm{PS})_c$ condition. To overcome this difficulty, by using an equivalent version of Cerami's condition and the symmetric mountain pass lemma (see [21]), we obtain the existence of multiple solutions for problem (1.1). To the best of our knowledge, our main results are new. Before stating the main results, we give some notations.

Set $E = H^2(\Omega) \cap H_0^1(\Omega)$, then $E$ is a Hilbert space with the following inner product and the norm:

$$\langle u, v\rangle_E = \int_\Omega (\Delta u \Delta v - a\nabla u \nabla v)dx, \qquad \|u\|_E = \langle u, u\rangle_E^{1/2}. \tag{1.4}$$

The corresponding energy functional of problem (1.1) is defined on $E$ by

$$I(u) = \frac{1}{2}\int_\Omega \left(|\Delta u|^2 - a|\nabla u|^2\right)dx - \int_\Omega F(x,u)dx, \tag{1.5}$$

where $F(x,u) = \int_0^u f(x,t)dt$. From $(f_1)$–$(f_3)$, it is easy to see that $I \in C^1(E, \mathbb{R})$, it is well known that the weak solutions of problem (1.1) are the critical points of the energy functional $I(u)$.

Our main results are stated as follows.

**Theorem 1.1.** *Assume that $f(x,u)$ satisfies assumptions $(f_1)$–$(f_4)$, and $\Lambda_k$ is given by* (2.8)*. Then the following hold.*

(i) *If $\ell \in (\Lambda_k, +\infty)$ is not an eigenvalue of problem* (2.4)*, then problem* (1.1) *has at least $k$ pairs of nontrivial solutions in $E$.*

(ii) *Suppose that $(f_5)$ is satisfied, then the conclusion of (i) holds even if $\ell$ is an eigenvalue of problem* (2.4)*.*

(iii) *If $\ell = +\infty$, and $(f_6)$ holds, then problem* (1.1) *has infinitely many nontrivial solutions in $E$.*

## 2. Preliminaries

In this section, we give some preliminary results which will be used to prove our main results.

Throughout this paper, we will denote by $|\Omega|$ the Lebesgue measure of $\Omega$, $B_\rho = \{u \in E : \|u\|_E < \rho\}$. $C$ will denote various positive constants, $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence. $o_m(1)$ denote $o_m(1) \to 0$ as $m \to \infty$. $L^s(\Omega)$, $(1 \le s < +\infty)$ denote Lebesgue spaces, the norm $L^s$ is denoted by $|\cdot|_s$ for $1 \le s < +\infty$. The dual space of a Banach space $E$ will be denoted by $E^{-1}$.

First, we recall an equivalent version of Cerami's condition as follows (see [22]).

*Definition 2.1.* Let $E$ be a Banach space. $I \in C^1(E, \mathbb{R})$ is said to satisfy condition $(C)$ at level $c \in \mathbb{R}$ ($(C)_c$ for short), if the following fact is true: any sequence $\{u_m\} \subset E$, which satisfies

$$I(u_m) \longrightarrow c, \qquad (1 + \|u_m\|_E)\|I'(u_m)\|_{E^{-1}} \longrightarrow 0, \quad (m \longrightarrow \infty) \tag{2.1}$$

possesses a convergent subsequence in $E$.

Next, we will state an abstract symmetric mountain pass lemma. For this purpose, we should first introduce the definition of genus (see [23–25]).

*Definition 2.2.* Let $E$ be a real Banach space and $A$ a subset of $E$. $A$ is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set $A$ which does not contain the origin, we define a genus $\gamma(A)$ of $A$ by the smallest integer $k$ such that there exists an odd continuous mapping from $A$ to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a $k$, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$.

Let $E$ be an infinite dimensional real Banach space, $I \in C^1(E, \mathbb{R})$, $\widehat{A}_0 = \{u \in E : I(u) \geq 0\}$, $\Gamma^* = \{h(0) = 0,\ h$ is an odd homeomorphism of $E$ and $h(B_1) \subset \widehat{A}_0\}$, $\Gamma_m = \{\mathcal{K} \subset E : \mathcal{K}$ is compact, symmetric with respect to the origin, and for any $h \in \Gamma^*$, there holds $\gamma(\mathcal{K} \cap h(\partial B_1)) \geq m\}$. If $\Gamma_m \neq \emptyset$, define

$$b_m = \inf_{\mathcal{K} \subset \Gamma_m} \max_{u \in \mathcal{K}} I(u). \tag{2.2}$$

Now, we recall an abstract symmetric mountain pass lemma, which can be found in [26, 27].

**Lemma 2.3.** *Let $e_1, e_2, \ldots, e_m, \ldots$ be linearly independent in $E$, and $E_i = \operatorname{span}\{e_1, e_2, \ldots, e_i\}$, $i = 1, 2, \ldots, m, \ldots$. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, $I(-u) = I(u)$, and $(C)_c$ condition for $c \geq 0$. Furthermore, there exists $\rho > 0$, $\alpha > 0$ such that $I(u) > 0$ in $B_\rho \setminus \{0\}$ and $I(u)|_{\partial B_\rho} \geq \alpha$. Then, if $E_m \cap \widehat{A}_0$ is bounded, then $\Gamma_m \neq \emptyset$ and $b_m \geq \alpha > 0$ is a critical value of $I$. Moreover, if $E_{m+i} \cap \widehat{A}_0$ is bounded for all $i = 1, \ldots r$, and*

$$b_{m+1} = \cdots = b_{m+r} = b, \tag{2.3}$$

*then $\gamma(K_b) \geq r$, where $K_b = \{u \in E : I(u) = b, I'(u) = 0\}$. If $E_m \cap \widehat{A}_0$ is bounded for all $m$, then $I(u)$ possesses infinitely many critical values.*

Let us consider the eigenvalue problem:

$$\begin{aligned} \Delta^2 u + a\Delta u &= \Lambda u, \quad x \in \Omega, \\ u = \Delta u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.4}$$

Set

$$\Phi(u) = \int_\Omega \left(|\Delta u|^2 - a|\nabla u|^2\right) dx, \qquad \Psi(u) = \int_\Omega |u|^2 dx. \tag{2.5}$$

For $a < \lambda_1$, $\Phi(u)$ and $\Psi(u)$ are well defined. Furthermore, $\Phi(u), \Psi(u) \in C^1(E, \mathbb{R})$, and a real value $\Lambda$ is an eigenvalue of problem (2.4) if and only if there exists $u \in E \setminus \{0\}$ such that $\Phi'(u) = \Lambda \Psi'(u)$. At this point, let us set

$$\mathcal{N} = \{u \in E : \Psi(u) = 1\}. \tag{2.6}$$

Then $\mathcal{N} \neq \emptyset$ and $\mathcal{N}$ is a $C^1$ manifold in $E$. It follows from the standard Lagrange multiples arguments that eigenvalues of (2.4) correspond to critical values of $\Phi|_{\mathcal{N}}$, and $\Phi$ satisfies the

(PS) condition on $\mathcal{N}$. Thus a sequence of critical values of $\Phi|_{\mathcal{N}}$ comes from the Ljusternik-Schnirelmann critical point theory on $C^1$ manifolds. For any $k \in N$, set

$$\Gamma_k = \{A \subset \mathcal{N} : A \text{ is compact, symmetric and } \gamma(A) \geq k\}. \tag{2.7}$$

Then values:

$$\Lambda_k := \inf_{A \in \Gamma_k} \max_{u \in A} \Phi(u) \tag{2.8}$$

are critical values and hence are eigenvalues of problem (2.4). Moreover, $0 < \Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots \leq \Lambda_k \leq \cdots \to +\infty$.

We prove some properties of functional $I(u)$ in the following lemma.

**Lemma 2.4.** *For the functional $I(u)$ defined by* (1.5)*, if assumptions $(f_1)$ and $(f_6)$ hold, and for any $\{u_m\} \subset E$ with $\langle I'(u_m), u_m\rangle \to 0$ as $m \to \infty$, then there is a subsequence, still denoted by $\{u_m\}$, such that*

$$I(tu_m) \leq \frac{1+t^2}{2m} + I(u_m) \tag{2.9}$$

*holds for all $t > 0$, $m \in N^+$.*

*Proof.* This lemma is essentially due to [27, 28]. For the sake of completeness, we prove it here.

By $\langle I'(u_m), u_m\rangle \to 0$ as $m \to \infty$, for a suitable subsequence, we may assume that

$$-\frac{1}{m} < \langle I'(u_m), u_m\rangle = \|u_m\|_E^2 - \int_\Omega f(x, u_m) u_m dx < \frac{1}{m}, \quad \forall m. \tag{2.10}$$

We claim that for any $t > 0$ and $m \in N^+$,

$$I(tu_m) < \frac{t^2}{2m} + \int_\Omega \left(\frac{1}{2} f(x, u_m) u_m - F(x, u_m)\right) dx. \tag{2.11}$$

Indeed, for any $t > 0$, at fixed $x \in \Omega$ and $m \in N^+$, we set

$$h(t) = \frac{t^2}{2} f(x, u_m) u_m - F(x, tu_m), \tag{2.12}$$

then

$$\begin{aligned} h'(t) &= t f(x, u_m) u_m - f(x, tu_m) u_m \\ &= tu_m \left( f(x, u_m) - \frac{1}{t} f(x, tu_m) \right) \quad \begin{cases} \geq 0, & 0 < t \leq 1, \\ \leq 0, & t \geq 1, \end{cases} \quad \text{by } (f_6), \end{aligned} \tag{2.13}$$

hence

$$h(t) \le h(1) \quad \forall t > 0. \tag{2.14}$$

Therefore,

$$\begin{aligned} I(tu_m) &= \frac{t^2}{2}\|u_m\|_E^2 - \int_\Omega F(x, tu_m)dx \\ &\le \frac{t^2}{2}\left(\frac{1}{m} + \int_\Omega f(x, u_m)u_m dx\right) - \int_\Omega F(x, tu_m)dx \text{ by } (2.10) \\ &\le \frac{t^2}{2m} + \int_\Omega \left(\frac{t^2}{2} f(x, u_m)u_m - F(x, tu_m)\right)dx \\ &\le \frac{t^2}{2m} + \int_\Omega \left(\frac{1}{2} f(x, u_m)u_m - F(x, u_m)\right)dx \text{ by } (2.14) \end{aligned} \tag{2.15}$$

and our claim (2.11) is proved.

On the other hand,

$$\begin{aligned} I(u_m) &= \frac{1}{2}\|u_m\|_E^2 - \int_\Omega F(x, u_m)dx \\ &\ge \frac{1}{2}\left(-\frac{1}{m} + \int_\Omega f(x, u_m)u_m dx\right) - \int_\Omega F(x, u_m)dx, \end{aligned} \tag{2.16}$$

that is,

$$\int_\Omega \left(\frac{1}{2} f(x, u_m)u_m - F(x, u_m)\right)dx \le \frac{1}{2m} + I(u_m). \tag{2.17}$$

Combining (2.11) and (2.17) we have that

$$I(tu_m) \le \frac{1+t^2}{2m} + I(u_m), \quad \forall t > 0, \; m \in N^+. \tag{2.18}$$

The proof is completed. □

## 3. Proof of the Main Results

We begin with the following lemma.

**Lemma 3.1.** *Let $c \ge 0$. Assume that $f(x,u)$ satisfies assumptions $(f_1)$–$(f_3)$. Then the following hold.*

(i) *$I(u)$ satisfies $(C)_c$ condition if $\ell < +\infty$ in assumption $(f_3)$, and $\ell$ is not an eigenvalue of problem* (2.4).

(ii) *If* $\ell < +\infty$ *is an eigenvalue of problem* (2.4) *and* $(f_5)$ *holds, then* $I(u)$ *satisfies* $(C)_c$ *condition.*

(iii) *If* $\ell = +\infty$, *and* $(f_6)$ *holds, then* $I(u)$ *satisfies* $(C)_c$ *condition.*

*Proof.* Suppose that $\{u_m\} \subset E$ is a $(C)_c$ sequence, that is, as $m \to \infty$, we have

$$I(u_m) \longrightarrow c \geq 0, \tag{3.1}$$

$$(1 + \|u_m\|_E)\left\|I'(u_m)\right\|_{E^{-1}} \longrightarrow 0, \quad \text{in } E^{-1}. \tag{3.2}$$

It is easy to see that (3.2) implies that as $m \to \infty$, there hold

$$\|u_m\|_E^2 - \int_\Omega f(x, u_m) u_m dx = o_m(1), \tag{3.3}$$

$$\int_\Omega (\Delta u_m \Delta\varphi - a\nabla u_m \nabla\varphi) dx - \int_\Omega f(x, u_m)\varphi dx = o_m(1), \qquad \forall\varphi \in E. \tag{3.4}$$

By Sobolev compact embedding, to show that $I(u)$ satisfies $(C)_c$ condition, it suffices to show the boundedness of $(C)_c$ sequence in $E$ for each case.

(i) Suppose that $0 < \ell < +\infty$ and $\ell$ is not an eigenvalue of problem (2.4). Arguing by contradiction, we suppose that there exists a subsequence, still denoted by $\{u_m\}$, such that as $m \to \infty$, there holds $\|u_m\|_E \to +\infty$. Define

$$p_m(x) = \begin{cases} \dfrac{f(x, u_m(x))}{u_m(x)}, & u_m(x) \neq 0, \\ 0, & u_m(x) = 0. \end{cases} \tag{3.5}$$

Then from assumptions $(f_1)$–$(f_3)$, there exists $M > 0$ such that

$$0 \leq p_m(x) \leq M. \tag{3.6}$$

Let

$$w_m = \frac{u_m}{\|u_m\|_E}. \tag{3.7}$$

Obviously, $w_m$ is bounded in $E$. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} w_m &\rightharpoonup w, && \text{weakly in } E, \\ w_m &\longrightarrow w, && \text{a.e. in } \Omega, \\ w_m &\longrightarrow w, && \text{strongly in } L^s(\Omega),\ \forall s \in [2, 2^*). \end{aligned} \tag{3.8}$$

It is easy to show that $w \not\equiv 0$. In fact, if $w \equiv 0$, then from (3.3), (3.6), (3.8) and the definitions of $p_m$ and $w_m$, as $m \to \infty$, we have

$$1 = \|w_m\|_E^2 = \int_\Omega p_m(x)|w_m|^2 dx + o_m(1) \leq M \int_\Omega |w_m|^2 dx + o_m(1) \longrightarrow 0, \tag{3.9}$$

which is a contradiction.

From (3.6), there exists $h(x) \in L^\infty(\Omega)$ with $0 \leq h(x) \leq M$ such that, up to a subsequence, as $m \to \infty$, there holds

$$p_m(x) \rightharpoonup h(x), \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{3.10}$$

Then from (3.8) it follows that

$$\begin{gathered} p_m(x)w_m \rightharpoonup h(x)w \quad \text{weakly in } L^2(\Omega), \\ \int_\Omega p_m(x)|w_m|^2 dx \longrightarrow \int_\Omega h(x)|w|^2 dx. \end{gathered} \tag{3.11}$$

On the other hand, from (3.3), (3.4), (3.5), and (3.7), we have

$$\int_\Omega (\Delta w_m \Delta\varphi - a\nabla w_m \nabla\varphi) dx = \int_\Omega p_m(x) w_m \varphi dx + o_m(1), \quad \forall \varphi \in E. \tag{3.12}$$

$$\int_\Omega \left(|\Delta w_m|^2 - a|\nabla w_m|^2\right) dx = \int_\Omega p_m(x)|w_m|^2 dx + o_m(1). \tag{3.13}$$

It follows from (3.11)–(3.13) that

$$\begin{gathered} \|w_m\|_E^2 = \int_\Omega h(x)|w|^2 dx + o_m(1), \\ \int_\Omega (\Delta w_m \Delta\varphi - a\nabla w_m \nabla\varphi) dx = \int_\Omega h(x) w\varphi dx + o_m(1), \quad \forall \varphi \in E. \end{gathered} \tag{3.14}$$

Therefore (3.14) implies that $w$ satisfies

$$\int_\Omega (\Delta w \Delta\varphi - a\nabla w \nabla\varphi) dx = \int_\Omega h(x) w\varphi dx, \quad \forall \varphi \in E. \tag{3.15}$$

Let

$$\begin{gathered} \Omega^0 = \{x \in \Omega : w(x) = 0\}, \\ \Omega^+ = \{x \in \Omega : w(x) > 0\}, \\ \Omega^- = \{x \in \Omega : w(x) < 0\}. \end{gathered} \tag{3.16}$$

Then $u_m(x) \to +\infty$ as $m \to \infty$ if $x \in \Omega^+$, and $u_m(x) \to -\infty$ as $m \to \infty$ if $x \in \Omega^-$. From assumption $(f_3)$, $h(x) \equiv \ell$ for all $x \in \Omega^+ \cup \Omega^-$. Thus (3.15) implies that $w$ satisfies

$$\int_{\Omega^0} (\Delta w \Delta \varphi - a \nabla w \nabla \varphi - h(x) w \varphi) dx + \int_{\Omega^+ \cup \Omega^-} (\Delta w \Delta \varphi - a \nabla w \nabla \varphi - h(x) w \varphi) dx = 0, \quad \forall \varphi \in E. \tag{3.17}$$

Therefore

$$\int_\Omega (\Delta w \Delta \varphi - a \nabla w \nabla \varphi) dx = \ell \int_\Omega w \varphi dx, \quad \forall \varphi \in E. \tag{3.18}$$

This means that $\ell$ is an eigenvalue of problem (2.4), which contradicts our assumption, so $\{u_m\}$ is bounded in $E$.

(ii) Suppose $\ell \in (0, +\infty)$ is an eigenvalue of problem (2.4), we need the additional assumption $(f_5)$.

From assumption $(f_5)$, there exists $T_0 > 0$ such that

$$f(x,u)u - 2F(x,u) \geq 0, \quad \forall |u| \geq T_0, \ x \in \Omega, \tag{3.19}$$

and there exists $C_0 = C_0(T_0) > 0$ such that

$$\int_{\{|u_m| \leq T_0\}} (f(x,u_m)u_m - 2F(x,u_m)) dx \geq -C_0. \tag{3.20}$$

Furthermore, under assumptions $(f_1)$–$(f_3)$, there exists $M > 0$ such that

$$|f(x,u)| \leq M|u|, \qquad |F(x,u)| \leq \frac{M}{2}|u|^2, \quad \forall x \in \Omega. \tag{3.21}$$

Let $K = (2c + C_0)(2MS)^{N/2}$, where $M > 0$ is given by (3.21), $S > 0$ is the best Sobolev constant such that

$$\left( \int_\Omega |u|^{2^*} dx \right)^{2/2^*} \leq S \int_\Omega \left( |\Delta u|^2 - a |\nabla u|^2 \right) dx, \quad \forall u \in E. \tag{3.22}$$

From assumption $(f_5)$, there exists $T = T(K) > T_0 > 0$ such that

$$f(x,u)u - 2F(x,u) \geq K, \quad \forall |u| \geq T, \ x \in \Omega. \tag{3.23}$$

For the above $T > 0$ and each $m \geq 1$, set

$$A_m = \{x \in \Omega : |u_m(x)| \geq T\}, \qquad B_m = \{x \in \Omega : |u_m(x)| \leq T\}. \tag{3.24}$$

From estimates (3.20), (3.1), (3.3), and (3.23), we get

$$\begin{aligned}
2c + o_m(1) &= \int_\Omega (f(x,u_m)u_m - 2F(x,u_m))dx \\
&\geq \int_{A_m} (f(x,u_m)u_m - 2F(x,u_m))dx - C_0 \\
&\geq K|A_m| - C_0,
\end{aligned} \tag{3.25}$$

where $|A_m|$ denotes the measure of $A_m$.

On the other hand, for any fixed $r > 2$, from (3.1) and (3.3), we have

$$\left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - \int_\Omega \left(F(x,u_m) - \frac{1}{r}f(x,u_m)u_m\right)dx = c + o_m(1). \tag{3.26}$$

Since $\Omega$ is bounded and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, there exists a constant $C = C(\Omega, f, T)$ such that

$$\left|\int_{B_m} \left(F(x,u_m) - \frac{1}{r}f(x,u_m)u_m\right)dx\right| \leq C, \quad \forall x \in \Omega. \tag{3.27}$$

Then, from (3.21)–(3.26), Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
c + o_m(1) &\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \int_{A_m} \left(F(x,u_m) - \frac{1}{r}f(x,u_m)u_m\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \int_{A_m} \left(\frac{1}{2}f(x,u_m)u_m - \frac{1}{r}f(x,u_m)u_m\right)dx \\
&= \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \left(\frac{1}{2} - \frac{1}{r}\right)\int_{A_m} f(x,u_m)u_m dx \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \left(\frac{1}{2} - \frac{1}{r}\right)M\int_{A_m} |u_m|^2 dx \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \left(\frac{1}{2} - \frac{1}{r}\right)M|u_m|_{2^*}^2|A_m|^{2/N} \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \left(\frac{1}{2} - \frac{1}{r}\right)MS\|u_m\|_E^2\left(\left|\frac{2c + C_0}{K}\right| + o_m(1)\right)^{2/N} \\
&\geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r}\right)\|u_m\|_E^2 - C - \left(\frac{1}{2} - \frac{1}{r}\right)MS\|u_m\|_E^2 \cdot o_m(1),
\end{aligned} \tag{3.28}$$

that is, $\{u_m\}$ is bounded in $E$.

(iii) Finally, we prove the case $\ell = +\infty$. Here the subcritical condition (1.2) is assumer as usual, but to make use of Lemma 2.4, $(f_6)$ is required in this case. Set

$$t_m = \frac{2\sqrt{c}}{\|u_m\|_E}, \qquad w_m = t_m u_m = \frac{2\sqrt{c}u_m}{\|u_m\|_E}. \tag{3.29}$$

Then $\|w_m\|_E = 2\sqrt{c}$ and $\{w_m\}$ is bounded in $E$. Hence, up to a subsequence, we may assume that: there exists $w \in E$ such that (3.8) also holds in this case. If $\|u_m\|_E \to +\infty$, we claim that

$$w(x) \not\equiv 0. \tag{3.30}$$

In fact, if $w(x) \equiv 0$ in $\Omega$, then (3.29) and (3.8) imply that

$$\int_\Omega F(x, w_m)dx \longrightarrow 0, \qquad I(w_m) = 4c + o_m(1). \tag{3.31}$$

However, applying Lemma 2.4 with $t = 2\sqrt{c}/\|u_m\|_E$, we have

$$I(w_m) \le \frac{1+t^2}{2m} + I(u_m) \longrightarrow c, \quad (m \longrightarrow \infty), \tag{3.32}$$

which contradicts (3.31), thus (3.30) holds.

On the other hand, similar to case (i), (3.13) holds. Let $\tilde{\Omega} = \Omega \setminus \{x \in \Omega : w(x) = 0\}$. Then $|\tilde{\Omega}| > 0$ by (3.30). From assumptions $(f_3)$ and $(f_4)$, $p_m(x) \ge 0$ and $p_m(x) \to +\infty$ as $m \to \infty$ in $\tilde{\Omega}$, where $p_m(x)$ is defined by (3.5). Hence, from (3.8) and (3.13), we have

$$\begin{aligned} 4c &= \liminf_{m\to\infty} \|w_m\|_E^2 = \liminf_{m\to\infty} \int_\Omega p_m(x)|w_m|^2 dx \\ &\ge \liminf_{m\to\infty} \int_{\tilde{\Omega}} p_m(x)|w_m|^2 dx \\ &\ge \int_{\tilde{\Omega}} \liminf_{m\to\infty} p_m(x)|w_m|^2 dx = +\infty, \end{aligned} \tag{3.33}$$

which is a contradiction, thus $\|u_m\|_E \nrightarrow +\infty$, that is, up to a subsequence, $\{u_m\}$ is bounded in $E$. □

*Proof of Theorem 1.1.* The proof of this theorem is divided in two steps.

*Step 1.* There exists $\rho > 0$, $\alpha > 0$ such that $I(u) > 0$ in $B_\rho(0)$ and $I(u)|_{\partial B_\rho} \ge \alpha$.

In fact, in each case, assumptions $(f_1)$–$(f_3)$ imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $u \in \mathbb{R}$, there holds

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{q-1}, \qquad |F(x,u)| \le \varepsilon|u|^2 + C_\varepsilon|u|^q, \tag{3.34}$$

where $q$ is the same as that in (1.2), from which, it is easy to see that there exists $\rho > 0, \alpha > 0$ such that $I(u) > 0$ in $B_\rho(0)$ and $I(u)|_{\partial B_\rho} \ge \alpha$.

*Step 2.* By the Symmetric Mountain Pass Lemma 2.3, to prove Theorem 1.1, it suffices to prove that for any $k \geq 1$, there exists a $k$-dimensional subspace $E_k$ of $E$ and $R_k > 0$ such that

$$I(u) \leq 0, \quad \forall u \in E_k \setminus B_{R_k}. \tag{3.35}$$

First, we prove (3.35) in the case $\ell \in (\Lambda_k, +\infty)$. Since $\ell > \Lambda_k$, there is $\varepsilon > 0$ such that $\ell - \varepsilon > \Lambda_k$. By the definition of $\Lambda_k$, there exists a $k$-dimensional subspace $E_k$ of $E$ such that, for the above $\varepsilon > 0$, there holds

$$\sup_{u \in E_k \setminus \{0\}} \frac{\Psi(u)}{\Phi(u)} \leq \Lambda_k + \frac{\varepsilon}{2} < l - \frac{\varepsilon}{2}, \tag{3.36}$$

that is,

$$\sup_{u \in E_k \setminus \{0\}} \frac{\Phi(u)}{\Psi(u)} > \frac{1}{\ell - \varepsilon/2}. \tag{3.37}$$

By assumption $(f_3)$, we have

$$\lim_{|u| \to +\infty} \frac{F(x,u)}{|u|^2} = \frac{\ell}{2}. \tag{3.38}$$

Then, for the above $\varepsilon > 0$, there exists $M > 0$ large enough such that

$$\frac{F(x,u)}{|u|^2} > \frac{1}{2}\left(\ell - \frac{\varepsilon}{4}\right), \quad \forall |u| > M. \tag{3.39}$$

Therefore, if $u \in E_k$ with $\|u\|_E = R$, by (3.39) and (3.37), we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}R^2 - \int_\Omega F(x,u)dx \\ &\leq \frac{1}{2}R^2 - \int_{|u|>M} F(x,u)dx - C(M,\Omega) \\ &\leq \frac{1}{2}R^2 - \frac{1}{2}\left(\ell - \frac{\varepsilon}{4}\right)\int_\Omega |u|^2 dx - C(M,\Omega) \\ &= \frac{R^2}{2}\left(1 - \left(\ell - \frac{\varepsilon}{4}\right)\int_\Omega \left(\frac{|u|}{R}\right)^2 dx\right) - C(M,\Omega) \\ &\leq \frac{R^2}{2}\left(1 - \frac{\ell - \varepsilon/4}{\ell - \varepsilon/2}\right) - C(M,\Omega) \\ &< 0, \end{aligned} \tag{3.40}$$

if $R \geq R_k$ and $R_k > 0$ large enough.

If $\ell = +\infty$, similar to (3.37), for any $k \geq 1$, there exists $E_k \subset E$ such that

$$\sup_{u \in E_k \setminus \{0\}} \frac{\Phi(u)}{\Psi(u)} > \frac{1}{\Lambda_k + 1/2}, \tag{3.41}$$

similar to (3.39), from assumption $(f_3)$ with $\ell = +\infty$ it follows that there exists $M_k > 0$ such that

$$\frac{2F(x,u)}{|u|^2} > \Lambda_k + 1, \quad \forall |u| > M_k. \tag{3.42}$$

Then, if $u \in E_k$ with $\|u\|_E = R$, we have

$$I(u) \leq \frac{R^2}{2}\left(1 - \frac{\Lambda_k + 1}{\Lambda_k + 1/2}\right) - C(M_k, k, \Omega) < 0, \tag{3.43}$$

if $R \geq R_k$ and $R_k > 0$ large enough. This completes the proof of Theorem 1.1.

□

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## References

[1] A. C. Lazer and P. J. McKenna, "Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis," *SIAM Review*, vol. 32, no. 4, pp. 537–578, 1990.
[2] A. C. Lazer and P. J. McKenna, "Global bifurcation and a theorem of Tarantello," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 3, pp. 648–655, 1994.
[3] P. J. McKenna and W. Reichel, "Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry," *Electronic Journal of Differential Equations*, vol. 37, pp. 1–13, 2003.
[4] G. Tarantello, "A note on a semilinear elliptic problem," *Differential and Integral Equations*, vol. 5, no. 3, pp. 561–565, 1992.
[5] A. M. Micheletti and A. Pistoia, "Multiplicity results for a fourth-order semilinear elliptic problem," *Nonlinear Analysis*, vol. 31, no. 7, pp. 895–908, 1998.
[6] A. M. Micheletti and A. Pistoia, "Nontrivial solutions for some fourth order semilinear elliptic problems," *Nonlinear Analysis*, vol. 34, no. 4, pp. 509–523, 1998.
[7] G. Xu and J. Zhang, "Existence results for some fourth-order nonlinear elliptic problems of local superlinearity and sublinearity," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 633–640, 2003.
[8] X. Liu and Y. Huang, "On sign-changing solution for a fourth-order asymptotically linear elliptic problem," *Nonlinear Analysis*, vol. 72, no. 5, pp. 2271–2276, 2010.
[9] Y. An and R. Liu, "Existence of nontrivial solutions of an asymptotically linear fourth-order elliptic equation," *Nonlinear Analysis*, vol. 68, no. 11, pp. 3325–3331, 2008.
[10] Y. Yang and J. Zhang, "Existence of solutions for some fourth-order nonlinear elliptic problems," *Journal of Mathematical Analysis and Applications*, vol. 351, no. 1, pp. 128–137, 2009.
[11] J. Zhang and S. Li, "Multiple nontrivial solutions for some fourth-order semilinear elliptic problems," *Nonlinear Analysis*, vol. 60, no. 2, pp. 221–230, 2005.

[12] Z. Jihui, “Existence results for some fourth-order nonlinear elliptic problems,” *Nonlinear Analysis*, vol. 45, pp. 29–36, 2001.

[13] J. Zhou and X. Wu, “Sign-changing solutions for some fourth-order nonlinear elliptic problems,” *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 542–558, 2008.

[14] J. Zhang and Z. Wei, “Multiple solutions for a class of biharmonic equations with a nonlinearity concave at the origin,” *Journal of Mathematical Analysis and Applications*, vol. 383, no. 2, pp. 291–306, 2011.

[15] Y. Liu and Z. Wang, “Biharmonic equations with asymptotically linear nonlinearities,” *Acta Mathematica Scientia. Series B*, vol. 27, no. 3, pp. 549–560, 2007.

[16] D. Lü, “Multiple solutions for a class of biharmonic elliptic systems with Sobolev critical exponent,” *Nonlinear Analysis*, vol. 74, no. 17, pp. 6371–6382, 2011.

[17] Y. Yin and X. Wu, “High energy solutions and nontrivial solutions for fourth-order elliptic equations,” *Journal of Mathematical Analysis and Applications*, vol. 375, no. 2, pp. 699–705, 2011.

[18] P. Mironescu and V. D. Rădulescu, “The study of a bifurcation problem associated to an asymptotically linear function,” *Nonlinear Analysis*, vol. 26, no. 4, pp. 857–875, 1996.

[19] P. Pucci and V. Rădulescu, “The impact of the mountain pass theory in nonlinear analysis: a mathematical survey,” *Bollettino della Unione Matematica Italiana. Serie 9*, vol. 3, no. 3, pp. 543–582, 2010.

[20] V. D. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods*, vol. 6 of *Contemporary Mathematics and Its Applications*, Hindawi Publishing Corporation, New York, NY, USA, 2008.

[21] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, Conference Board of the Mathematical Sciences, Washington, DC, USA, 1986.

[22] D. G. Costa and C. A. Magalhães, “Existence results for perturbations of the $p$-Laplacian,” *Nonlinear Analysis*, vol. 24, no. 3, pp. 409–418, 1995.

[23] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[24] P. Mironescu and V. D. Rădulescu, “A multiplicity theorem for locally Lipschitz periodic functionals,” *Journal of Mathematical Analysis and Applications*, vol. 195, no. 3, pp. 621–637, 1995.

[25] A. Ambrosetti and P. H. Rabinowitz, “Dual variational methods in critical point theory and applications,” vol. 14, pp. 349–381, 1973.

[26] G. Li and H.-S. Zhou, “Multiple solutions to $p$-Laplacian problems with asymptotic nonlinearity as $u^{p-1}$ at infinity,” *Journal of the London Mathematical Society. Second Series*, vol. 65, no. 1, pp. 123–138, 2002.

[27] G. Li and H.-S. Zhou, “Asymptotically linear Dirichlet problem for the $p$-Laplacian,” *Nonlinear Analysis*, vol. 43, pp. 1043–1055, 2001.

[28] C. A. Stuart and H. S. Zhou, “Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbf{R}^N$,” *Communications in Partial Differential Equations*, vol. 24, no. 9-10, pp. 1731–1758, 1999.