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Research Article

Solution of Second-Order IVP and BVP of Matrix Differential Models Using Matrix DTM

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We introduce a matrix form of differential transformation method (DTM) and apply for nonlinear second-order initial value problems (IVPs) and boundary value problems (BVPs) of matrix models which are given by $\mathbf{u}''(t) = f(t, \mathbf{u}(t), \mathbf{u}'(t))$ and subject to initial conditions $\mathbf{u}(a) = \mathbf{u}_0, \mathbf{u}'(a) = \mathbf{u}_1$ and boundary conditions $\mathbf{u}(a) = \mathbf{u}_0, \mathbf{u}(b) = \mathbf{u}_1$, where $\mathbf{u}_0$, $\mathbf{u}_1 \in R^{r\times q}$. Also the convergence of present method is established. Several illustrative examples are given to demonstrate the effectiveness of the present method.

1. Introduction

In this work, we study two important cases of nonlinear second-order matrix models given by the following.

(1) Matrix initial-value problems are of the following form:

$$\begin{aligned} \mathbf{u}''(t) &= f(t, \mathbf{u}(t), \mathbf{u}'(t)), \\ \mathbf{u}(a) = \mathbf{u}_0, &\qquad \mathbf{u}'(a) = \mathbf{u}_1, \end{aligned} \qquad a \le t \le b,\ [a,b] \subset \mathbb{R}. \tag{1.1}$$

(2) Matrix boundary-value problems are of the following form:

$$\begin{aligned} \mathbf{u}''(t) &= f(t, \mathbf{u}(t), \mathbf{u}'(t)), \\ \mathbf{u}(a) = \mathbf{u}_0, &\qquad \mathbf{u}(b) = \mathbf{u}_1, \end{aligned} \qquad a \le t \le b,\ [a,b] \subset \mathbb{R}. \tag{1.2}$$

Matrices $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}(t) \in \mathbb{R}^{r\times q}$ and matrix function $f : [a,b] \times \mathbb{R}^{r\times q} \times \mathbb{R}^{r\times q} \mapsto \mathbb{R}^{r\times q}$ are frequent in different fields in physics and engineering. Note that (1.1) could be the statement of Newton second law of motion for a coupled mechanical system. These problems generally arise frequently in many areas of science and engineering, for example, fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics, and so forth, where one solves scalar or vectorial problems with boundary-value conditions [1–3].

Many authors studied (1.1), (1.2), and other similar forms of these equation, by different numerical methods, such as Jodar et al. [4–6], Al-Said [7], Al-Said and Noor [8], and Hargreaves and Higham [9]. In [10], (1.1) studied was subject to initial conditions using Cubic Matrix Splines method.

The differential transform method (DTM) is a seminumerical/analytic technique that formalizes the Taylor series in a totally different manner. It was first introduced by Zhou in a study about electrical circuits [11]. The DTM obtains an analytical solution in the form of a polynomial. It is different from the traditional high-order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. DTM has been successfully applied to solve many nonlinear problems arising in engineering, physics, mechanics, biology, and so forth. Ayaz [12] applied DTM for solution of system of ordinary differential equations. Arikoglu and Ozkol [13] employed DTM on differential-difference equations. Furthermore, the method may be employed for the solution of partial differential equations. Abazari et al. employed DTM on some partial differential equations (PDEs) and their coupled version in [14–18] and extended DTM to solve the first and second kinds of the Riccati matrix differential equations in [19]. Recently, Abazari and Ganji [20] extended to the reduced differential transform method (RDTM) to study the partial differential equation with-proportional delay, and [21] applied RDTM on simulation of generalized Hirota-Satsuma coupled KdV equation. We also note that the differential transformation method was applied for solution of fractional order nonlinear boundary value problems in [22] and the nonlinear higher-order boundary value problems; see [23].

The purpose of this paper is extended DTM, mentioned in [19], to apply for (1.1) subject to initial conditions and (1.2) subject to boundary conditions.

The layout of the paper is as follows. In Section 2, the matrix differential transform method will be introduced. In Section 3, the convergence of present method is illustrated. In Section 4, some numerical results are given to clarify the method and a comparison is made with the existing results. Section 5 is the brief conclusion of this paper. Finally some references are listed in the end. Note that we have computed the numerical results by Maple programming.

2. Basic Definitions

With reference to [5–9, 13, 14], we introduce in this section the basic definition of the matrix form of one-dimensional DTM.

Definition 2.1. If $\mathbf{u}(t) \in \mathbb{R}^{n\times n}$ is matrix analytical function in the domain T, then it will be differentiated continuously with respect to time $t \in T$,

$$\frac{d^k \mathbf{u}(t)}{dt^k} = \phi(t,k), \tag{2.1}$$

where k belongs to the set of nonnegative integer, denoted by the K domain. Therefore, for $t = t_i$, (2.1) can be written as

$$\mathbf{U}_i(k) = \phi(t_i,k) = \left[\frac{d^k \mathbf{u}(t)}{dt^k}\right]_{t=t_i}, \quad \forall k \in K, \tag{2.2}$$

where $\mathbf{U}_i(k) \in \mathbb{R}^{n\times n}$ is called the spectrum of $\mathbf{u}(t)$ at $t = t_i$, in the K domain.

Definition 2.2. If $\mathbf{u}(t) \in \mathbb{R}^{n\times n}$ can be expressed by Taylor's series about fixed point t_i, then $\mathbf{u}(t)$ can be represented as

$$\mathbf{u}(t) = \sum_{k=0}^{\infty} \frac{\mathbf{u}^{(k)}(t_i)}{k!}(t-t_i)^k. \tag{2.3}$$

If $\mathbf{u}_n(t) = \sum_{k=0}^{n}(\mathbf{u}^{(k)}(t_i\ )/k!)(t-t_i)^k$ is the n-partial sums of a Taylor's series (2.3), then

$$\mathbf{u}(t) = \sum_{k=0}^{n} \frac{\mathbf{u}^{(k)}(t_i)}{k!}(t-t_i)^k + R_n(t), \tag{2.4}$$

where $\mathbf{u}_n(t)$ is called the nth Taylor polynomial for $\mathbf{u}(t)$ about t_i and $R_n(t)$ is remainder term. If $\mathbf{U}(k)$ is defined as

$$\mathbf{U}(k) = \frac{\mathbf{u}^{(k)}(t_i)}{k!}, \quad k = 0, 1, \ldots, \tag{2.5}$$

then (2.3) reduces to

$$\mathbf{u}(t) = \sum_{k=0}^{\infty} \mathbf{U}(k)(t-t_i)^k, \tag{2.6}$$

and the n-partial sums of a Taylor's series (2.6) reduce to

$$\mathbf{u}_n(t) = \sum_{k=0}^{n} \mathbf{U}(k)(t-t_i)^k. \tag{2.7}$$

Table 1: The fundamental operations of one-dimensional matrix differential transform method.

Original matrix function	Matrix-transformed function
$\mathbf{w}(x) = \mathbf{u}(x) \pm \mathbf{v}(x)$	$\mathbf{W}(k) = \mathbf{U}(k) \pm \mathbf{V}(k)$
$\mathbf{w}(x) = c\mathbf{u}(x)$	$\mathbf{W}(k) = c\mathbf{U}(k)$
$\mathbf{w}(x) = \dfrac{d}{dx}\mathbf{u}(x)$	$\mathbf{W}(k) = (k+1)\mathbf{U}(k+1)$
$\mathbf{w}(x) = \dfrac{d^m}{dx^m}\mathbf{u}(x)$	$\mathbf{W}(k) = \dfrac{(k+m)!}{k!}\mathbf{U}(k+m)$
$\mathbf{w}(x) = \mathbf{u}(x)\mathbf{v}(x)$	$\mathbf{W}(k) = \sum_{l=0}^{k}\mathbf{U}(k-l)\mathbf{V}(l)$

The $\mathbf{U}(k)$ defined in (2.5) is called the matrix differential transform of matrix function $u(t)$. For a special case, when $t_0 = 0$, then solution (2.6) reduces to

$$\mathbf{u}(t) = \sum_{k=0}^{n}\mathbf{U}(k)t^k + R_n(t). \tag{2.8}$$

From the aforementioned definitions, it can be found that the concept of the one-dimensional matrix differential transform is derived from the Taylor series expansion. With (2.5) and (2.6), the fundamental mathematical operations performed by one-dimensional matrix differential transform can readily be obtained and listed in Table 1.

3. Convergence Analysis

In this section, we show that the presented matrix differential transformation method is convergence.

Theorem 3.1. *Let the matrix function* $\mathbf{u} : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ *and its first* n *derived functions be relatively continuous and finite on an interval* T *and differentiable on* $T - Q$ *(*Q *countable). Let* $t_0, t \in T$*. Then* (2.3) *and* (2.4) *hold, with*

$$\begin{aligned} R_n(x) &= \frac{1}{n!}\int_{t_0}^{x}\mathbf{u}^{(n+1)}(t)(x-t)^n dt, \\ \|R_n(x)\|_\infty &\le \frac{|x-t_0|^{n+1}}{(n+1)!}\left\|\mathbf{u}^{(n+1)}(t)\right\|_\infty, \end{aligned} \tag{3.1}$$

where $\|\cdot\|_\infty$ *is infinity matrix norm.*

Proof. By Definition 2.2, we get

$$R_n(x) = \mathbf{u}(x) - \mathbf{u}(t_0) - \sum_{k=1}^{n}\mathbf{u}^{(k)}(t_0)\frac{(x-t_0)^k}{k!}. \tag{3.2}$$

We use the right-hand side as a "pattern" to define a function $\mathbf{h} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. This time, we keep x fixed (say, $x = a \in T$) and replace t_0 by a variable t. Thus we set

$$\mathbf{h}(t) = \mathbf{u}(a) - \mathbf{u}(t) - \frac{\mathbf{u}'(t)}{1!}(a-t) - \cdots - \frac{\mathbf{u}^{(n)}(t)}{n!}(a-t)^n, \quad t \in \mathbb{R}. \tag{3.3}$$

Then $\mathbf{h}(t_0) = R_n(a)$ and $\mathbf{h}(a) = 0$. Our assumptions imply that $\mathbf{h}$ is relatively continuous and finite on T and differentiable on $T - Q$. Differentiating (3.3), we see that all cancels out except for one term:

$$\mathbf{h}'(t) = -\frac{\mathbf{u}^{(n+1)}(t)}{n!}(a-t)^n, \quad t \in T - Q. \tag{3.4}$$

Then for $t \in T$ we get $-\mathbf{h}(t) = \int_t^a (\mathbf{u}^{(n+1)}\ (t)/n!)(a-t)^n dt$, and

$$\int_{t_0}^{a} \frac{\mathbf{u}^{(n+1)}(t)}{n!}(a-t)^n dt = -\mathbf{h}(a) + \mathbf{h}(t_0) = R_n(a). \tag{3.5}$$

As $x = a$, (3.1) is proved. Next, let $M = \|\mathbf{u}^{(n+1)}(t)\|_\infty$. If $M = +\infty$, (3.1) is valid. If $M < +\infty$, we define $g(t) = M((t-a)^{n+1}/(n+1)!)$ for $t \ge a$, and $g(t) = -M((a-t)^{n+1}/(n+1)!)$ for $t \le a$. In both cases, for $t \in T - Q$ we have $g'(t) = M(|t-a|^n/n!) \ge \|\mathbf{h}'(t)\|_\infty$; then we get $\|\mathbf{h}(t_0) - \mathbf{h}(a)\|_\infty \le \|g(t_0) - g(a)\|_\infty$, or $\|R_n(a)\|_\infty \le M(|a-t_0|^{n+1}/(n+1)!)$ Thus (3.1) follows, because a is arbitrary value. □

From Theorem 3.1, we get that if $n \to \infty$, then $\|R_n(x)\|_\infty \to 0_{n\times n}$. Then by this theorem, the convergence of this method is investigated.

4. Applications and Numerical Results

This section is devoted to computational results. We applied the method presented in this paper and solved six examples. The first two examples are allotted to matrix initial-value problem and the next examples are about matrix boundary-value form. These examples are chosen such that there exist exact solutions for them. All calculations are implemented by Maple.

Example 4.1. In the first example, we consider the following non linear second-order matrix initial-value problem:

$$\mathbf{u}''(t) + \mathbf{u}'(t)\mathbf{u}(t) = \mathbf{c}(t),$$
$$\mathbf{u}(0) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{u}'(0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.1}$$

where $\mathbf{c}(t) = \begin{pmatrix} 2t^3-6t^2+4t-2 & 2t-2 \\ 0 & e^t+e^{2t} \end{pmatrix}$. By applying matrix differential transform operator Table 1 on nonlinear system (4.1), for $k = 0, 1, 2, \ldots, n$, we get

$$\frac{(k+2)!}{k!}\mathbf{U}(k+2) + \sum_{l=0}^{k}(k-l+1)\mathbf{U}(k-l+1)\mathbf{U}(l) = \mathbf{C}(k),$$
$$\mathbf{U}(0) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{U}(1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

where $\mathbf{U}(k)$ and $\mathbf{C}(k)$ are the differential transform of $\mathbf{u}(t)$ and $\mathbf{c}(t)$, respectively. From (4.2), and for $k = 0, 1, 2, \ldots$, we get

$$\mathbf{U}(k+2) = \frac{-k!}{(k+2)!}, \left\{ \sum_{l=0}^{k}(k-l+1)\mathbf{U}(k-l+1)\mathbf{U}(l) - \mathbf{C}(k) \right\}, \tag{4.3}$$

we substitute the initial condition (4.2), in recursive equation (4.3), for $k = 0, 1, 2$, we get $\mathbf{U}(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}, \mathbf{U}(3) = \begin{pmatrix} 0 & 0 \\ 0 & 1/6 \end{pmatrix}$, and $\mathbf{U}(4) = \begin{pmatrix} 0 & 0 \\ 0 & 1/24 \end{pmatrix}$, and then from the inverse differential transform operator (2.7), we get

$$\mathbf{U}_4(t) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} t + \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} t^2 + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6} \end{pmatrix} t^3 + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{24} \end{pmatrix} t^4. \tag{4.4}$$

The closed form of above solution is $\mathbf{u}(t) = \begin{pmatrix} 2t-t^2 & -1 \\ 0 & e^t \end{pmatrix}$, which is exactly the same as the exact solution.

Example 4.2. In this example, we consider the following nonlinear second-order matrix differential equations:

$$\mathbf{u}''(t) + \mathbf{u}(t)\mathbf{u}'(t) - \mathbf{u}^2(t) = c(t),$$
$$\mathbf{u}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{u}'(0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{4.5}$$

where $c(t) = \begin{pmatrix} e^t-e^{2t} & -e^t \\ 2e^t+te^t+e^{2t} & e^t \end{pmatrix}$. The differential transform version of nonlinear system (4.5), for $k = 0, 1, 2, \ldots, n$, is

$$\frac{(k+2)!}{k!}\mathbf{U}(k+2) + \sum_{l=0}^{k}(l+1)\mathbf{U}(k-l)\mathbf{U}(l+1) - \sum_{l=0}^{k}\mathbf{U}(k-l)\mathbf{U}(l) = \mathbf{C}(k),$$
$$\mathbf{U}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{U}(1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{4.6}$$

where $\mathbf{U}(k)$ and $\mathbf{C}(k)$ are the matrix differential transform of $\mathbf{u}(t)$ and $\mathbf{c}(t)$, respectively. From (4.6), and for $k = 0, 1, 2, \ldots$, we get following iteration equation:

$$\mathbf{U}(k+2) = \frac{-k!}{(k+2)!}\left\{\sum_{l=0}^{k}(l+1)\mathbf{U}(k-l)\mathbf{U}(l+1) - \sum_{l=0}^{k}\mathbf{U}(k-l)\mathbf{U}(l) - \mathbf{C}(k)\right\}. \tag{4.7}$$

By utilizing the initial values $\mathbf{U}(0)$ and $\mathbf{U}(1)$ in recursive equations (4.7), for $k = 0, 1, 2$, the first four terms of $\mathbf{U}(k)$ are obtained as follows: $\mathbf{U}(2) = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 1/2 \end{pmatrix}$, $\mathbf{U}(3) = \begin{pmatrix} 1/6 & -1/6 \\ 1/2 & 1/6 \end{pmatrix}$, and $\mathbf{U}(4) = \begin{pmatrix} 1/24 & -1/2 \\ 1/6 & 1/24 \end{pmatrix}$, and then from (2.7), we get

$$\mathbf{U}_4(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}t + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}t^2 + \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix}t^3 + \begin{pmatrix} \frac{1}{24} & -\frac{1}{2} \\ \frac{1}{6} & \frac{1}{24} \end{pmatrix}t^4. \tag{4.8}$$

The closed form of the previous solution is $\mathbf{u}(t) = \begin{pmatrix} e^t & -e^t \\ te^t & e^t \end{pmatrix}$, which is exactly the same as the exact solution.

Example 4.3. In the third example, consider the following linear second-order boundary-value problem:

$$\begin{gathered} \mathbf{u}''(t) + \mathbf{a}(t)\mathbf{u}'(t) + \mathbf{b}(t)\mathbf{u}(t) = \mathbf{c}(t), \\ \mathbf{u}(0) = \mathbf{u}(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{gathered} \tag{4.9}$$

where $\mathbf{a}(t) = \begin{pmatrix} -3t & -3 \\ -3 & 0 \end{pmatrix}$, $\mathbf{b}(t) = \begin{pmatrix} 9 & 0 \\ -t & 1 \end{pmatrix}$, and $\mathbf{c}(t)\begin{pmatrix} 9t^2-2 & -6t^2-6t+5 \\ -t^4-8t^2+2 & t^2-t+3 \end{pmatrix}$. By applying matrix differential transform operator listed in Table 1, on linear system (4.9), for $k = 0, 1, 2, \ldots, n$, we obtain

$$\begin{gathered} \frac{(k+2)!}{k!}\mathbf{U}(k+2) + \sum_{l=0}^{k}(l+1)\mathbf{A}(k-l)\mathbf{U}(l+1) + \sum_{l=0}^{k}\mathbf{B}(k-l)\mathbf{U}(l) = \mathbf{C}(k), \\ \mathbf{U}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sum_{k=0}^{n}\mathbf{U}(k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{gathered} \tag{4.10}$$

where $\mathbf{U}(k)$, $\mathbf{A}(k)$, $\mathbf{B}(k)$, and $\mathbf{C}(k)$ are the matrix differential transform of $\mathbf{u}(t)$, $\mathbf{a}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$, respectively. For $k = 0, 1, 2, \ldots$, (4.10) can be rewriten as in the following iterative equation:

$$\mathbf{U}(k+2) = \frac{-k!}{(k+2)!}\left\{\sum_{l=0}^{k}(l+1)\mathbf{A}(k-l)\mathbf{U}(l+1) - \sum_{l=0}^{k}\mathbf{B}(k-l)\mathbf{U}(l) - \mathbf{C}(k)\right\}. \tag{4.11}$$

Then from (4.11), for $k = 0, 1, 2$, we get

$$\begin{aligned}
\mathbf{U}(2) &= -\frac{1}{2}(\mathbf{A}(0)\mathbf{U}(1) + \mathbf{B}(0)\mathbf{U}(0) - \mathbf{C}(0)),\\
\mathbf{U}(3) &= -\frac{1}{6}(2\mathbf{A}(0)\mathbf{U}(2) + \mathbf{A}(1)\mathbf{U}(1) + \mathbf{B}(0)\mathbf{U}(1) + \mathbf{B}(1)\mathbf{U}(0) - \mathbf{C}(1)),\\
\mathbf{U}(4) &= -\frac{1}{12}(\mathbf{A}(0)\mathbf{U}(3) + 2\mathbf{A}(1)\mathbf{U}(2) + \mathbf{A}(2)\mathbf{U}(1) + \mathbf{B}(0)\mathbf{U}(2) + \mathbf{B}(1)\mathbf{U}(1)\\
&\qquad + \mathbf{B}(2)\mathbf{U}(0) - \mathbf{C}(2)).
\end{aligned} \tag{4.12}$$

In the same manner, the rest of components were obtained using the MAPLE Package. From boundary condition (4.10) we get

$$\mathbf{U}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sum_{k=0}^{n} \mathbf{U}(k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.13}$$

To obtain the remain coefficient component of $\mathbf{U}(k)$, for $k \geq 1$, it is enough to find $\mathbf{U}(1)$ from differential transform version of boundary conditions (4.13). Therefore, assume that $\mathbf{Y}(1) = \left(\begin{smallmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{smallmatrix}\right)$; then for $n = 3$, from $\sum_{k=0}^{3} \mathbf{U}(k) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we get

$$\begin{pmatrix} \frac{3}{2}a_{1,1} + \frac{3}{2}a_{2,1} & \frac{3}{2}a_{1,2} + \frac{3}{2}a_{2,2} + 3 \\ \frac{7}{3}a_{2,1} + \frac{3}{2}a_{1,1} & \frac{7}{3}a_{2,2} + \frac{3}{2}a_{1,2} + \frac{23}{6} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.14}$$

By solving (4.14), we get $\mathbf{U}(1) = \left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. Note that for $n \geq 3$, by solving $\sum_{k=0}^{n} \mathbf{U}(k) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we always get $\mathbf{U}(1) = \left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. By substituting $\mathbf{U}(0)$ and $\mathbf{U}(1)$ in list (4.12), we get $\mathbf{U}(2) = \left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{U}(3) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, and for $n \geq 4$, we have $\mathbf{U}(n) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore from inverse differential transform operator (2.7), the four-term approximation solutions is obtained as follows:

$$\mathbf{U}_4(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} t + \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} t^2 + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} t^3, \tag{4.15}$$

which is exactly the same as the exact solution $\mathbf{u}(t) = \left(\begin{smallmatrix} t^3-t^2 & t^2-t \\ -t^3+t^2 & t^3-t \end{smallmatrix}\right)$.

Example 4.4. Finally, we consider the following nonlinear second-order matrix boundary-value problem:

$$\begin{aligned}
\mathbf{u}''(t) - \mathbf{u}'(t)\mathbf{u}(t) - \mathbf{u}(t) &= \mathbf{c}(t),\\
\mathbf{u}(0) = \mathbf{u}(1) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad 0 \leq t \leq 1,
\end{aligned} \tag{4.16}$$

where $\mathbf{c}(t) = \begin{pmatrix} 2+4t^2-7t^3+3t^4 & -2+6t-2t^2+5t^3-3t^5 \\ -2+t-3t^2-t^3+3t^4 & 6t+t^2+2t^4-3t^5 \end{pmatrix}$. Then for $k = 0, 1, 2, \ldots, n$, we have

$$\frac{(k+2)!}{k!}\mathbf{U}(k+2) - \sum_{l=0}^{k}(k-l+1)\mathbf{U}(k-l+1)\mathbf{U}(l) - \mathbf{U}(k) = \mathbf{C}(k),$$
$$\mathbf{U}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sum_{k=0}^{n}\mathbf{U}(k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.17}$$

Therefore from (4.17), some of the first quantities are obtained as follows:

$$\mathbf{U}(2) = \frac{1}{2}\mathbf{U}(1)\mathbf{U}(0) + \frac{1}{2}\mathbf{U}(0) + \frac{1}{2}\mathbf{C}(0),$$
$$\mathbf{U}(3) = \frac{1}{3}\mathbf{U}(2)\mathbf{U}(0) + \frac{1}{6}\mathbf{U}(1)^2 + \frac{1}{6}\mathbf{U}(1) + \frac{1}{6}\mathbf{C}(1),$$
$$\mathbf{U}(4) = \frac{1}{4}\mathbf{U}(3)\mathbf{U}(0) + \frac{1}{4}\mathbf{U}(2)\mathbf{U}(1) + \frac{1}{12}\mathbf{U}(2) + \frac{1}{12}\mathbf{C}(2), \tag{4.18}$$
$$\mathbf{U}(5) = \frac{1}{5}\mathbf{U}(4)\mathbf{U}(0) + \frac{1}{5}\mathbf{U}(3)\mathbf{U}(1) + \frac{1}{10}\mathbf{U}(2)^2 + \frac{1}{20}\mathbf{U}(3) + \frac{1}{20}\mathbf{C}(3).$$

In the same manner, the rest of components were obtained using the MAPLE Package. Similar to the previous examples, assume that $\mathbf{U}(1) = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$; then by assuming the various $n = 3$, in the boundary condition (4.17), we get

$$\begin{pmatrix} 1 + \frac{7}{6}a_{1,1} + \frac{1}{6}a_{1,1}^2 + \frac{1}{6}a_{1,2}a_{2,1} & \frac{7}{6}a_{1,2} + \frac{1}{6}a_{1,1}a_{1,2} + \frac{1}{6}a_{1,2}a_{2,2} \\ -\frac{5}{6} + \frac{7}{6}a_{2,1} + \frac{1}{6}a_{1,1}a_{2,1} + \frac{1}{6}a_{2,1}a_{2,2} & 1 + \frac{7}{6}a_{2,2} + \frac{1}{6}a_{1,2}a_{2,1} + \frac{1}{6}a_{2,2}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.19}$$

From the *common* solution of (4.19), we obtain $\mathbf{U}(1) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, and by setting $n = 4$ we get

$$\begin{pmatrix} \frac{17}{12}a_{1,1} + \frac{17}{12} + \frac{1}{6}a_{1,1}^2 & \frac{4}{3}a_{1,2} + \frac{1}{6}a_{1,1}a_{1,2} + \frac{1}{6}a_{1,2}a_{2,2} \\ +\frac{1}{6}a_{1,2}a_{2,1} - \frac{1}{6}a_{2,1} - \frac{1}{12}a_{1,2} & -\frac{1}{6}a_{2,2} - \frac{1}{12}a_{1,1} - \frac{1}{4} \\ \frac{5}{4}a_{2,1} - \frac{7}{6} + \frac{1}{6}a_{2,1}a_{1,1} & \frac{7}{6}a_{2,2} + \frac{1}{6}a_{1,2}a_{2,1} + \frac{1}{6}a_{2,2}^2 \\ +\frac{1}{6}a_{2,2}a_{2,1} - \frac{1}{6}a_{1,1} - \frac{1}{12}a_{2,2} & +\frac{13}{12} - \frac{1}{6}a_{1,2} - \frac{1}{12}a_{2,1} \end{pmatrix} = [0]_{2\times 2}. \tag{4.20}$$

In this case also from the *common* solution of (4.20), we get $\mathbf{U}(1) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. By a sample computation for $n = 5, 6, \ldots$, it is easy to obtain that the *common* solution of $\sum_{k=0}^{n}\mathbf{U}(k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is $\mathbf{U}(1) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. By substituting the $\mathbf{U}(0)$ and $\mathbf{U}(1)$ in (4.18) and utilizing the obtained quantities coefficients $\mathbf{U}(:)$ in (2.7), the series form of solution of equation (4.16) is obtained as follows:

$$\mathbf{U}_4(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} t + \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} t^3. \quad (4.21)$$

The closed form of previous solution is $\mathbf{u}(t) = \begin{pmatrix} -t+t^2 & -t^2+t^3 \\ t-t^2 & -t+t^3 \end{pmatrix}$, which is exactly the same as the exact solution.

5. Conclusions

In this paper, we have shown that DTM can successfully be used for solving the linear and nonlinear second-order Matrix IVPs and Matrix BVPs. This method is simple and easy to use and solves the problem without any need for discretizing the variables. Also this method is useful for finding an accurate approximation of the exact solution. A symbolic calculation software package, Maple, is used in the derivations. The method gives rapidly converging series solutions. The accuracy of the obtained solution can be improved by taking more terms in the solution. In many cases, the series solutions obtained with DTM can be written in exact closed form. Further we note that DTM can also be applied to solve the nonlinear matrix differential Riccati equations (first and second kinds of Riccati matrix differential equations) where DTM technique provides a sequence of matrix functions which converges to the exact solution of the problem; see [19]. In fact the present work is the extension of [19].

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