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Research Article

On Oscillations of Solutions of Third-Order Dynamic Equation

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We are proving the new oscillation theorems for the solutions of third-order linear nonautonomous differential equation with complex coefficients. In the case of real coefficients we derive the oscillation criterion that is invariant with respect to the adjoint transformation. Our main tool is a new version of Levinson's asymptotic theorem.

1. Introduction

Consider an ordinary nonautonomous differential equation of the third order

$$Lv = v'''(t) - 3a_2(t)v''(t) + 6a_1(t)v'(t) + 2a_0(t)v(t) = 0 \tag{1.1}$$

with complex valued variable coefficients $a_0(t)$, $a_1(t)$, and $a_2(t)$.

A solution of (1.1) is said to be oscillatory if it has an infinite sequence of zeros in (t_0, ∞), and nonoscillatory, otherwise. Equation (1.1) is said to be *non-oscillatory* if all solutions are non-oscillatory and is said to be *oscillatory* if there exists at least one oscillatory solution.

Oscillation theorems for ordinary differential equation of the third order in the case of real variable coefficients have been studied in [1–7]. To the best of the author's knowledge, the oscillations of the solutions of nonautonomous third order equations with complex coefficients have not been studied yet.

Let $C^k(t_0,\infty)$ be the set of k times differentiable functions on (t_0,∞). By substitution $v(t)=u(t)e^{\int_{t_0}^{t} a_2(s)ds}$ equation (1.1) with $a_2(t)\in C^2(t_0,\infty)$ turns to the following equation:

$$Pu = u'''(t) + 3I_1(t)u'(t) + 2I_2(t)u(t) = 0, \tag{1.2}$$

where the functions $I_1(t)$ and $I_2(t)$ are given by

$$I_1(t) = 2a_1(t) + a_2'(t) - a_2^2(t), \qquad I_2(t) = a_0(t) + 3a_1(t)a_2(t) - a_2^3(t) + \frac{a_2''(t)}{2}. \tag{1.3}$$

If $0 < |e^{\int_{t_0}^{\infty} a_2(s)ds}| < \infty$ or

$$-\infty < \int_{t_0}^{\infty} \Re[a_2(s)]ds < \infty, \tag{1.4}$$

then the solutions of (1.1) and (1.2) have the same oscillation properties; that is, (1.1) is oscillatory if and only if (1.2) is oscillatory.

Define characteristic (Weierstrass) function of (1.2) depending on a phase function $\eta_j(t)$

$$\mathrm{Char}_j(t) = \mathrm{Char}(\eta_j) = e^{-\int_{t_0}^{t}\eta_j(s)ds} P\left(e^{\int_{t_0}^{t}\eta_j(s)ds}\right), \quad j = 1,2,3. \tag{1.5}$$

By direct calculations

$$\mathrm{Char}_j(t) = \eta_j''(t) + 3\eta_j'(t)\eta_j(t) + \eta_j^3 + 3\eta_j(t)I_1(t) + 2I_2(t), \quad j = 1,2,3. \tag{1.6}$$

To consider the case of complex coefficients, we are using asymptotic solutions of (1.2) in Euler form $u(t) = e^{\int_{t_0}^{t}\eta(s)ds}$ with phase functions $\eta_j(t)$, $j = 1,2,3$, that are approximate solutions of the characteristic equation $\mathrm{Char}(\eta_j) = 0$.

Theorem 1.1 (see [4]). *If Mammana's condition*

$$M(t) = I_2(t) - \frac{3I_1'(t)}{4} > 0 \quad \text{or} \quad M(t) < 0 \tag{1.7}$$

is satisfied except at isolated points at which $M(t)$ may vanish, then (1.2) *is oscillatory if and only if its adjoint is oscillatory.*

We will show that Mammana's condition (1.7) is connected with the dichotomy condition of Levinson, and it has a topological character (see condition (2.24) below).

Theorem 1.2 (see Lazer [5]). *Assume that conditions*

$$I_1(t) \le 0, \qquad I_2(t) > 0, \quad t > t_0, \tag{1.8}$$

$$\int_{t_0}^{\infty} \left(I_2(t) - (-I_1(t))^{3/2} \right) dt = \infty \tag{1.9}$$

are satisfied. Then (1.2) *with the real coefficients is oscillatory.*

The adjoint transformation $I_2(t) \to (3I_1'(t)/2) - I_2(t)$, or $M(t) \to -M(t)$ transforms (1.2) to its adjoint equation $-w'''(t) - 3I_1(t)w'(t) + (2I_2(t) - 3I_1'(t))w(t) = 0$. Note that condition (1.9) is not invariant with respect to the adjoint transformation $M(t) \to -M(t)$. In the case of real coefficients under some restrictions, we will give the criterion of oscillations of solutions of (1.2) that is invariant with respect to the adjoint transformation (see Theorem 2.9 below).

2. Main Theorems

Let $W[t,a,b] = a(t)b'(t) - a'(t)b(t)$ be the Wronskian of two differentiable functions $a(t)$ and $b(t)$. The following asymptotic theorem is proved by using Levinson's asymptotic theorem [8].

Theorem 2.1. *Assume that there exists complex-valued phase functions $\eta_j(t) \in C^2(t_0,\infty)$, $j = 1,2,3$, such that expressions $\Re[\eta_j(t) - \eta_k(t)]$, $k \le j$ do not change a sign, that is,*

$$\Re\left[\eta_j(t) - \eta_k(t)\right] \le 0 \quad \text{or} \quad \Re\left[\eta_j(t) - \eta_k(t)\right] \ge 0, \quad k,j = 1,2,3,\ k \le j,\ t > t_0, \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{\left(\left|\eta_{13}(t)\right| + \left|\eta_{23}(t)\right|\right)|\mathrm{Char}_2(t)|}{|G(t)|} dt < \infty, \qquad \eta_{jk}(t) = \eta_j(t) - \eta_k(t),\ k,j = 1,2,3, \tag{2.2}$$

$$\int_{t_0}^{\infty} \frac{\left(\left|\eta_{13}(t)\right|^2 + \left|\eta_{23}(t)\right|^2\right)|\mathrm{Char}_2(t) - \mathrm{Char}_3(t)|}{\left|\eta_{23}(t)G(t)\right|} dt < \infty, \tag{2.3}$$

where

$$G(t) = W\left[t,\eta_{12},\eta_{13}\right] - \eta_{12}(t)\eta_{13}(t)\eta_{23}(t), \qquad \frac{G'(t)}{G(t)} + \eta_1(t) + \eta_2(t) + \eta_3(t) = 0. \tag{2.4}$$

Then solutions of (1.2) *may be represented in the form*

$$u_k(t) = \sum_{j=1}^{3} \varphi_j(t)\left(\delta_{jk} + \varepsilon_{jk}(t)\right)C_j, \qquad \lim_{t\to\infty} \varepsilon_{jk}(t) = 0, \quad j,k = 1,2,3, \tag{2.5}$$

where $\mathrm{Char}_j(t)$, $j = 1,2,3$ *are defined in* (1.5), (1.6), *and*

$$\varphi_j(t) = e^{\int_{t_0}^{t} \eta_j(s)ds}, \qquad \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \ne k. \end{cases} \tag{2.6}$$

Note that conditions (2.2) and (2.3) are given in terms of characteristic functions.

We will say that (1.2) has asymptotic solutions $e^{\int_{t_0}^{t}\eta_j(s)ds}$ corresponding to the phase functions $\eta_j(t)\in C^2(t_0,\infty)$, $j=1,2,3$ if (2.1)–(2.3) are satisfied.

Theorem 2.2. *The solution of* (1.2) *corresponding to the asymptotic solution with the phase $\eta_k(t)$ is oscillatory if and only if*

$$\int_{t_0}^{\infty}\Im\left[\eta_k(t)\right]dt=\infty. \tag{2.7}$$

The following theorem we deduce from Theorem 2.2 by choosing

$$\eta_1(t)=-\frac{a'(t)}{a(t)},\qquad \eta_2(t)=a(t)-\frac{a'(t)}{a(t)},\qquad \eta_3(t)=-a(t)-\frac{a'(t)}{a(t)}. \tag{2.8}$$

Theorem 2.3. *Assume that there exists a complex-valued function $a(t)\in C^3(t_0,\infty)$ such that $a^{-1/2}(t)\in C^2(t_0,\infty)$, and*

$$\Re[a(t)] \textit{ does not change the sign on } (t_0,\infty), \tag{2.9}$$

$$\int_{t_0}^{\infty}\left|3I_1(t)+a^2(t)+4a^{1/2}(t)\left(a^{-1/2}(t)\right)''\right|\frac{dt}{|a(t)|}<\infty, \tag{2.10}$$

$$\int_{t_0}^{\infty}\left|2I_2(t)+3I_1(t)\left(a(t)-\frac{a'(t)}{a(t)}\right)+\frac{\left(a^{-1}(t)e^{\int_{t_0}^{t}a(s)ds}\right)'''}{a^{-1}(t)e^{\int_{t_0}^{t}a(s)ds}}\right|\frac{dt}{a^2(t)}<\infty. \tag{2.11}$$

Then (1.2) *with complex coefficients has one nonoscillatory solution and two linearly independent oscillatory solutions if and only if*

$$\int_{t_0}^{\infty}\Im[a(t)]dt=\infty,\quad \text{or}\quad \int_{t_0}^{\infty}\Im[a(t)]dt=-\infty. \tag{2.12}$$

By taking $a(t)=\lambda/t$ from Theorem 2.3, we get the following corollary.

Corollary 2.4. *Assume that for some complex number $\lambda\neq 0$*

$$\int_{t_0}^{\infty}t^2\left|2I_2(t)+\frac{1-\lambda^2}{t^3}\right|dt<\infty,\qquad \int_{t_0}^{\infty}t\left|3I_1(t)+\frac{\lambda^2-1}{t^2}\right|dt<\infty. \tag{2.13}$$

Then (1.2) *with complex coefficients has one non-oscillatory solution and two linearly independent oscillatory solutions if and only if*

$$\Im[\lambda]>0. \tag{2.14}$$

By taking $\lambda=1$ from Corollary 2.4, we obtain well-known result [4].

Corollary 2.5. *Assume that conditions*

$$\int_{t_0}^{\infty} t^2 |I_2(t)| dt < \infty, \qquad \int_{t_0}^{\infty} t |I_1(t)| dt < \infty \tag{2.15}$$

are satisfied. Then (1.2) *with complex coefficients is non-oscillatory.*

By taking $a(t) = -1/t \ln(t)$ from Theorem 2.3 we get another corollary.

Corollary 2.6. *Assume that conditions*

$$\int_{t_0}^{\infty} t^2 \ln^2(t) \left| 2I_2(t) + \frac{3I_1(t)}{t} \right| dt < \infty, \qquad \int_{t_0}^{\infty} t \ln(t) \left| 3I_1(t) - \frac{1}{t^2} \right| dt < \infty \tag{2.16}$$

are satisfied. Then (1.2) *with complex coefficients is non-oscillatory.*

Example 2.7. From Corollary 2.6, (1.2) with

$$I_1(t) = \frac{1}{3t^2}, \qquad I_2(t) = -\frac{1}{2t^3} \tag{2.17}$$

is non-oscillatory. Note that Corollary 2.5 is not applicable for this example since condition (2.15) fails.

In the case $M(t) \equiv 0$ by taking $a(t) = i\sqrt{3I_1(t)}$, from Theorem 2.3, we deduce the following theorem.

Theorem 2.8. *Assume that* $I_1(t) \in C^3(t_0, \infty)$, $I_1(t) \geq \beta > 0$, *and*

$$M(t) = I_2(t) - \frac{3I_1'(t)}{4} \equiv 0, \quad t \in (t_0, \infty), \quad \int_{t_0}^{\infty} \left| \left(I_1^{-1/4}(t) \right)'' I_1^{-1/4}(t) \right| dt < \infty, \tag{2.18}$$

$$\int_{t_0}^{\infty} \left| \left(\left(I_1^{-1/4}(t) \right)'' I_1^{-3/4}(t) \right)' \right| dt < \infty. \tag{2.19}$$

Then (1.2) *with the real coefficients has one non-oscillatory solution and two linearly independent oscillatory solutions if and only if*

$$\int_{t_0}^{\infty} \sqrt{I_1(t)} dt = \infty. \tag{2.20}$$

Another result may be proved by the different choice of the phase functions as follows:

$$\eta_j(t) = d^{1/3}(t) e^{-i\pi(2j+1)/3} - \frac{I_1(t)}{d^{1/3}(t)} e^{i\pi(2j+1)/3} - \frac{G'(t)}{3G(t)}, \quad j = 1, 2, 3, \tag{2.21}$$

where $G(t)$ is defined in (2.4), and

$$d(t) = M(t) + \sqrt{M^2(t) + I_1^3(t)}, \qquad M(t) = I_2(t) - \frac{3I_1'(t)}{4}. \tag{2.22}$$

Define 3 auxiliary regions on the real plane

$$\begin{aligned} R_1 &= \left\{(I_1, M) \in R^2 \mid I_1 \leq 0,\ M < 0,\ I_1^3 + M^2 < 0\right\}, \\ R_2 &= \left\{(I_1, M) \in R^2 \mid I_1 < 0,\ M \leq 0,\ I_1^3 + M^2 \geq 0\right\}, \\ R_3 &= R^2 \setminus (R_1 \cup R_2) = \left((I_1, M) \in R^2,\ M > 0,\ \text{or } I_1 > 0\right). \end{aligned} \tag{2.23}$$

Theorem 2.9. *Assume that $I_1(t) \in C^1(t_0, \infty), R_0$ is simply connected region $R_0 \subset R_j$ for some $j = 1, 2, 3$, and conditions* (2.2) *and* (2.3)*,*

$$(I_1(t), M(t)) \in R_0, \quad t > t_0 \tag{2.24}$$

are satisfied. Then (1.2) *has one non-oscillatory solution and two linearly independent oscillatory solutions if and only if*

$$\int_{t_0}^{\infty} \Re\left[\left(\sqrt{M^2(t) + I_1^3(t)} + M(t)\right)^{1/3} + \left(\sqrt{M^2(t) + I_1^3(t)} - M(t)\right)^{1/3}\right] dt = \infty \tag{2.25}$$

for at least one of cubic roots.

Note that condition (2.25) is invariant with respect to the adjoint transformation $M(t) \to -M(t)$.

For the case of the real constant coefficients $I_1(t) = I_1$, $I_2(t) = I_2$ from Theorem 2.9, one can deduce the obvious result that (1.2) is oscillatory if and only if $I_1^3 + I_2^2 > 0$. Indeed in this case condition (2.25) turns to $I_1^3 + I_2^2 > 0$, and conditions (2.2), (2.3), and (2.24) could be dropped.

Remark 2.10. Levinson's dichotomy condition (2.24) is satisfied if the modified Mammana's condition is satisfied as follows:

$$M(t) \geq 0, \quad \text{or} \quad M(t) \leq 0, \quad I_1(t) \geq 0, \quad t > t_0. \tag{2.26}$$

If $I_1(t) \geq 0$, then $M(t)$ under condition (2.24) may change the sign.

Theorem 2.9 does not exclude the case $I_2(t) < 0$, but Theorem 1.2 does. In the case $I_1(t) \equiv 0$ conditions of Theorem 2.9 are simplified.

Theorem 2.11. *Assume that $I_2(t)$ is real, it does not change the sign, and conditions*

$$\int_{t_0}^{\infty} \left|\left(I_2^{-1/3}\right)'''(t) I_2^{-1/3}\right| dt < \infty, \qquad \int_{t_0}^{\infty} \left|\left(I_2^{-1/6}\right)''(t) I_2^{-1/6}\right| dt < \infty \tag{2.27}$$

are satisfied. Then equation

$$u'''(t) + 2I_2(t)u(t) = 0 \tag{2.28}$$

has one non-oscillatory solution and two linearly independent oscillatory solutions if and only if

$$\int_{t_0}^{\infty} |I_2(t)|^{1/3} dt = \infty. \tag{2.29}$$

In the case $M(t) = 0$ (self-adjoint equation (1.2)), condition (2.25) turns to (2.20) (see Theorem 2.8 above), which is Leighton's (see [9]) necessary condition of oscillations for solutions of the second-order equation $u''(t) + I_1(t)u(t) = 0$.

Example 2.12. Equation

$$u'''(t) + \left(\frac{1}{t^3} + \frac{1-(i+\mu)^2}{t^2}\right)u'(t) + \left(\frac{1}{t^4} + \frac{(i+\mu)^2 - 1}{t^3}\right)u(t) = 0, \tag{2.30}$$

where μ is a real number and is oscillatory by Corollary 2.4 since conditions (2.13) and (2.14) are satisfied with $\lambda = i+\mu$. Note that for this example Theorem 1.2 is not applicable since both conditions (1.8) and (1.9) fail even when $\mu = 0$.

3. Proofs

Our main tool is Levinson's asymptotic theorem.

Theorem 3.1 (see [8]). *Let $\Lambda(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t))$ be an $n \times n$ diagonal matrix function which satisfies dichotomy condition.*

For each pair of integers i and j in $[1, n]$ $(i \neq j)$ exist constants K_1, K_2 such that for all x and t, $t_0 \le t \le x < \infty$

$$\int_t^x \Re[\lambda_i(s) - \lambda_j(s)]ds \le K_1, \quad \text{or} \quad \int_t^x \Re[\lambda_i(s) - \lambda_j(s)]ds \ge K_2. \tag{3.1}$$

Let the $n \times n$ matrix $N(t)$ satisfy $N(t) \in L_1(t_0, \infty)$ or

$$\int_t^x |N(t)|ds < \infty, \tag{3.2}$$

by which we mean that each entry in $N(t)$ has an absolutely convergent infinite integral. Then the system

$$Y'(t) = (\Lambda(t) + N(t))Y(t) \tag{3.3}$$

has a vector solution $Y(t)$ with the asymptotic form

$$Y(t) = (E + \varepsilon(t)) e^{\int_{t_0}^{t} \Lambda(s) ds} C, \qquad \lim_{t\to\infty} \varepsilon(t) = 0, \tag{3.4}$$

where E is the identity matrix, $\varepsilon(t)$ is the $n \times n$ error matrix-function, and $C = (C_1, \dots C_n)^{\mathrm{tr}}$ is a constant column vector.

Proof of Theorem 2.1. Rewrite (1.2) as a system

$$\begin{gathered} y'(t) = A(t) y(t), \\ A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2I_2(t) & -3I_1(t) & 0 \end{pmatrix}, \quad y(t) = \begin{pmatrix} u(t) \\ u'(t) \\ u''(t) \end{pmatrix}. \end{gathered} \tag{3.5}$$

By transformation

$$y(t) = \Phi(t) z(t), \tag{3.6}$$

where matrix function $\Phi(t)$ is defined via phase functions $\eta_j(t)$ as follows:

$$\Phi(t) = \begin{pmatrix} \dfrac{e^{\int_{t_0}^{t} \eta_1(s) ds}}{\mu_1(t)} & \dfrac{e^{\int_{t_0}^{t} \eta_2(s) ds}}{\mu_2(t)} & \dfrac{e^{\int_{t_0}^{t} \eta_3(s) ds}}{\mu_3(t)} \\ \dfrac{\eta_1(t)}{\mu_1(t)} e^{\int_{t_0}^{t} \eta_1(s) ds} & \dfrac{\eta_2(t)}{\mu_2(t)} e^{\int_{t_0}^{t} \eta_2(s) ds} & \dfrac{\eta_3(t)}{\mu_3(t)} e^{\int_{t_0}^{t} \eta_3(s) ds} \\ \dfrac{\left(\eta_1'(t) + \eta_1^2(t)\right)}{\mu_1(t)} e^{\int_{t_0}^{t} \eta_1(s) ds} & \dfrac{\left(\eta_2'(t) + \eta_2^2(t)\right)}{\mu_2(t)} e^{\int_{t_0}^{t} \eta_2(s) ds} & \dfrac{\left(\eta_3'(t) + \eta_3^2(t)\right)}{\mu_3(t)} e^{\int_{t_0}^{t} \eta_3 ds} \end{pmatrix}, \tag{3.7}$$

we get the following:

$$z'(t) = \Phi^{-1}(t) \left(A(t) \Phi(t) - \Phi'(t) \right) z(t), \quad \text{or} \quad z'(t) = (D(t) + B(t)) z(t), \tag{3.8}$$

where

$$B(t)=\frac{1}{G(t)}\begin{pmatrix}\eta_{23}(t)\mathrm{Char}_1(t) & \dfrac{\mu_1\eta_{23}\mathrm{Char}_2(t)}{\mu_2(t)}e^{\int_{t_0}^{t}\eta_{21}ds} & \dfrac{\mu_1\eta_{23}\mathrm{Char}_3(t)}{\mu_3(t)}e^{\int_{t_0}^{t}\eta_{31}ds}\\ \dfrac{\mu_2\eta_{31}\mathrm{Char}_1(t)}{\mu_1(t)}e^{\int_{t_0}^{t}\eta_{12}ds} & \eta_{31}(t)\mathrm{Char}_2(t) & \dfrac{\mu_2\eta_{31}\mathrm{Char}_3(t)}{\mu_3(t)}e^{\int_{t_0}^{t}\eta_{32}ds}\\ \dfrac{\mu_3\eta_{12}\mathrm{Char}_1(t)}{\mu_1(t)}e^{\int_{t_0}^{t}\eta_{13}ds} & \dfrac{\mu_3\eta_{12}\mathrm{Char}_2(t)}{\mu_2(t)}e^{\int_{t_0}^{t}\eta_{23}ds} & \eta_{12}(t)\mathrm{Char}_3(t)\end{pmatrix},$$

$$D(t)=\begin{pmatrix}\dfrac{\mu_1'(t)}{\mu_1(t)} & 0 & 0\\ 0 & \dfrac{\mu_2'(t)}{\mu_2(t)} & 0\\ 0 & 0 & \dfrac{\mu_3'(t)}{\mu_3(t)}\end{pmatrix},\qquad \eta_{jk}(t)=\eta_j(t)-\eta_k(t). \tag{3.9}$$

Choosing specific auxiliary functions

$$\mu_1(t)=1,\qquad \mu_2(t)=e^{\int_{t_0}^{t}\eta_{21}(s)ds},\qquad \mu_3(t)=e^{\int_{t_0}^{t}\eta_{31}(s)ds}, \tag{3.10}$$

we have

$$D(t)=\begin{pmatrix}0&0&0\\0&\eta_{21}&0\\0&0&\eta_{31}\end{pmatrix},\qquad B(t)=\frac{1}{G(t)}\begin{pmatrix}\eta_{32}\mathrm{Char}_1 & \eta_{23}\mathrm{Char}_2 & \eta_{23}\mathrm{Char}_3\\ \eta_{31}\mathrm{Char}_1 & \eta_{31}\mathrm{Char}_2 & \eta_{31}\mathrm{Char}_3\\ \eta_{12}\mathrm{Char}_1 & \eta_{12}\mathrm{Char}_2 & \eta_{12}\mathrm{Char}_3\end{pmatrix}. \tag{3.11}$$

Here and further we suppress the time variable t for the simplicity.

From Liouville's formula $\det\Phi(t) = Ce^{\int_{t_0}^{t}\mathrm{Tr}(A(s))ds} = C$ applied to (3.7) with the assumption that $\eta_j(t)$ are solutions of $\mathrm{Char}_j(t)=0$, $\mu_1=\mu_2=\mu_3=1$, we get the following

$$C=\det(\Phi(t))=G(t)e^{\int_{t_0}^{t}(\eta_1+\eta_2+\eta_3)(s)ds}. \tag{3.12}$$

The Liouville's formula may be written in the form

$$\frac{G'(t)}{G(t)}+\eta_1(t)+\eta_2(t)+\eta_3(t)=0. \tag{3.13}$$

We always are choosing the phase functions $\eta_j(t)$ such that (3.13) is satisfied (see (2.4)). From (3.13), we get that

$$\frac{(\eta_1-\eta_3)\mathrm{Char}_{12}(t)-(\eta_1-\eta_2)\mathrm{Char}_{13}(t)}{G(t)}=-\eta_1(t)-\eta_2(t)-\eta_3(t)-\frac{G'(t)}{G(t)}, \tag{3.14}$$

$$\frac{\mathrm{Char}_1(t)-\mathrm{Char}_2(t)}{\eta_1(t)-\eta_2(t)}=\frac{\mathrm{Char}_1(t)-\mathrm{Char}_3(t)}{\eta_1(t)-\eta_3(t)}=\frac{\mathrm{Char}_2(t)-\mathrm{Char}_3(t)}{\eta_2(t)-\eta_3(t)}. \tag{3.15}$$

To apply Theorem 3.1 to system (3.8) note that from (2.1) it follows dichotomy condition (3.1) of Theorem 3.1:

$$\int_t^x \Re[\eta_{kj}]\,ds \geq 0, \quad \text{or} \quad \int_t^x \Re[\eta_{kj}]\,ds \leq 0, \quad x \geq t,\ j \neq k,\ k, j = 1, 2, 3. \tag{3.16}$$

Condition (3.2) of Theorem 3.1 turns to $B(t) \in L_1(t_0, \infty)$, and it is followed from

$$\frac{\eta_{13}(t)\mathrm{Char}_j(t)}{G(t)}, \frac{\eta_{23}(t)\mathrm{Char}_j(t)}{G(t)} \in L_1(t_0, \infty), \quad j = 1, 2, 3. \tag{3.17}$$

One can drop condition $\eta_{23}(t)\mathrm{Char}_1(t)/G(t) \in L_1(t_0, \infty)$ since from (3.15) we have

$$\frac{\eta_{23}(t)\mathrm{Char}_1(t)}{G(t)} = \frac{\eta_{13}(t)\mathrm{Char}_2(t)}{G(t)} - \frac{\eta_{12}\mathrm{Char}_3}{G}. \tag{3.18}$$

Assuming that

$$\frac{\eta_{13}^2(t)(\mathrm{Char}_2(t) - \mathrm{Char}_3(t))}{\eta_{23}(t)G(t)} \in L_1(t_0, \infty), \tag{3.19}$$

condition $\eta_{13}\mathrm{Char}_1/G \in L_1(t_0, \infty)$ may be dropped as well since

$$\frac{\eta_{13}\mathrm{Char}_1}{G} = \frac{\eta_{13}(\eta_{13}\mathrm{Char}_2 - \eta_{12}\mathrm{Char}_3)}{G\eta_{23}} = \frac{\eta_{13}(\eta_{13}(\mathrm{Char}_2 - \mathrm{Char}_3) + \eta_{23}\mathrm{Char}_3)}{\eta_{23}G}. \tag{3.20}$$

So condition (3.2) of Theorem 3.1 turns to

$$\frac{(|\eta_{23}(t)| + |\eta_{13}(t)|)|\mathrm{Char}_j(t)|}{|G(t)|}, \frac{|\eta_{13}(t)|^2|\mathrm{Char}_2 - \mathrm{Char}_3(t)|}{|\eta_{23}(t)G(t)|} \in L_1(t_0, \infty), \quad j = 2, 3, \tag{3.21}$$

or (2.2) and (2.3). From Theorem 3.1 applied to system (3.8) and we get that

$$\begin{gathered} z(t) = z_0(t)(E + \varepsilon(t))C, \qquad z_0(t) = e^{\int_{t_0}^t D(s)ds} = \begin{pmatrix} \mu_1(t) & 0 & 0 \\ 0 & \mu_2(t) & 0 \\ 0 & 0 & \mu_3(t) \end{pmatrix} C_1, \\ y(t) = \Phi(t)z(t) = \Phi(t)z_0(t)(E + \varepsilon(t))C, \end{gathered} \tag{3.22}$$

or representation (2.5). □

Proof of Theorem 2.2. Theorem 2.2 is followed from Theorem 2.1 since in representation (2.5) one may choose asymptotic solutions as follows:

$$\varphi_2(t) = e^{\int_{t_0}^t \Re[\eta_2(s)]ds} \sin \int_{t_0}^t \Im[\eta_2(s)]\,ds\,ds, \qquad \varphi_3(t) = e^{\int_{t_0}^t \Re[\eta_2(s)]ds} \cos \int_{t_0}^t \Im[\eta_2(s)]\,ds \tag{3.23}$$

which are oscillating if and only if condition (2.7) is satisfied. □

Proof of Theorem 2.3. Theorem 2.3 is deduced from Theorem 2.2 by choosing phase functions as in (2.8). From

$$2\Re[\eta_{21}] = 2\Re[\eta_{13}] = \Re[\eta_{23}] = 2\Re[a(t)], \tag{3.24}$$

condition (2.1) turns to condition (2.9). From (2.4), we get that

$$G(t) = 2a^3(t), \qquad \eta_1 + \eta_2 + \eta_3 + \frac{G'(t)}{G(t)} = 0. \tag{3.25}$$

Since in conditions (2.2)-(2.3) the function $G(t)$ appears in denominator we should assume that $G(t) \neq 0$, or $a(t) \neq 0$. By direct calculations, we get that

$$\begin{gathered}\frac{2\eta_{13}\mathrm{Char}_2}{G} = \frac{1}{a^2}\left(2I_2 + 3I_1\left(a - \frac{a'}{a}\right) + \frac{\left(a^{-1}(t)e^{\int_{t_0}^{t} a(s)ds}\right)'''}{a^{-1}(t)e^{\int_{t_0}^{t} a(s)ds}}\right), \\ \frac{\eta_{13}(\mathrm{Char}_2 - \mathrm{Char}_3)}{G} = \frac{3I_1(t)}{a(t)} + a(t) + 4\left(a^{-1/2}\right)'' a^{-1/2}.\end{gathered} \tag{3.26}$$

In view of

$$\frac{2\eta_{13}}{G} = \frac{\eta_{23}}{G} = \frac{4\eta_{13}^2}{\eta_{23}G} = \frac{1}{a^2}, \tag{3.27}$$

conditions (2.2) and (2.3) of Theorem 2.2 turn to (2.10) and (2.11).

Further the asymptotic solution

$$e^{\int_{t_0}^{t}\eta_2(s)ds} + e^{\int_{t_0}^{t}\eta_3(s)ds} = \frac{2}{a(t)}e^{\int_{t_0}^{t}\Re[a(s)]ds}\cos\int_{t_0}^{t}\Im[a(s)]ds \tag{3.28}$$

is oscillating if and only if (2.12) is satisfied. Indeed, the solution corresponding to the asymptotic solution $e^{\int_{t_0}^{t}\eta_1 ds} = C/a(t)$ is non-oscillatory ($1/a(t)$ does not have zeros; otherwise $a(t) \in C^3(t_0, \infty)$ is undefined at some points). □

Proof of Corollaries 2.4 and 2.6. We deduce Corollaries 2.4 and 2.6 from Theorem 2.3 by the special choice of function $a(t)$ as follows:

$$a(t) = \frac{\lambda}{t\ln^{\gamma}(t)}, \quad \lambda \neq 0. \tag{3.29}$$

From (2.10) and (2.11) we get that

$$t\ln^{\gamma}(t)\left|3I_1+\frac{\gamma(\gamma-2)}{t^2\ln^2(t)}+\frac{\lambda^2}{t^2\ln^{2\gamma}(t)}-\frac{1}{t^2}\right|\in L_1(t_0,\infty),$$
$$rlt^2\ln^{2\gamma}(t)\left|2I_2+\frac{3I_1}{t}\left(1+\frac{\gamma}{\ln(t)}+\frac{\lambda}{\ln^{\gamma}(t)}\right)+\frac{\gamma(\gamma-1)(\gamma-2)}{t^3\ln^3(t)}\right.$$
$$\left.+\frac{\lambda\gamma(\gamma-2)}{t^3\ln^{2+\gamma}(t)}-\frac{\gamma}{t^3\ \ln(t)}+\frac{\lambda^3}{t^3\ln^{3\gamma}(t)}-\frac{\lambda}{t^3\ln^{\gamma}(t)}\right|\in L_1(t_0,\infty), \tag{3.30}$$

or in the case $\gamma=0$

$$\int_{t_0}^{\infty}t\left(3I_1(t)+\frac{\lambda^2-1}{t^2}\right)dt<\infty,$$
$$\int_{t_0}^{\infty}t^2\left|2I_2(t)+\frac{3(1+\lambda)I_1(t)}{t}+\frac{\lambda(\lambda^2-1)}{t^3}\right|dt<\infty, \tag{3.31}$$

which is equivalent to (2.13).

Further from (2.12) we get condition (2.14) in the case $\gamma=0$:

$$\int_{t_0}^{\infty}\frac{\Im[\lambda]dt}{t\ln^{\gamma}(t)}=\infty,\quad \text{or}\quad \Im[\lambda]>0,\quad \gamma=0. \tag{3.32}$$

The proof of Corollary 2.6 is followed from (3.30) to (3.27) by choosing $\gamma=-\lambda=1$. □

Proof of Theorem 2.8. Theorem 2.8 is followed from Theorem 2.3. Indeed from $M(t)\equiv 0, t>t_0$, $a(t)=i\sqrt{3I_1(t)}$, we have $2I_2(t)=-a'(t)a(t)$. Condition (2.10) turns to (2.18) as follows:

$$\frac{3I_1+a^2}{a}+4\left(a^{-1/2}\right)''a^{-1/2}=Q(t)\in L_1(t_0,\infty),\qquad Q(t)=4\left(a^{-1/2}\right)''a^{-1/2}, \tag{3.33}$$

and condition (2.11) turns to (2.19) since

$$\frac{1}{a^2}\left(2I_2+3I_1\left(a-\frac{a'(t)}{a}\right)+\frac{v'''(t)}{v}\right)=\frac{v'''(t)}{a^2v}-a=\left(\frac{Q(t)}{2a(t)}\right)'+Q(t), \tag{3.34}$$

where

$$v(t)=a^{-1}(t)e^{\int_{t_0}^{t}a(s)ds},\qquad Q(t)=4\left(a^{-1/2}\right)''a^{-1/2}=4\left(I_1^{-1/4}\right)''I_1^{-1/4}. \tag{3.35}$$

□

Proof of Theorem 2.9. Let choose the phase functions $\eta_j(t)$ as in (2.21). We deduce Theorem 2.9 from Theorem 2.2. By calculations

$$G(t) = \frac{\sqrt{3}\left(3I_1^3 + 3d^2 + 3dI_1' - 2d'I_1\right)}{id} = -i3\sqrt{3}d\left[1 + \frac{I_1^3}{d^2} + \frac{I_1}{3d}\left(\ln\frac{I_1^3}{d^2}\right)'\right], \tag{3.36}$$

$$\eta_{12} = \frac{3}{2}\left(\frac{I_1}{d^{1/3}} - d^{1/3}\right) - \frac{i\sqrt{3}}{2}\left(\frac{I_1}{d^{1/3}} + d^{1/3}\right), \qquad \eta_{23} = i\sqrt{3}\left(\frac{I_1}{d^{1/3}} + d^{1/3}\right), \tag{3.37}$$

to deduce dichotomy conditions (2.1) from (2.24), it is enough to show that

$$\begin{aligned} \Re[\eta_2 - \eta_3] &> 0, \quad \text{if } (I_1, M) \in R_1, \\ \Re[\eta_2 - \eta_3] &\leq 0, \quad \text{if } (I_1, M) \in R^2 \setminus R_1, \\ \Re[\eta_1 - \eta_2] &> 0, \quad \text{if } (I_1, M) \in R_2, \\ \Re[\eta_1 - \eta_2] &\leq 0, \quad \text{if } (I_1, M) \in R^2 \setminus R_2. \end{aligned} \tag{3.38}$$

□

Case 1 ($I_1^3(t) + M^2(t) \geq 0, t > t_0$). In this case d and η_1 are real, $\eta_{2,3}$ are complex conjugate and from (3.37)

$$\Re[\eta_1 - \eta_2] = \Re[\eta_1 - \eta_3] = \frac{3}{2}\left(\frac{I_1}{d^{1/3}} - d^{1/3}\right), \qquad \Re[\eta_2 - \eta_3] = 0. \tag{3.39}$$

If $I_1 < 0, M \leq 0$, then $d = M + \sqrt{M^2 + I_1^3} < M + |M| = 0$, $\Re[\eta_1 - \eta_2] > 0$.

If $I_1 \leq 0, M > 0$, then $d = M + \sqrt{M^2 + I_1^3} \geq M > 0$, $\Re[\eta_1 - \eta_2] < 0$.

Otherwise if $I_1 > 0$, then $d > M + \sqrt{M^2} \geq 0$, $d^{1/3} > 0$ and

$$\begin{gathered} d^2 = \left(M + \sqrt{M^2 + I_1^3}\right)^2 = I_1^3 + 2M^2\left(1 + \sqrt{1 + \frac{I_1^3}{M^2}}\right) \geq I_1^3, \quad d^{2/3} \geq I_1 > 0, \\ \Re[\eta_1 - \eta_2] = \frac{3}{2}\left(\frac{I_1}{d^{1/3}} - d^{1/3}\right) \leq 0. \end{gathered} \tag{3.40}$$

Further condition (2.7) turns to (2.25) since $\Im[\eta_2] = (1/2)\Im[\eta_2 - \eta_3]$.

Case 2 ($I_1^3(t)+M^2(t)<0, t>t_0$). In this case $d=M+i\sqrt{-I_1^3-M^2}$ is complex valued, $\eta_{1,2,3}, G$ are real. In view of

$$\begin{gathered}
d\overline{d}=\left(M+\sqrt{M^2+I_1^3}\right)\left(M-\sqrt{M^2+I_1^3}\right)=-I_1^3, \\
d^{1/3}\left(\overline{d}\right)^{1/3}=-I_1, \quad \overline{d}^{1/3}=-\frac{I_1}{d^{1/3}}, \quad \left(re^{-ia}\right)^{1/3}=\overline{r^{1/3}e^{ia/3}}, \quad \overline{d}^{1/3}=\overline{d^{1/3}},
\end{gathered} \tag{3.41}$$

we get that

$$\begin{gathered}
\eta_2=e^{i\pi/3}d^{1/3}-\frac{e^{-i\pi/3}I_1}{d^{1/3}}-\frac{G'(t)}{3G(t)}=2\Re\left[e^{i\pi/3}d^{1/3}\right]-\frac{G'(t)}{3G(t)}, \qquad \Im\left[\eta_2\right]=0, \\
\eta_1=-2\Re\left[d^{1/3}\right]-\frac{G'(t)}{3G(t)}, \qquad \eta_3=2\Re\left[e^{-i\pi/3}d^{1/3}\right]-\frac{G'(t)}{3G(t)}.
\end{gathered} \tag{3.42}$$

Condition (2.25) fails in this case: $\Re[d^{1/3}-\overline{d}^{1/3}]=0$ and (1.2) is non-oscillatory. Further

$$\begin{gathered}
d=M+i\sqrt{-M^2-I_1^3}=re^{ia}, \qquad \tan(a)=\frac{\sqrt{-M^2-I_1^3}}{M}, \quad a\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \\
\eta_1-\eta_2=\sqrt{3}\left(\Im\left[d^{1/3}\right]-\sqrt{3}\Re\left[d^{1/3}\right]\right)=3\Im\left[d^{1/3}\right]\left(\frac{1}{\sqrt{3}}-\cot\left(\frac{a}{3}\right)\right), \\
\eta_1-\eta_3=-\sqrt{3}\left(\Im\left[d^{1/3}\right]+\sqrt{3}\Re\left[d^{1/3}\right]\right)=-3\Im\left[d^{1/3}\right]\left(\frac{1}{\sqrt{3}}+\cot\left(\frac{a}{3}\right)\right), \\
\eta_2-\eta_3=-2\sqrt{3}\Im\left[d^{1/3}\right]=-2\sqrt{3}r^{1/3}\sin\left(\frac{a}{3}\right).
\end{gathered} \tag{3.43}$$

If $M>0$ then $\tan(a)>0$, $0<a<\pi/2$, $a=\tan^{-1}\sqrt{-1-(I_1^3/M^2)}$, and we get the following:

$$\begin{gathered}
0<\frac{a}{3}<\frac{\pi}{6}, \quad \sin\left(\frac{a}{3}\right)>0, \quad \cos\left(\frac{a}{3}\right)>0, \quad -\infty<1+\frac{I_1^3}{M^2}\le 0, \\
\Im\left[d^{1/3}\right]=r^{1/3}\sin\left(\frac{a}{3}\right)>0, \quad \cot\left(\frac{a}{3}\right)>\frac{1}{\sqrt{3}}, \\
\eta_1-\eta_2<0, \quad \eta_1-\eta_3<0, \quad \eta_2-\eta_3<0.
\end{gathered} \tag{3.44}$$

If $M < 0$ then $\tan(a) < 0$, $-\pi/2 < a < 0$, $a = -\tan^{-1}\sqrt{-1-(I_1^3/M^2)}$, and we get the following:

$$-\frac{\pi}{6} < \frac{a}{3} < 0, \quad \sin\left(\frac{a}{3}\right) < 0, \quad \cos\left(\frac{a}{3}\right) > 0,$$
$$\Im\left[d^{1/3}\right] < 0, \quad \cot\left(\frac{a}{3}\right) < -\frac{1}{\sqrt{3}},$$
$$\eta_1 - \eta_2 < 0, \quad \eta_1 - \eta_3 < 0, \quad \eta_2 - \eta_3 > 0. \tag{3.45}$$

Proof of Theorem 2.11. By taking $I_1(t) \equiv 0$, we get the following:

$$d(t) = 2I_2(t), \quad G(t) = -6i\sqrt{3}I_2(t), \quad \eta_1(t) = -d^{1/3} - \frac{G'}{3G} = -d^{1/3} - \frac{d'}{3d}$$
$$\eta_{2,3}(t) = e^{\pm i\pi/3}d^{1/3} - \frac{d'}{3d} = \frac{1 \pm i\sqrt{3}}{2}d^{1/3} - \frac{d'}{3d}, \quad \Im\left[\eta_{2,3}\right] = \pm\frac{\sqrt{3}}{2}d^{1/3}, \tag{3.46}$$

and Theorem 2.9 turns to Theorem 2.11. □

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