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Research Article

Fixed Points and Generalized Hyers-Ulam Stability

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In this paper we prove a fixed-point theorem for a class of operators with suitable properties, in very general conditions. Also, we show that some recent fixed-points results in Brzdęk et al., (2011) and Brzdęk and Ciepliński (2011) can be obtained directly from our theorem. Moreover, an affirmative answer to the open problem of Brzdęk and Ciepliński (2011) is given. Several corollaries, obtained directly from our main result, show that this is a useful tool for proving properties of generalized Hyers-Ulam stability for some functional equations in a single variable.

1. Introduction

The study of functional equations stability originated from a question of Ulam [1], concerning the stability of group homomorphisms. In 1941 Hyers [2] gave an affirmative answer to the question of Ulam for Cauchy equation in Banach spaces. The Hyers result was generalized by Aoki [3] for additive mappings and independently by Rassias [4] for linear mappings, by considering the unbounded Cauchy differences. A further generalization was obtained by Găvruţa [5] in 1994, by replacing the Cauchy differences by a control mapping φ, in the spirit of Rassias approach. See also [6] for more generalizations. We mention that the proofs of the results in the above mentioned papers used the direct method (of Hyers): the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution. We refer the reader to the expository papers [7, 8] and to the books [9–11] (see also the papers [12–17], for supplementary details).

On the other hand, in 1991 Baker [18] used the Banach fixed-point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu [19] proposed a new method, successively developed in [20], to obtaining the existence of the exact solutions and the error estimations, based on the fixed-point alternative. Concerning the stability of some functionals equations in a single variable, we mention the articles of Cădariu and Radu [21], of Miheţ [22], which applied the Luxemburg-Jung fixed-point theorem in generalized metric spaces, as well as the paper of Găvruţa [23] which used the Matkowski's fixed-point theorem. Also, Găvruţa introduced a new method in [24], called the *weighted space method*, for the generalized Hyers-Ulam stability (see, also [25]). It is worth noting that two fixed-point alternatives together with the error estimations for generalized contractions of type Bianchini-Grandolfi and Matkowski are pointed out by Cădariu and Radu, and then used as fundamental tools for proving stability of Cauchy functional equation in β-normed spaces [26], as well as of the monomial functional equation [27]. We also mention the new survey of Ciepliński [28], where some applications of different fixed-point theorems to the theory of the Hyers-Ulam stability of functional equations are presented.

Very recently, Brzdęk et al. proved in [29] a fixed-point theorem for (not necessarily) linear operators and they used it to obtain Hyers-Ulam stability results for a class of functional equations in a single variable. A fixed-point result of the same type was proved by Brzdęk and Ciepliński [30], in complete non-Archimedean metric spaces as well as in complete metric spaces. Also, they formulated an open problem concerning the uniqueness of the fixed point of the operator $\mathcal{T}$, which will be defined in the next section.

Our principal purpose is to obtain a fixed point theorem for a class of operators with suitable properties, in very general conditions. After that, we will show that some recent results in [29, 30] can be obtained as particular cases of our theorem. Moreover, by using our outcome, we will give an affirmative answer to the open problem of Brzdęk and Ciepliński, posed in the end of the paper [30]. Finally, we will show that main Theorem 2.2 is an efficient tool for proving generalized Hyers-Ulam stability results of several functional equations in a single variable.

2. Results

We consider a nonempty set X, a complete metric space (Y, d), and the mappings $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ and $\mathcal{T} : Y^X \to Y^X$. We recall that Y^X is the space of all mappings from X into Y.

Definition 2.1. One says that $\mathcal{T}$ is Λ-contractive if for $u, v : X \to Y$ and $\delta \in \mathbb{R}_+^X$ with

$$d(u(t), v(t)) \leq \delta(t), \quad \forall t \in X, \tag{2.1}$$

it follows

$$d((\mathcal{T}u)(t), (\mathcal{T}v)(t)) \leq (\Lambda\delta)(t), \quad \forall t \in X. \tag{2.2}$$

In the following, we assume that Λ satisfies the condition:

(C_1) for every sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}_+^X$ and every $t \in X$,

$$\lim_{n\to\infty} \delta_n(t) = 0 \Longrightarrow \lim_{n\to\infty} (\Lambda\delta_n)(t) = 0. \tag{2.3}$$

Also, we suppose that $\varepsilon \in \mathbb{R}_+^X$ is a given function such that

(C_2)

$$\varepsilon^*(t) := \sum_{k=0}^{\infty}\left(\Lambda^k\varepsilon\right)(t) < \infty, \quad t \in X. \tag{2.4}$$

Theorem 2.2. *One supposes that the operator $\mathcal{T}$ is Λ-contractive and the conditions (C_1) and (C_2) hold. One considers a mapping $f \in Y^X$ such that*

$$d\big((\mathcal{T}f)(t), f(t)\big) \le \varepsilon(t), \quad \forall t \in X. \tag{2.5}$$

Then, for every $t \in X$, the limit

$$g(t) := \lim_{n\to\infty}(\mathcal{T}^n f)(t) \tag{2.6}$$

exists and the mapping g is the unique fixed point of $\mathcal{T}$ with the property

$$d\big((\mathcal{T}^m f)(t), g(t)\big) \le \sum_{k=m}^{\infty}\left(\Lambda^k\varepsilon\right)(t), \quad t \in X,\ m \in \mathbb{N} = \{0,1,2,\dots\}. \tag{2.7}$$

Moreover, if one has

(C_3)

$$\lim_{n\to\infty}(\Lambda^n\varepsilon^*)(t) = 0, \quad \forall t \in X, \tag{2.8}$$

then g is the unique fixed point of $\mathcal{T}$ with the property

$$d\big(f(t), g(t)\big) \le \varepsilon^*(t), \quad \forall t \in X. \tag{2.9}$$

Proof. We have

$$d\Big(\left(\mathcal{T}^{n+1}f\right)(t), (\mathcal{T}^n f)(t)\Big) \le (\Lambda^n\varepsilon)(t), \quad t \in X. \tag{2.10}$$

Indeed, for $n = 0$, the relation (2.10) is (2.5).

We suppose that (2.10) holds. Since $\mathcal{T}$ is Λ-contractive, we have

$$d\Big(\left(\mathcal{T}^{n+2}f\right)(t), \left(\mathcal{T}^{n+1}f\right)(t)\Big) \le (\Lambda(\Lambda^n\varepsilon))(t), \quad t \in X. \tag{2.11}$$

By using the triangle inequality and (2.10), we obtain, for $n > m$

$$d\big((\mathcal{T}^n f)(t), (\mathcal{T}^m f)(t)\big) \le \sum_{k=m}^{n-1}\left(\Lambda^k\varepsilon\right)(t), \quad t \in X. \tag{2.12}$$

Hence the sequence $\{\mathcal{T}^n f(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (Y, d) is complete, it results that there exists $g \in Y^X$ defined by

$$g(t) := \lim_{n\to\infty} (\mathcal{T}^n f)(t). \tag{2.13}$$

Then, in view of (2.12), we get (2.7).

Now, we prove that g is a fixed point for the operator $\mathcal{T}$. To this end, we show that $\mathcal{T}$ is a pointwise continuous. Indeed, if $h_m(t) \xrightarrow[m\to\infty]{} h(t)$, $t \in X$, then

$$|h_m, h|(t) := d(h_m(t), h(t)) \xrightarrow[m\to\infty]{} 0, \quad t \in X. \tag{2.14}$$

By using condition (C_1) we have $(\Lambda|h_m, h|)(t) \xrightarrow[m\to\infty]{} 0, t \in X$. But

$$d((\mathcal{T}h_m)(t), (\mathcal{T}h)(t)) \le (\Lambda|h_m, h|)(t), \tag{2.15}$$

so it follows that $d((\mathcal{T}h_m)(t), (\mathcal{T}h)(t)) \xrightarrow[m\to\infty]{} 0$.

Since $\mathcal{T}$ is a pointwise continuous, we obtain $(\mathcal{T}(\mathcal{T}^n f))(t) \xrightarrow[n\to\infty]{} (\mathcal{T}g)(t)$. Hence $g(t) = (\mathcal{T}g)(t)$ for $t \in X$.

It is easy to prove that g is the unique point of $\mathcal{T}$, which satisfies (2.7): for $n \to \infty$ in (2.12), it results

$$d(g(t), (\mathcal{T}^m f)(t)) \le \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X. \tag{2.16}$$

If g_1 is another fixed point of $\mathcal{T}$ such that (2.7) holds, then we have

$$d\left(g_1(t), (\mathcal{T}^m f)(t)\right) \le \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X. \tag{2.17}$$

Hence

$$d\left(g_1(t), g(t)\right) \le 2\sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \quad t \in X, \tag{2.18}$$

so letting $m \to \infty$ we obtain $d(g_1(t), g(t)) = 0$ for $t \in X$. Thus $g_1 = g$.

To prove the last part of the theorem, we take $m = 0$ in (2.7) and we obtain (2.9). Moreover, if (C_3) holds and g_2 is another fixed point of $\mathcal{T}$ such that (2.9) is satisfied, then we have

$$d\left((\mathcal{T}^n g_2)(t), (\mathcal{T}^n f)(t)\right) \le (\Lambda^n \varepsilon^*)(t), \quad t \in X, \tag{2.19}$$

hence

$$d\big(g_2(t),(\mathcal{T}^n f)(t)\big)\le(\Lambda^n\varepsilon^*)(t),\quad t\in X. \tag{2.20}$$

Letting $n\to\infty$, we obtain $d(g_2(t),g(t))=0$, for $t\in X$, so $g=g_2$. □

Corollary 2.3. *Let X be a nonempty set, (Y,d) a complete metric space, and let $\Lambda:\mathbb{R}_+^X\to\mathbb{R}_+^X$ be a nondecreasing operator satisfying the hypothesis (C_1).*

If $\mathcal{T}:Y^X\to Y^X$ is an operator satisfying the inequality

$$d\big((\mathcal{T}\xi)(x),(\mathcal{T}\mu)(x)\big)\le\Lambda\big(d\big(\xi(x),\mu(x)\big)\big),\quad \xi,\mu\in Y^X,\ x\in X, \tag{2.21}$$

and the functions $\varepsilon:X\to\mathbb{R}_+$ and $\varphi:X\to Y$ are such that

$$\begin{gathered} d\big((\mathcal{T}\varphi)(x),\varphi(x)\big)\le\varepsilon(x),\quad x\in X,\\ \varepsilon^*(x):=\sum_{k=0}^{\infty}\Big(\Lambda^k\varepsilon\Big)(x)<\infty,\quad x\in X, \end{gathered} \tag{2.22}$$

then, for every $x\in X$, the limit

$$\psi(x):=\lim_{n\to\infty}(\mathcal{T}^n\varphi)(x) \tag{2.23}$$

exists and the function $\psi\in Y^X$, defined in this way, is a fixed point of $\mathcal{T}$, with

$$d\big(\varphi(x),\psi(x)\big)\le\varepsilon^*(x),\quad x\in X. \tag{2.24}$$

Moreover, if the condition (C_3) holds, then the mapping ψ is the unique fixed point of $\mathcal{T}$ with the property

$$d\big(\varphi(x),\psi(x)\big)\le\varepsilon^*(x),\quad x\in X. \tag{2.25}$$

Proof. To apply Theorem 2.2 it is sufficient to show that the operator $\mathcal{T}$ from the above corollary is Λ-contractive, in the sense of the Definition 2.1. To this end, let us suppose that ξ, $\mu\in Y^X$, $\delta\in\mathbb{R}_+^X$ and

$$d\big(\xi(x),\mu(x)\big)\le\delta(x),\quad x\in X. \tag{2.26}$$

By using (2.21) and the non-decreasing property of Λ, we obtain that

$$d\big((\mathcal{T}\xi)(x),(\mathcal{T}\mu)(x)\big)\le\Lambda\big(d\big(\xi(x),\mu(x)\big)\big)\le\Lambda(\delta(x)),\quad x\in X. \tag{2.27}$$

Hence $\mathcal{T}$ is Λ-contractive. The uniqueness follows from Theorem 2.2. □

The results of Corollary 2.3 (except for the uniqueness of ψ) have been proved recently by Brzdęk and Ciepliński [30]. Actually, the authors have stated there an open question concerning the uniqueness of ψ.

Another recent result proved in [29], by Brzdęk et al., can be obtained from Theorem 2.2.

Corollary 2.4 (Corollary [see [29], Theorem 1]). *Let X be a nonempty set, (Y, d) a complete metric space, $f_1, \ldots, f_s : X \to X$, and let $L_1, \ldots, L_s : X \to \mathbb{R}_+$ be given maps. Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by*

$$(\Lambda\delta)(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)) \tag{2.28}$$

for $\delta : X \to \mathbb{R}_+$ and $x \in X$. If $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \le \sum_{i=1}^{s} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))), \quad \xi, \mu \in Y^X,\ x \in X, \tag{2.29}$$

and the functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ are such that

$$\begin{gathered} d((\mathcal{T}\varphi)(x), \varphi(x)) \le \varepsilon(x), \quad x \in X, \\ \varepsilon^*(x) := \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) < \infty, \quad x \in X, \end{gathered} \tag{2.30}$$

then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n \to \infty} (\mathcal{T}^n \varphi)(x) \tag{2.31}$$

exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathcal{T}$, with

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x), \quad x \in X. \tag{2.32}$$

Proof. We apply Theorem 2.2. Therefore, it is necessary to prove that the operator $\mathcal{T}$, defined in (2.28), is Λ-contractive. To this end, let us suppose that $\xi, \mu \in Y^X$, $\delta \in \mathbb{R}_+^X$ and

$$d(\xi(x), \mu(x)) \le \delta(x), \quad \forall x \in X. \tag{2.33}$$

By using (2.28) and (2.29), we obtain that

$$\begin{aligned} d((\mathcal{T}\xi)(x),(\mathcal{T}\mu)(x)) &\le \sum_{i=1}^{s} L_i(x) d(\xi(f_i(x)),\mu(f_i(x))) \\ &\le \sum_{i=1}^{s} L_i(x)\delta(f_i(x)) \\ &= \Lambda(\delta(x)), \quad x \in X, \end{aligned} \tag{2.34}$$

so $\mathcal{T}$ is Λ-contractive.

On the other hand, from definition of Λ, it results immediately that the relation (C_1) holds.

The uniqueness of φ results also from Theorem 2.2. To this end, we prove that the linear operator Λ satisfy the hypotheses (C_3):

$$\begin{aligned} \Lambda^n(\varepsilon^*(x)) &= \Lambda^n\left(\sum_{k=0}^{\infty}\left(\Lambda^k\varepsilon\right)(x)\right) \\ &= \sum_{k=0}^{\infty}\left(\Lambda^{n+k}\varepsilon\right)(x) = \sum_{m=n}^{\infty}(\Lambda^m\varepsilon)(x). \end{aligned} \tag{2.35}$$

Thus

$$\lim_{n\to\infty}\Lambda^n(\varepsilon^*(x)) = 0, \quad x \in X. \tag{2.36}$$

□

The following result of generalized Hyers-Ulam stability for the functional equation:

$$\Theta(x,\varphi(f_1(x)),\ldots,\varphi(f_s(x))) = \varphi(x), \quad x \in X, \tag{2.37}$$

can be also derived from Theorem 2.2. (The unknown mapping is φ; the others are given functions.)

Corollary 2.5. *Let X be a nonempty set, let (Y,d) be a complete metric space, and let the operators $\Theta : X \times Y^s \to Y$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$. We suppose that Θ is Λ-contractive, the conditions (C_1) and (C_2) hold, and let one consider a function $\varphi \in Y^X$ such that*

$$d(\varphi(x),\Theta(x,\varphi(f_1(x)),\ldots,\varphi(f_s(x)))) \le \varepsilon(x), \quad x \in X, \tag{2.38}$$

for the given mappings $f_1,\ldots,f_s : X \to X$. Then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty}(\mathcal{T}^n\varphi)(x), \tag{2.39}$$

where $(\mathcal{T}\varphi)(x) = \Theta(x, \varphi(f_1(x)), \ldots, \varphi(f_s(x)))$, *exists and the function* $\psi \in Y^X$, *above defined, is the unique solution of the functional equation* (2.37) *with property*

$$d\left((\mathcal{T}^m\varphi)(x), \psi(x)\right) \leq \sum_{k=m}^{\infty}\left(\Lambda^k\varepsilon\right)(x), \quad x \in X,\ m \in \mathbb{N} = \{0,1,2,\ldots\}. \tag{2.40}$$

Moreover, if one has

$$\lim_{n\to\infty}(\Lambda^n\varepsilon^*)(x) = 0, \quad \forall x \in X, \tag{2.41}$$

then ψ *is the unique solution of* (2.37), *with the property*

$$d\left(\psi(x), \varphi(x)\right) \leq \varepsilon^*(x), \quad \forall x \in X. \tag{2.42}$$

Remark 2.6. It is easy to see that if we take in the above result

$$(\Lambda\delta)(x) := \sum_{i=1}^{s} L_i(x)\delta\left(f_i(x)\right), \quad \forall x \in X \tag{2.43}$$

for the given mappings $L_1, \ldots, L_s : X \to \mathbb{R}_+$ and $\delta : X \to \mathbb{R}_+$, we obtain the Corollary 3 in [29].

From Theorem 2.2 we obtain the following fixed-point result.

Corollary 2.7. *Let* (Y, d) *be a metric space and let* $c : [0,\infty) \to [0,\infty)$ *be a function, with the property: for every sequence* $\varepsilon_n \in [0,\infty)$, *with* $\lim_{n\to\infty}\varepsilon_n = 0 \Rightarrow \lim_{n\to\infty}c(\varepsilon_n) = 0$. *Let one consider an operator* $T : Y \to Y$ *such that, for* $u, v \in Y$ *and* $\lambda \geq 0$, *with* $d(u,v) \leq \lambda$, *it follows* $d(Tu, Tv) \leq c(\lambda)$. *Moreover, let* $\varepsilon > 0$ *and* $f \in Y$ *be such that*

$$\varepsilon^* = \sum_{n=0}^{\infty} c^n(\varepsilon) < \infty \tag{2.44}$$

and $d(Tf, f) \leq \varepsilon$. *Then there exists*

$$g := \lim_{n\to\infty} T^n f, \tag{2.45}$$

which is the unique fixed point of T, *with*

$$d(T^m f, g) \leq \sum_{k=m}^{\infty} c^k(\varepsilon), \quad \forall m \in \mathbb{N} = \{0,1,2,\ldots\}. \tag{2.46}$$

Moreover, if

$$\lim_{n \to \infty} c^n(\varepsilon^*) = 0 \tag{2.47}$$

holds, then g is the unique fixed point of T, with the property $d(f, g) \leq \varepsilon^*$.

Proof. The result follows immediately from Theorem 2.2 by taking X to be the set with a single element. □

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