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Research Article

Two General Algorithms for Computing Fixed Points of Nonexpansive Mappings in Banach Spaces

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Recently, Yao et al. (2011) introduced two algorithms for solving a system of nonlinear variational inequalities. In this paper, we consider two general algorithms and obtain the extension results for computing fixed points of nonexpansive mappings in Banach spaces. Moreover, the fixed points solve the same system of nonlinear variational inequalities.

1. Introduction

Let X be a real Banach space and let C be a nonempty closed convex subset of X. Recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

Recently, Yao et al. [1] considered the following algorithms:

$$x_t = \Pi_C(I - tF)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_t, \tag{1.1}$$

and for an arbitrary point $x_0 \in C$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\Pi_C(I - \alpha_n F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_n, \quad n \ge 0, \tag{1.2}$$

where $\Pi_C : X \to C$ is a sunny nonexpansive retraction, $F : C \to X$ is a strongly positive bounded linear operator and $A, B : C \to X$ are α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively. They proved that the $\{x_t\}$ defined by (1.1) and $\{x_n\}$ defined by (1.2) converge strongly to a unique solution $\overline{x}$ of the variational inequality

$\langle F(\overline{x}), j(\overline{x}-z)\rangle \leq 0$. Furthermore, they proved that the above algorithms converge strongly to some solutions of a system of nonlinear inequalities, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{aligned} \tag{1.3}$$

For related works, please see [2–5] and the references therein.

In this paper, we introduce two general algorithms (3.3) and (3.22) (defined below) and prove that the proposed algorithms strongly converge to $x^* \in \mathrm{Fix}(T)$ which solves the variational inequality $\langle \overline{F}x^*, j(x^* - u)\rangle \leq 0$, $u \in \mathrm{Fix}(T)$, where $\overline{F} : C \to X$ is a β-Lipschitzian and η-strongly accretive operator. It is worth pointing out that our proofs contain some new techniques.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual space. The value of $f \in X^*$ and $x \in X$ will be denoted by $\langle x, f\rangle$. For the sequence $\{x_n\}$ in X, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ means that $\{x_n\}$ converges strongly to x.

Let $\eta > 0$, a mapping $\overline{F}$ of C into X is said to be η-strongly accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle \overline{F}x - \overline{F}y, j(x-y)\right\rangle \geq \eta\|x-y\|^2, \tag{2.1}$$

for all $x, y \in C$. A mapping $\overline{F}$ from C into X is said to be β-Lipschitzian if, for $\beta > 0$,

$$\left\|\overline{F}x - \overline{F}y\right\| \leq \beta\|x-y\|, \tag{2.2}$$

for all $x, y \in C$. From the definition of F (see [1]), we note that a strongly positive bounded linear operator F is a $\|F\|$-Lipschitzian and $\overline{\gamma}$-strongly accretive operator.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \left\|\frac{x+y}{2}\right\| \leq 1 - \delta. \tag{2.3}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}, \tag{2.4}$$

exists for all $x, y \in U$. It is said to be uniformly smooth if the limit (2.4) is attained uniformly for $x, y \in U$. Also, we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau \right\}. \tag{2.5}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [6]). *Let q be a given real number with $1 < q \leq 2$ and let X be a q-uniformly smooth Banach space. Then*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q, \tag{2.6}$$

for all $x, y \in X$, where K is the q-uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1} \right\}, \tag{2.7}$$

for all $x \in X$.

Lemma 2.2 (see [7]). *Let C be a closed convex subset of a smooth Banach space X, let D be a nonempty subset of C and Π be a retraction from C onto D. Then Π is sunny and nonexpansive if and only if*

$$\langle u - \Pi(u), j(y - \Pi(u))\rangle \leq 0, \tag{2.8}$$

for all $u \in C$ and $y \in D$.

Lemma 2.3 (see [8]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

Lemma 2.4 (see [9, 10]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0, \tag{2.9}$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$, (ii) $\limsup_{n\to\infty}\delta_n \leq 0$ or $\sum_{n=0}^{\infty}\lambda_n\delta_n < \infty$, and (iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$. Then $\lim_{n\to\infty}s_n = 0$.

Lemma 2.5 (see [11]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E and $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1. \tag{2.10}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n$, $n \geq 0$, and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

In addition, we need the following extension of Lemma 2.5 in Wang and Hu [2] in a 2-uniformly smooth Banach space.

Lemma 2.6. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let $\overline{F} : C \to X$ be a β-Lipschitzian and η-strongly accretive operator with $0 < \eta \leq \sqrt{2}\beta K$ and $0 < t < \eta/2\beta^2 K^2$. Then $S = (I - t\overline{F}) : C \to X$ is a contraction with contraction coefficient $\tau_t = \sqrt{1 - 2t(\eta - t\beta^2 K^2)}$.*

Proof. By Lemma 2.1, we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \left\|(x-y) - t\left(\overline{F}x - \overline{F}y\right)\right\|^2 \\ &= \|x-y\|^2 - 2t\left\langle \overline{F}x - \overline{F}y, j(x-y)\right\rangle + 2t^2K^2\left\|\overline{F}x - \overline{F}y\right\|^2 \\ &\leq \|x-y\|^2 - 2t\eta\|x-y\|^2 + 2t^2\beta^2K^2\|x-y\|^2 \\ &= \left[1 - 2t\left(\eta - t\beta^2K^2\right)\right]\|x-y\|^2, \end{aligned} \tag{2.11}$$

for all $x, y \in C$. From $0 < \eta \leq \sqrt{2}\beta K$ and $0 < t < \eta/2\beta^2 K^2$, we have

$$\|Sx - Sy\| \leq \tau_t \|x - y\|, \tag{2.12}$$

where $\tau_t = \sqrt{1 - 2t(\eta - t\beta^2 K^2)} \in (0,1)$. Hence S is a contraction with contraction coefficient τ_t. □

3. Main Results

Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\overline{F} : C \to X$ be a β-Lipschitzian and η-strongly accretive operator with $0 < \eta \leq \sqrt{2}\beta K$. Let $t \in (0, \eta/2\beta^2 K^2)$ and $\tau_t = \sqrt{1 - 2t(\eta - t\beta^2 K^2)}$, consider a mapping S_t on C defined by

$$S_t x = \Pi_C\left(I - t\overline{F}\right)Tx, \quad x \in C, \tag{3.1}$$

where Π_C is a sunny nonexpansive retraction from X onto C. It is easy to see that S_t is a contraction. Indeed, from Lemma 2.6, we have

$$\begin{aligned}
\|S_t x - S_t y\| &\leq \left\|\Pi_C\left(I - t\overline{F}\right)Tx - \Pi_C\left(I - t\overline{F}\right)Ty\right\| \\
&\leq \left\|\left(I - t\overline{F}\right)Tx - \left(I - t\overline{F}\right)Ty\right\| \\
&\leq \tau_t\|Tx - Ty\| \\
&\leq \tau_t\|x - y\|,
\end{aligned} \tag{3.2}$$

for all $x, y \in C$. Therefore, the following implicit method is well defined:

$$x_t = \Pi_C\left(I - t\overline{F}\right)Tx_t, \quad x_t \in C. \tag{3.3}$$

Theorem 3.1. *The net $\{x_t\}$ generated by the implicit method* (3.3) *converges in norm, as $t \to 0^+$ to the unique solution $x^* \in \operatorname{Fix}(T)$ of the variational inequality:*

$$\left\langle \overline{F}x^*, j(x^* - u)\right\rangle \leq 0, \quad u \in \operatorname{Fix}(T). \tag{3.4}$$

Proof. We first show that the solution set of (3.4) is singleton. As a matter of fact, we assume that $x^* \in \operatorname{Fix}(T)$ and $\tilde{x} \in \operatorname{Fix}(T)$ both are solutions to (3.4), then

$$\left\langle \overline{F}x^*, j(x^* - \tilde{x})\right\rangle \leq 0, \tag{3.5}$$

$$\left\langle \overline{F}\tilde{x}, j(\tilde{x} - x^*)\right\rangle \leq 0. \tag{3.6}$$

Adding (3.5) to (3.6), we get

$$\left\langle \overline{F}x^* - \overline{F}\tilde{x}, j(x^* - \tilde{x})\right\rangle \leq 0. \tag{3.7}$$

The strong accretive of $\overline{F}$ implies that $x^* = \tilde{x}$, and the uniqueness is proved. Below we use $x^* \in \operatorname{Fix}(T)$ to denote the unique solution of (3.4).

Next, we prove that $\{x_t\}$ is bounded. Taking $u \in \operatorname{Fix}(T)$, from (3.3) and using Lemma 2.6, we have

$$\begin{aligned}
\|x_t - u\| &= \left\|\Pi_C\left(I - t\overline{F}\right)Tx_t - \Pi_C u\right\| \\
&\leq \left\|\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tu - t\overline{F}Tu\right\| \\
&\leq \left\|\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tu\right\| + t\left\|\overline{F}u\right\| \\
&\leq \tau_t\|x_t - u\| + t\left\|\overline{F}u\right\|,
\end{aligned} \tag{3.8}$$

that is,

$$\|x_t - u\| \leq \frac{t}{1-\tau_t}\left\|\overline{F}u\right\|. \tag{3.9}$$

Observe that

$$\lim_{t\to 0^+}\frac{t}{1-\tau_t} = \frac{1}{\eta}. \tag{3.10}$$

From $t \to 0^+$, we may assume, without loss of generality, that $t \leq \eta/2\beta^2K^2 - \epsilon$, where ϵ is an arbitrarily small positive number. Thus, we have $t/(1-\tau_t)$ to be continuous, for all $t \in [0, \eta/2\beta^2K^2 - \epsilon]$. Therefore, we obtain

$$M_1 = \sup\left\{\frac{t}{1-\tau_t} : t \in \left(0, \frac{\eta}{2\beta^2K^2} - \epsilon\right]\right\} < +\infty. \tag{3.11}$$

From (3.9) and (3.11), we have $\{x_t\}$ bounded and so is $\{\overline{F}Tx_t\}$.

On the other hand, from (3.3), we obtain

$$\|x_t - Tx_t\| = \left\|\Pi_C\left(I - t\overline{F}\right)Tx_t - \Pi_C Tx_t\right\| \leq \left\|\left(I - t\overline{F}\right)Tx_t - Tx_t\right\| = t\left\|\overline{F}Tx_t\right\| \longrightarrow 0 \quad (t \longrightarrow 0^+). \tag{3.12}$$

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume that $\{t_n\} \in (0, \eta/2\beta^2K^2)$ such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$. It follows from (3.12) that

$$\|x_n - Tx_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.13}$$

For a given $u \in \mathrm{Fix}(T)$, by (3.3) and using Lemma 2.2, we have

$$\left\langle x_t - \left(I - t\overline{F}\right)Tx_t, j(x_t - u)\right\rangle \leq 0. \tag{3.14}$$

By (3.14) and using Lemma 2.6, we have

$$\begin{aligned}
\|x_t - u\|^2 &= \langle x_t - u, j(x_t - u)\rangle \\
&= \left\langle x_t - \left(I - t\overline{F}\right)Tx_t, j(x_t - u)\right\rangle + \left\langle\left(I - t\overline{F}\right)Tx_t - u, j(x_t - u)\right\rangle \\
&\leq \left\langle\left(I - t\overline{F}\right)Tx_t - u, j(x_t - u)\right\rangle \\
&\leq \left\langle\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tu, j(x_t - u)\right\rangle + t\left\langle\overline{F}u, j(u - x_t)\right\rangle \\
&\leq \tau_t\|x_t - u\|^2 + t\left\langle\overline{F}u, j(u - x_t)\right\rangle,
\end{aligned} \tag{3.15}$$

that is,

$$\|x_t - u\|^2 \le \frac{t}{1-\tau_t}\left\langle \overline{F}u, j(u - x_t)\right\rangle \le M_1\left\langle \overline{F}u, j(u - x_t)\right\rangle. \tag{3.16}$$

In particular,

$$\|x_n - u\|^2 \le M_1\left\langle \overline{F}u, j(u - x_n)\right\rangle. \tag{3.17}$$

Since $\{x_t\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x}$. Noticing (3.13) we can use Lemma 2.3 to get $\tilde{x} \in \mathrm{Fix}(T)$. Therefore we can substitute $\tilde{x}$ for u in (3.17) to get

$$\|x_n - \tilde{x}\| \le M_1\left\langle \overline{F}\tilde{x}, j(\tilde{x} - x_n)\right\rangle. \tag{3.18}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0^+$.

We next show that $\tilde{x}$ solves the variational inequality (3.4). Observe that

$$\begin{aligned} x_t &= \Pi_C\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)x_t + \left(I - t\overline{F}\right)Tx_t + x_t - t\overline{F}(x_t) \\ &\Longrightarrow \overline{F}(x_t) = \frac{1}{t}\left[\Pi_C\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)x_t + \left(I - t\overline{F}\right)Tx_t\right]. \end{aligned} \tag{3.19}$$

For any $u \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} \langle \overline{F}x_t, j(x_t - u)\rangle &= \frac{1}{t}\left\langle \Pi_C\left(I - t\overline{F}\right)Tx_t - \left(I - t\overline{F}\right)Tx_t, j(x_t - u)\right\rangle \\ &\quad - \frac{1}{t}\left\langle \left(I - t\overline{F}\right)x_t - \left(I - t\overline{F}\right)Tx_t, j(x_t - u)\right\rangle \\ &\le -\frac{1}{t}\langle x_t - Tx_t, j(x_t - u)\rangle + \left\langle \overline{F}x_t - \overline{F}Tx_t, j(x_t - u)\right\rangle \\ &\le -\frac{1}{t}\langle (I - T)x_t - (I - T)u, j(x_t - u)\rangle + \beta\|x_t - Tx_t\|\|x_t - u\| \\ &\le \beta M_2\|x_t - Tx_t\|, \end{aligned} \tag{3.20}$$

where $M_2 = \sup\{\|x_t - u\|, t \in (0, \eta/2\beta^2K^2)\}$.

Now replacing t in (3.20) with t_n and letting $n \to \infty$, we have

$$\left\langle \overline{F}\tilde{x}, j(\tilde{x} - u)\right\rangle \le 0. \tag{3.21}$$

That is, $\tilde{x} \in \mathrm{Fix}(T)$ is a solution of (3.4), hence $\tilde{x} = x^*$ by uniqueness. In summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals x^*. Therefore, $x_t \to x^*$ as $t \to 0$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X with a weakly sequentially continuous duality mapping j. Let $\overline{F} : C \to X$ be a β-Lipschitzian and η-strongly accretive operator with $0 < \eta \leq \sqrt{2}\beta K$. Suppose that $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$ and satisfy the conditions:*

(A1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(A2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= \Pi_C\left(I - \alpha_n\overline{F}\right)Tx_n, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)y_n, \quad n \geq 0. \end{aligned} \tag{3.22}$$

Then the sequence $\{x_n\}$ strongly converges to $x^* \in \mathrm{Fix}(T)$ which solves the variational inequality (3.4).

Proof. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. From $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $0 < \alpha_n \leq \eta/2\beta^2K^2 - \epsilon$ for all n. In fact, let $u \in \mathrm{Fix}(T)$, from (3.22) and using Lemma 2.6, we have

$$\begin{aligned} \|y_n - u\| &= \left\|\Pi_C\left(I - \alpha_n\overline{F}\right)Tx_n - \Pi_C u\right\| \\ &\leq \left\|\left(I - \alpha_n\overline{F}\right)Tx_n - \left(I - \alpha_n\overline{F}\right)Tu - \alpha_n\overline{F}u\right\| \\ &\leq \tau_{\alpha_n}\|x_n - u\| + \alpha_n\left\|\overline{F}u\right\|, \end{aligned} \tag{3.23}$$

where $\tau_{\alpha_n} = \sqrt{1 - 2\alpha_n(\eta - \alpha_n\beta^2K^2)} \in (0,1)$. Then from (3.22) and (3.23), we obtain

$$\begin{aligned} \|x_{n+1} - u\| &\leq \beta_n\|x_n - u\| + (1-\beta_n)\|y_n - u\| \\ &\leq \beta_n\|x_n - u\| + (1-\beta_n)\left(\tau_{\alpha_n}\|x_n - u\| + \alpha_n\left\|\overline{F}u\right\|\right) \\ &\leq \left[1 - (1-\beta_n)(1-\tau_{\alpha_n})\right]\|x_n - u\| + (1-\beta_n)\alpha_n\left\|\overline{F}u\right\| \\ &\leq \max\left\{\|x_n - u\|, \frac{\alpha_n\left\|\overline{F}u\right\|}{1-\tau_{\alpha_n}}\right\}. \end{aligned} \tag{3.24}$$

By induction, we have

$$\|x_n - u\| \leq \max\left\{\|x_1 - u\|, M_3\left\|\overline{F}u\right\|\right\}, \tag{3.25}$$

where $M_3 = \sup\{\alpha_n/(1-\tau_{\alpha_n}) : 0 < \alpha_n \leq \eta/2\beta^2K^2 - \epsilon\} < +\infty$. Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$ and $\{\overline{F}Tx_n\}$ are bounded.

Step 2. We claim that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Observe that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \left\|\Pi_C\left(I - \alpha_{n+1}\overline{F}\right)Tx_{n+1} - \Pi_C\left(I - \alpha_n\overline{F}\right)Tx_n\right\| \\ &\le \left\|\left(I - \alpha_{n+1}\overline{F}\right)Tx_{n+1} - \left(I - \alpha_n\overline{F}\right)Tx_n\right\| \\ &\le \|Tx_{n+1} - Tx_n\| + \alpha_{n+1}\left\|\overline{F}Tx_{n+1}\right\| + \alpha_n\left\|\overline{F}Tx_n\right\| \\ &\le \|x_{n+1} - x_n\| + \alpha_{n+1}\left\|\overline{F}Tx_{n+1}\right\| + \alpha_n\left\|\overline{F}Tx_n\right\|.\end{aligned} \tag{3.26}$$

Therefore, we have

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.27}$$

From (3.22), (3.27), and using Lemma 2.5, we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Step 3. We claim that $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$. Observe that

$$\begin{aligned}\|y_n - Ty_n\| &= \left\|\Pi_C\left(I - \alpha_n\overline{F}\right)Tx_n - \Pi_C Ty_n\right\| \\ &\le \|Tx_n - Ty_n\| + \alpha_n\left\|\overline{F}Tx_n\right\| \\ &\le \|x_n - y_n\| + \alpha_n\left\|\overline{F}Tx_n\right\|.\end{aligned} \tag{3.28}$$

Hence, from Step 2 and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{3.29}$$

Step 4. We claim that $\limsup_{n\to\infty}\langle\overline{F}x^*, j(x^* - y_n)\rangle \le 0$, where $x^* = \lim_{t\to 0}x_t$ and x_t is defined by (3.3). Since y_n is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges weakly to ω. From Step 3, we obtain $Ty_{n_k} \rightharpoonup \omega$. From Lemma 2.3, we have $\omega \in \operatorname{Fix}(T)$. Hence, using Theorem 3.1, we have $x^* \in \operatorname{Fix}(T)$ and

$$\limsup_{n\to\infty}\left\langle\overline{F}x^*, j(x^* - y_n)\right\rangle = \lim_{k\to\infty}\left\langle\overline{F}x^*, j(x^* - y_{n_k})\right\rangle = \left\langle\overline{F}x^*, j(x^* - \omega)\right\rangle \le 0. \tag{3.30}$$

Step 5. We claim that $\{x_n\}$ converges strongly to $x^* \in \operatorname{Fix}(T)$. From (3.22) and using Lemma 2.2, we have

$$\left\langle\Pi_C\left(I - \alpha_n\overline{F}\right)Tx_n - \left(I - \alpha_n\overline{F}\right)Tx_n, j(y_n - x^*)\right\rangle \le 0. \tag{3.31}$$

Observe that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \left\langle \Pi_C\left(I - \alpha_n \overline{F}\right)Tx_n - x^*, j(y_n - x^*) \right\rangle \\
&= \left\langle \Pi_C\left(I - \alpha_n \overline{F}\right)Tx_n - \left(I - \alpha_n \overline{F}\right)Tx_n, j(y_n - x^*) \right\rangle \\
&\quad + \left\langle \left(I - \alpha_n \overline{F}\right)Tx_n - x^*, j(y_n - x^*) \right\rangle \\
&\le \left\langle \left(I - \alpha_n \overline{F}\right)Tx_n - x^*, j(y_n - x^*) \right\rangle \\
&\le \left\langle \left(I - \alpha_n \overline{F}\right)Tx_n - \left(I - \alpha_n \overline{F}\right)Tx^*, j(y_n - x^*) \right\rangle + \alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle \\
&\le \left\| \left(I - \alpha_n \overline{F}\right)Tx_n - \left(I - \alpha_n \overline{F}\right)Tx^* \right\| \|y_n - x^*\| + \alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle \\
&\le \tau_{\alpha_n} \|x_n - x^*\| \|y_n - x^*\| + \alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle \\
&\le \frac{\tau_{\alpha_n}^2}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|y_n - x^*\|^2 + \alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle,
\end{aligned}
\tag{3.32}
$$

that is,

$$
\|y_n - x^*\|^2 \le \tau_{\alpha_n} \|x_n - x^*\|^2 + 2\alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle. \tag{3.33}
$$

By (3.22) and (3.33), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|y_n - x^*\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left[\tau_{\alpha_n} \|x_n - x^*\|^2 + 2\alpha_n \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle \right] \\
&\le \left[1 - (1 - \beta_n)(1 - \tau_{\alpha_n})\right] \|x_n - x^*\|^2 + 2M_3 (1 - \beta_n)(1 - \tau_{\alpha_n}) \left\langle \overline{F}x^*, j(x^* - y_n) \right\rangle \\
&= (1 - \lambda_n) \|x_n - x^*\|^2 + \lambda_n \delta_n,
\end{aligned}
\tag{3.34}
$$

where $\lambda_n = (1 - \beta_n)(1 - \tau_{\alpha_n})$, $\delta_n = 2M_3 \langle \overline{F}x^*, j(x^* - y_n) \rangle$. It is easy to see that $\lambda_n \to 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\limsup_{n \to \infty} \delta_n \le 0$. Hence, by Lemma 2.4, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$. From $x^* = \lim_{t \to 0} x_t$ and Theorem 3.1, we have x^* to be the unique solution of the variational inequality (3.4). □

Taking $T = \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ and $\overline{F} = F$, where $0 < \lambda \le \alpha/K^2$ and $0 < \mu \le \eta/K^2$, we obtain the following theorems immediately.

Corollary 3.3 (see [1, Theorem 3.5]). *The net $\{x_t\}$ generated by the implicit method* (1.1) *converges in norm, as $t \to 0^+$, to the unique solution $\widetilde{x}$ of variational inequality*

$$
\widetilde{x} \in \Omega, \quad \langle F(\widetilde{x}), j(\widetilde{x} - z) \rangle \le 0, \quad z \in \Omega. \tag{3.35}
$$

Corollary 3.4 (see [1, Theorem 3.7]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X and let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively. Let $F : C \to H$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by* (1.2). *Suppose that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (A1) and (A2), then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega$ which solves the variational inequality* (3.35).

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