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Research Article

Existence of $2m - 1$ Positive Solutions for Sturm-Liouville Boundary Value Problems with Linear Functional Boundary Conditions on the Half-Line

Yanmei Sun[1] and Zengqin Zhao[2]

[1] *Department of Mathematics and Information Sciences, Weifang University, Shandong, Weifang 261061, China*
[2] *Department of Mathematics, Qufu Normal University, Shandong, Qufu 273165, China*

Correspondence should be addressed to Yanmei Sun, sunyanmei2009@126.com

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By using the Leggett-Williams fixed theorem, we establish the existence of multiple positive solutions for second-order nonhomogeneous Sturm-Liouville boundary value problems with linear functional boundary conditions. One explicit example with singularity is presented to demonstrate the application of our main results.

1. Introduction

In this paper, we consider the following Sturm-Liouville boundary value problems on the half-line

$$\begin{gathered}(p(t)u'(t))' + \Phi(t) f(t, u(t), u'(t)) = 0, \quad 0 < t < +\infty, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0^+} p(t)u'(t) = \mathbf{T}(u), \\ \alpha_2 \lim_{t\to +\infty} u(t) + \beta_2 \lim_{t\to +\infty} p(t)u'(t) = \mathbf{K}(u),\end{gathered} \tag{1.1}$$

where $f : R^+ \times R^+ \times R \to R^+$ is a continuous function, $f \not\equiv 0$ on any subinterval of R^+, here $R^+ = [0, +\infty)$; $\Phi : R^+ \to R^+$ is a Lebesgue integrable function and may be singular at

$t = 0$; $p \in C(R^+, R^+) \bigcap C^1(R^+), \int_0^{+\infty} ds/p(s) < +\infty$; $\alpha_i, \beta_i \geq 0$ $(i = 1, 2)$ with $\rho = \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\alpha_2 \int_0^{+\infty} ds/p(s)$; $\mathbf{T}$, $\mathbf{K}$ are linear positive functionals on $C(R^+)$ ($\mathbf{T}$, $\mathbf{K}$ which are called positive if $\mathbf{T}(u)$, $\mathbf{K}(u) \geq 0$ for $u \in C(R^+)$).

The theory of nonlocal boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics. There are many studies for nonlocal, including three-point, m-point, and integral boundary value problems on finite interval by applying different methods [1–3]. It is well known that boundary value problems on infinite interval arise in the study of radial solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium [4–6]. But the theory of Sturm-Liouville nonhomogeneous boundary value problems on infinite interval is yet rare.

The linear functional boundary conditions cover some nonlocal three-point, m-point, and integral boundary conditions. In [7], Zhao and Li investigated some nonlinear singular differential equations with linear functional boundary conditions. However, the differential equations were defined only in a finite interval. Recently, Liu et al. [6] studied multiple positive solutions for Sturm-Liouville boundary value problems on the half-line

$$\begin{gathered}(p(t)u'(t))' + m(t)f(t, u(t)) = 0, \quad 0 \leq t < +\infty, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0^+} p(t)u'(t) = 0, \\ \alpha_2 \lim_{t\to +\infty} u(t) + \beta_2 \lim_{t\to +\infty} p(t)u'(t) = 0.\end{gathered} \tag{1.2}$$

However, the authors did not consider the case when Sturm-Liouville boundary value problems are nonhomogeneous. Therefore BVP(1.1) is the direct extension of [7]. So it is worthwhile to investigate BVP(1.1).

We denote

$$a(t) = \beta_1 + \alpha_1 \int_0^t \frac{ds}{p(s)}, \qquad b(t) = \beta_2 + \alpha_2 \int_t^{+\infty} \frac{ds}{p(s)}, \tag{1.3}$$

$$u(t) = \rho^{-1}[1 + a(t)b(t)]y_1(t), \qquad u'(t) = y_2(t), \qquad f(t, u, u') = \Psi(t, y_1, y_2),$$

$$\begin{gathered}a(0) = \lim_{t\to 0^+} a(t) = \beta_1, \qquad a(\infty) = \lim_{t\to +\infty} a(t) = \beta_1 + \alpha_1 \int_0^{+\infty} \frac{ds}{p(s)}, \\ b(0) = \lim_{t\to 0^+} b(t) = \beta_2 + \alpha_2 \int_0^{+\infty} \frac{ds}{p(s)}, \qquad b(\infty) = \lim_{t\to +\infty} b(t) = \beta_2.\end{gathered} \tag{1.4}$$

In this paper, we always assume that the following conditions hold.

(H_1) $\Psi(t, y_1, y_2) \leq q(t)Q(y_1, y_2)$, $q(t) \in C(R^+, R^+)$, $Q(y_1, y_2) \in C(R^+ \times R, R^+)$ and $\int_0^{+\infty} \Phi(s)q(s)ds < +\infty$.

(H_2) For any constant $\tau \in [0, +\infty)$, $0 < \mathbf{T}(a(\tau)) < \rho$, $0 < \mathbf{K}(b(\tau)) < \rho$ and

$$\Delta = \begin{vmatrix} \rho - \mathbf{T}(b(\tau)) & \mathbf{T}(a(\tau)) \\ \mathbf{K}(b(\tau)) & \rho - \mathbf{K}(a(\tau)) \end{vmatrix} > 0. \tag{1.5}$$

Motivated and inspired by [5–9], we are concerned with the existence of multiple positive solutions for BVP(1.1) by applying Leggett-Williams fixed theorem. The main new features presented in this paper are as follows. Firstly, Sturm-Liouville nonhomogeneous boundary value problems with linear functional boundary conditions are seldom researched, it brings about many difficulties when we imply the integral equations of BVP(1.1). To solve the problem, we use a new method of undetermined coefficient to obtain the integral equations of boundary value problems with nonhomogeneous boundary conditions. Secondly, we discuss the existence of triple positive solutions and $2m-1$ positive solutions of BVP(1.1). Finally, the methods used in this paper are different from [1, 6, 7] and the results obtained in this paper generalize and involve some results in [5].

The rest of paper is organized as follows. In Section 2, we present some preliminaries and lemmas. We state and prove the main results in Section 3. Finally, in Section 4, one example with a singular nonlinearity is presented to demonstrate the application of Theorem 3.1.

2. Preliminary

In order to discuss the main results, we need the following lemmas.

Lemma 2.1. *Under the condition $\int_0^{+\infty} ds/p(s) < +\infty$ and $\rho > 0$, the boundary value problem*

$$\begin{gathered}(p(t)u'(t))' + y(t) = 0, \quad 0 < t < +\infty, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0^+} p(t)u'(t) = \mathbf{T}(u), \\ \alpha_2 \lim_{t\to+\infty} u(t) + \beta_2 \lim_{t\to+\infty} p(t)u'(t) = \mathbf{K}(u),\end{gathered} \tag{2.1}$$

has a unique solution for any $y \in L[0,+\infty)$. Moreover, this unique solution can be expressed in the form

$$u(t) = \int_0^{+\infty} G(t,s)y(s)ds + A(y)a(t) + B(y)b(t), \tag{2.2}$$

where $G(t,s)$, $A(y)$, and $B(y)$ are defined by

$$G(t,s) = \rho^{-1}\begin{cases} a(t)b(s), & 0 \le t \le s < \infty, \\ a(s)b(t), & 0 \le s < t < \infty, \end{cases} \tag{2.3}$$

$$A(y) = \frac{1}{\Delta}\begin{vmatrix} \mathbf{T}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) & \rho - \mathbf{T}(b(\tau)) \\ -\mathbf{K}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) & \mathbf{K}(b(\tau)) \end{vmatrix}, \tag{a}$$

$$B(y) = \frac{1}{\Delta}\begin{vmatrix} \mathbf{K}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) & \rho - \mathbf{K}(b(\tau)) \\ -\mathbf{T}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) & \mathbf{T}(b(\tau)) \end{vmatrix}. \tag{b}$$

Proof. $a(t)$ and $b(t)$ in (1.3) are two linear independent solutions of the equation $(p(t)u'(t))' = 0$, so the general solutions for the equation $(p(t)u'(t))' + y(t) = 0$ can be expressed in the form

$$u(t) = \int_0^{+\infty} G(t,s)y(s)ds + Ca(t) + Db(t), \tag{2.4}$$

where C, D are undetermined constants. Through verifying directly, when C and D satisfy (a) and (b) separately, $u(t)$ in (2.4) is a solution of BVP(2.1).

Now we need to prove that when $u(t)$ in (2.4) is a solution of BVP(2.1), C and D satisfy (a) and (b) separately.

Let $u(t) = \int_0^{+\infty} G(t,s)y(s)ds + Ca(t) + Db(t)$ be a solution of BVP(2.1), then

$$\begin{aligned} u(t) &= \int_0^t \frac{1}{\rho}a(s)b(t)y(s)ds + \int_t^{+\infty} \frac{1}{\rho}a(t)b(s)y(s)ds + Ca(t) + Db(t), \\ u'(t) &= \frac{b'(t)}{\rho}\int_0^t a(s)y(s)ds + \frac{a'(t)}{\rho}\int_t^{+\infty} b(s)y(s)ds + \frac{C\alpha_1}{p(t)} - \frac{D\alpha_2}{p(t)} \\ &= \frac{1}{p(t)}\left(\frac{-\alpha_2}{\rho}\int_0^t a(s)y(s)ds + \frac{\alpha_1}{\rho}\int_t^{+\infty} b(s)y(s)ds + C\alpha_1 - D\alpha_2\right), \\ (p(t)u'(t))' &= \frac{-\alpha_2}{\rho}a(t)y(t) - \frac{\alpha_1}{\rho}b(t)y(t) = -\frac{\alpha_2 a(t) + \alpha_1 b(t)}{\rho}y(t) = -y(t). \end{aligned} \tag{2.5}$$

That is, $(p(t)u'(t))' + y(t) = 0$.

By (2.4), we have

$$\begin{aligned} u(0) &= \frac{\beta_1}{\rho}\int_0^{+\infty} b(s)y(s)ds + C\beta_1 + D\beta_2 + D\alpha_2\int_0^{+\infty}\frac{ds}{p(s)}, \\ u'(0) &= \frac{1}{p(0)}\left(\frac{\alpha_1}{\rho}\int_0^{+\infty} b(s)y(s)ds + C\alpha_1 - D\alpha_2\right), \\ u(\infty) &= \frac{\beta_2}{\rho}\int_0^{+\infty} a(s)y(s)ds + C\beta_1 + D\beta_2 + C\alpha_1\int_0^{+\infty}\frac{ds}{p(s)}, \\ u'(\infty) &= \frac{1}{p(\infty)}\left(-\frac{\alpha_2}{\rho}\int_0^{+\infty} a(s)y(s)ds + C\alpha_1 - D\alpha_2\right), \end{aligned} \tag{2.6}$$

then

$$\begin{aligned} D\rho &= \mathbf{T}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) + C\mathbf{T}(a(\tau)) + D\mathbf{T}(b(\tau)), \\ C\rho &= \mathbf{K}\left(\int_0^{+\infty} G(\tau,s)y(s)ds\right) + C\mathbf{K}(a(\tau)) + D\mathbf{K}(b(\tau)). \end{aligned} \tag{2.7}$$

From (2.7), we obtain that C and D satisfy (a) and (b) separately. The proof is completed. □

Remark 2.2. Assume that (H_2) holds. Then $0 \leq A(y) < +\infty$, $0 \leq B(y) < +\infty$ for any $y \geq 0$ and any solution $u(t)$ of BVP(2.1) is nonnegative.

Lemma 2.3. *From* (1.3) *and* (2.3), *it is easy to get the following properties.*

(1) $G(t,s)/\rho^{-1}[1+a(t)b(t)] \leq 1$, $a(t)/1+a(t)b(t) < 1/b(t) \leq 1/\beta_2$, $b(t)/1+a(t)b(t) < 1/a(t) \leq 1/\beta_1$.

(2) $\overline{G}(s) = \lim_{t\to+\infty} G(t,s) = (\beta_2/\rho)a(s) < +\infty$.

(3) $G(t,s) \leq G(s,s) \leq a(s)b(s)/\rho < +\infty$.

Lemma 2.4. *For any constant* $0 < a^* < b^* < \infty$, *there exists* $0 < c^* < 1$, *such that, for* $\tau, s \in [0,\infty)$, $G(t,s)/\rho^{-1}[1+a(t)b(t)] \geq c^* G(\tau,s)/\rho^{-1}[1+a(\tau)b(\tau)]$, $a(t)/\rho^{-1}[1+a(t)b(t)] \geq c^* a(\tau)/\rho^{-1}[1+a(\tau)b(\tau)]$, $b(t)/\rho^{-1}[1+a(t)b(t)] \geq c^* b(\tau)/\rho^{-1}[1+a(\tau)b(\tau)]$, $t \in [a^*, b^*]$.

Proof. By (1.3), it is obvious that $a(t)$ is increasing, and $b(t)$ is decreasing on $t \in [0,+\infty)$; therefore, by (2.3), we have

$$\frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]} = \begin{cases} \dfrac{a(t)b(s)}{1+a(t)b(t)} \geq \dfrac{a(a^*)\beta_2}{1+a(b^*)b(a^*)}, & t \leq s, \\ \dfrac{a(s)b(t)}{1+a(t)b(t)} \geq \dfrac{b(b^*)\beta_1}{1+a(b^*)b(a^*)}, & s \leq t. \end{cases} \tag{2.8}$$

We take $c^* = \min\{a(a^*)\beta_2/(1+a(b^*)b(a^*)), b(b^*)\beta_1/(1+a(b^*)b(a^*))\}$, then $0 < c^* < 1$; this is because that

$$\frac{a(a^*)\beta_2}{1+a(b^*)b(a^*)} \leq \frac{a(b^*)b(a^*)}{1+a(b^*)b(a^*)} < 1, \qquad \frac{b(b^*)\beta_1}{1+a(b^*)b(a^*)} \leq \frac{a(b^*)b(a^*)}{1+a(b^*)b(a^*)} < 1. \tag{2.9}$$

By Lemma 2.3(1), we have $G(\tau,s)/\rho^{-1}[1+a(\tau)b(\tau)] \leq 1$, then

$$\begin{gathered} \frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]} \geq c^* \geq c^* \frac{G(\tau,s)}{\rho^{-1}[1+a(\tau)b(\tau)]}, \\ \frac{a(t)}{1+a(t)b(t)} \geq \frac{a(a^*)}{1+a(b^*)b(a^*)} = \frac{a(a^*)\beta_2}{1+a(b^*)b(a^*)}\frac{1}{\beta_2} \geq c^* \frac{1}{b(t)} > c^* \frac{a(\tau)}{1+a(\tau)b(\tau)}. \end{gathered} \tag{2.10}$$

Similarly, we can obtain that $b(t)/(1+a(t)b(t)) \geq c^*(b(\tau)/(1+a(\tau)b(\tau)))$. The proof is completed. □

In this paper, we use the space

$$E = \left\{ u \in C^1(R^+) : \sup_{t\in[0,+\infty)} \frac{|u(t)|}{\rho^{-1}[1+a(t)b(t)]} < +\infty, \sup_{t\in[0,+\infty)} \left|u'(t)\right| < +\infty \right\} \tag{2.11}$$

with the norm $\|u\| = \max\{\|u\|_1, \|u'\|_\infty\}$, where $\|u\|_1 = \sup_{t\in[0,+\infty)} |u(t)|/\rho^{-1}[1+a(t)b(t)]$ and $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$, then $(E, \|u\|)$ is a Banach space.

Let

$$P = \left\{ u \in E : u(t) \geq 0, \min_{t \in [a^*, b^*]} \frac{u(t)}{\rho^{-1}[1 + a(t)b(t)]} \geq c^* \frac{u(\tau)}{\rho^{-1}[1 + a(\tau)b(\tau)]}, \tau \in R^+ \right\}. \quad (2.12)$$

Clearly P is a cone of E.

Lemma 2.5 (see [10]). *Let $M \subseteq C_l(R^+, R) = \{x \in C(R^+, R) \mid \lim_{t \to +\infty} x(t)$ exists$\}$, then M is precompact if the following conditions hold:*

(a) *M is bounded in C_l;*

(b) *the functions belonging to M are locally equicontinuous on any interval of R^+;*

(c) *the functions from M are equiconvergent; that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

We shall consider nonnegative continuous and concave functional α on P; that is, $\alpha : P \to [0, \infty)$ is continuous and satisfies

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \quad \forall x, \ y \in P, \ 0 \leq t \leq 1. \quad (2.13)$$

We denote the set $\{x \in P \mid a \leq \alpha(x), \|x\| \leq b\} (b > a > 0)$ by $P(\alpha, a, b)$ and

$$P_r = \{x \in P \mid \|x\| < r\}. \quad (2.14)$$

The key tool in our approach is the following Leggett-Williams fixed point theorem.

Theorem 2.6 (see [11]). *Let $T : \overline{P_c} \to P_c$ be completely continuous and α a nonnegative continuous concave functional on P with $\alpha(x) \leq \|x\|$ for any $x \in \overline{P_c}$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(c_1) *$\{x \in P(\alpha, b, d) \mid \alpha(x) > b\} \neq \phi$, and $\alpha(Tx) > b$, for $x \in P(\alpha, b, d)$;*

(c_2) *$\|Tx\| < a$, for $x \in \overline{P_a}$;*

(c_3) *$\alpha(Tx) > b$ for $x \in P(\alpha, b, c)$ with $\|Tx\| > d$.*

Then T has at least three fixed points x_1, x_2, x_3, with

$$\|x_1\| < a, \qquad b < \alpha(x_2), \qquad \|x_3\| > a, \qquad \alpha(x_3) < b. \quad (2.15)$$

3. Existence Results

Define the operator $T : P \to P$ by

$$(Tu)(t) = \int_0^{+\infty} G(t,s)\Phi(s) f(s, u(s), u'(s)) ds + A(\Phi f)a(t) + B(\Phi f)b(t), \quad 0 < t < \infty. \quad (3.1)$$

Then $u(t)$ is a fixed point of operator T if and only if $u(t)$ is a solution of BVP(1.1).

For convenience, we denote $\delta,\ \alpha(x)$ by

$$0<\delta\leq\frac{a(a^*)b(b^*)}{1+a(b^*)b(a^*)}\int_{a^*}^{b^*}\Phi(s)ds,\qquad \alpha(u)=\min_{t\in[a^*,b^*]}\frac{u(t)}{\rho^{-1}[1+a(t)b(t)]},\quad \forall u\in P. \tag{3.2}$$

Theorem 3.1. *Suppose that* $(H_1),(H_2)$ *hold, and assume there exist* $0<r_1<b_1<l_1<r_2$ *with* $l_1=\max\{b_1/c^*,\sup_{t\in[0,+\infty)}(b_1/c^*p(t))\}$, *such that*

(H$_3$) $Q(y_1,y_2)\leq\min\{r_2/\int_0^{+\infty}\Phi(s)q(s)ds+A(\Phi q)/\rho^{-1}\beta_2+B(\Phi q)/\rho^{-1}\beta_1, r_2/\sup_{t\in[0,+\infty)}(1/p(t))(\int_0^{+\infty}\Phi(s)q(s)ds+A(\Phi q)\alpha_1+B(\Phi q)\alpha_2)\}$, $0\leq y_1\leq r_2$, $|y_2|\leq r_2$,

(H$_4$) $\Psi(t,y_1,y_2)>b_1/\delta,\ t\in[a^*,b^*],\ b_1\leq y_1\leq r_2,\ |y_2|\leq r_2$,

(H$_5$) $Q(y_1,y_2)<\min\{r_1/\int_0^{+\infty}\Phi(s)q(s)ds+(A(\Phi q)/\rho^{-1}\beta_2)+(B(\Phi q)/\rho^{-1}\beta_1), r_1/\sup_{t\in[0,+\infty)}(1/p(t))(\int_0^{+\infty}\Phi(s)q(s)ds+A(\Phi q)\alpha_1+B(\Phi q)\alpha_2)\}$, $0\leq y_1\leq r_1$, $|y_2|\leq r_1$.

Then BVP(1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *with*

$$\|u_1\|<r_1,\quad b_1<\alpha(u_2),\qquad \|u_3\|>r_1,\qquad \alpha(u_3)<b_1. \tag{3.3}$$

Proof. Firstly we prove that $T:P\to P$ is continuous.

We will show that $T:P\to P$ is well defined and $T(P)\subset P$. For all $u(t)\in P$, by (H$_2$), $\Phi(t)$ and f are nonnegative functions, and we have $Tu(t)\geq 0$. From (H$_1$), (H$_2$), we obtain

$$\begin{aligned}A(\Phi f)&=\frac{1}{\Delta}\begin{vmatrix}\mathbf{T}\left(\int_0^{+\infty}G(\tau,s)\Phi(s)f(s,u(s),u'(s))ds\right) & \rho-\mathbf{T}(b(\tau))\\ -\mathbf{K}\left(\int_0^{+\infty}G(\tau,s)\Phi(s)f(s,u(s),u'(s))ds\right) & \mathbf{K}(b(\tau))\end{vmatrix}\\ &\leq\frac{\max_{y_1\in[0,\|u\|],|y_2|\leq\|u\|}Q(y_1,y_2)}{\Delta}\begin{vmatrix}\mathbf{T}\left(\int_0^{+\infty}G(\tau,s)\Phi(s)q(s)ds\right) & \rho-\mathbf{T}(b(\tau))\\ -\mathbf{K}\left(\int_0^{+\infty}G(\tau,s)\Phi(s)q(s)ds\right) & \mathbf{K}(b(\tau))\end{vmatrix}\\ &=A(\Phi q)\max_{y_1\in[0,\|u\|],|y_2|\leq\|u\|}Q(y_1,y_2).\end{aligned} \tag{A}$$

In the same way, we have

$$B(\Phi f)\leq B(\Phi q)\max_{y_1\in[0,\|u\|],|y_2|\leq\|u\|}Q(y_1,y_2). \tag{B}$$

By Lemma 2.3(1), (A), (B), and (H_1), for all $u(t) \in P$, we have

$$\begin{aligned}
\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]} &= \int_0^{+\infty} \frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]}\Phi(s)f(s,u(s),u'(s))ds + \frac{A(\Phi f)a(t)}{\rho^{-1}[1+a(t)b(t)]} \\
&\quad + \frac{B(\Phi f)b(t)}{\rho^{-1}[1+a(t)b(t)]} \\
&\le \int_0^{+\infty} \Phi(s)f(s,u(s),u'(s))ds + \frac{A(\Phi f)}{\rho^{-1}\beta_2} + \frac{B(\Phi f)}{\rho^{-1}\beta_1} \\
&\le \max_{y_1\in[0,\|u\|],|y_2|\le\|u\|} Q(y_1,y_2)\left(\int_0^{+\infty}\Phi(s)q(s)ds + \frac{A(\Phi q)}{\rho^{-1}\beta_2} + \frac{B(\Phi q)}{\rho^{-1}\beta_1}\right) \\
&< +\infty,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\left|(Tu)'(t)\right| &= \frac{1}{p(t)}\left|\int_0^t \frac{-\alpha_2 a(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds \right.\\
&\qquad \left. + \int_t^{+\infty} \frac{\alpha_1 b(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds + A(\Phi f)\alpha_1 - B(\Phi f)\alpha_2\right| \\
&\le \sup_{t\in[0,+\infty)}\frac{1}{p(t)}\max_{y_1\in[0,\|u\|],|y_2|\le\|u\|} Q(y_1,y_2) \\
&\quad \times\left(\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right) \\
&< +\infty.
\end{aligned} \tag{3.5}$$

Hence, $T : P \to P$ is well defined. By (3.1), (H_1), the Lebesgue dominated convergence theorem and the continuity of $p(t)$, for any $u \in P$, $t_1, t_2 \in R^+$, we have

$$\begin{aligned}
\left|(Tu)'(t_1) - (Tu)'(t_2)\right| &\le \frac{\alpha_2 a(\infty)}{\rho}\left|\frac{1}{p(t_1)} - \frac{1}{p(t_2)}\right|\int_0^{t_1}\Phi(s)f(s,u(s),u'(s))ds \\
&\quad + \frac{\alpha_2 a(\infty)}{\rho p(t_2)}\int_{t_1}^{t_2}\Phi(s)f(s,x(s),x'(s))ds \\
&\quad + \frac{\alpha_1 b(0)}{\rho}\left|\frac{1}{p(t_1)} - \frac{1}{p(t_2)}\right|\int_0^{+\infty}\Phi(s)f(s,u(s),u'(s))ds \\
&\quad + \frac{\alpha_1 b(0)}{\rho p(t_2)}\int_{t_1}^{t_2}\Phi(s)f(s,x(s),x'(s))ds \\
&\quad + (A(\Phi f)\alpha_1 + B(\Phi f)\alpha_2)\left|\frac{1}{p(t_1)} - \frac{1}{p(t_2)}\right| \\
&\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2.
\end{aligned} \tag{3.6}$$

That is, $(Tu)(t) \in C^1(R_0^+)$; therefore, $(Tu)(t) \in E$.

By Lemma 2.4, we have

$$\begin{aligned}
\min_{t\in[a^*,b^*]}\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]} &= \min_{t\in[a^*,b^*]}\left(\int_0^{+\infty}\frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]}\Phi(s)f(s,u(s),u'(s))ds\right.\\
&\qquad\left.+\frac{a(t)A(\Phi f)}{\rho^{-1}[1+a(t)b(t)]}+\frac{b(t)B(\Phi f)}{\rho^{-1}[1+a(t)b(t)]}\right)\\
&\geq c^*\left(\int_0^{+\infty}\frac{G(\tau,s)}{\rho^{-1}[1+a(\tau)b(\tau)]}\Phi(s)f(s,u(s),u'(s))ds\right.\\
&\qquad\left.+\frac{a(\tau)A(\Phi f)}{\rho^{-1}[1+a(t)b(t)]}+\frac{b(\tau)B(\Phi f)}{\rho^{-1}[1+a(t)b(t)]}\right)\\
&= c^*\frac{(Tu)(\tau)}{\rho^{-1}[1+a(\tau)b(\tau)]},
\end{aligned}\tag{3.7}$$

therefore $T : P \to P$.

We show that $T : P \to P$ is continuous. In fact suppose $\{u_m\} \subseteq P, u_0 \in P$ and $u_m \to u_0(m \to +\infty)$, then there exists $M > 0$, such that $\|u_m\| \leq M$. By (H_1), we have

$$\begin{aligned}
\int_0^{+\infty}\Phi(s)\left|f(s,u_m(s),u_m'(s))-f(s,u_0(s),u_0'(s))\right|ds &\leq 2\int_0^{+\infty}\Phi(s)f(s,u(s),u'(s))ds\\
&\leq 2\max_{y_1\in[0,M],|y_2|\leq M}Q(y_1,y_2)\\
&\qquad\times\int_0^{+\infty}\Phi(s)q(s)ds\\
&< +\infty.
\end{aligned}\tag{3.8}$$

Therefore, by Lemma 2.3(1), the continuity of f and the Lebesgue dominated convergence theorem imply that

$$\begin{aligned}
\left|\frac{(Tu_m)(t)-(Tu_0)(t)}{\rho^{-1}[1+a(t)b(t)]}\right| &= \left|\int_0^{+\infty}\frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]}\right.\\
&\qquad\left.\times\Phi(s)\left[f(s,u_m(s),u_m'(s))-f(s,u_0(s),u_0'(s))\right]ds\right|\\
&\leq\int_0^{+\infty}\Phi(s)\left|f(s,u_m(s),u_m'(s))-f(s,u_0(s),u_0'(s))\right|ds\longrightarrow 0,\\
&\qquad m\longrightarrow+\infty,
\end{aligned}$$

$$\left|(Tu_{\mathrm{m}})'(t)-(Tu_0)'(t)\right| \leq \sup_{t\in[0,+\infty)}\frac{1}{p(t)}\int_0^{+\infty}\Phi(s)\left|f\left(s,u_m(s),u_m'(s)\right)-f\left(s,u_0(s),u_0'(s)\right)\right|ds$$
$$\longrightarrow 0,\quad m\longrightarrow +\infty. \tag{3.9}$$

Thus, $\|Tu_m - Tu_0\| \to 0(m\to +\infty)$. Therefore $T: P\to P$ is continuous.

Secondly we show that $T: P\to P$ is compact operator.

For any bounded set $B\subset P$, there exists a constant $L>0$ such that $\|u\|\leq L$, for all $u\in B$. By Lemma 2.3(1), (A), (B), and (H_1), we have

$$\begin{aligned}
(Tu)(t) &= \rho^{-1}[1+a(t)b(t)]\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]}\\
&\leq \rho^{-1}[1+a(\infty)b(0)]\left(\int_0^{+\infty}\frac{G(t,s)}{\rho^{-1}[1+a(t)b(t)]}\Phi(s)f\left(s,u(s),u'(s)\right)ds\right.\\
&\quad\left.+\frac{A(\Phi f)a(t)}{\rho^{-1}[1+a(t)b(t)]}+\frac{B(\Phi f)b(t)}{\rho^{-1}[1+a(t)b(t)]}\right)\\
&\leq \rho^{-1}[1+a(\infty)b(0)]\left(\int_0^{+\infty}\Phi(s)f\left(s,u(s),u'(s)\right)ds+\frac{A(\Phi f)}{\rho^{-1}\beta_2}+\frac{B(\Phi f)}{\rho^{-1}\beta_1}\right)\\
&\leq \rho^{-1}[1+a(\infty)b(0)]\max_{y_1\in[0,L],|y_2|\leq L}Q(y_1,y_2)\left(\int_0^{+\infty}\Phi(s)q(s)ds+\frac{A(\Phi q)}{\rho^{-1}\beta_2}+\frac{B(\Phi q)}{\rho^{-1}\beta_1}\right)\\
&< +\infty,\\
(Tu)(\infty) &= \int_0^{+\infty}\overline{G}(s)\Phi(s)f\left(s,u(s),u'(s)\right)ds+A(\Phi f)a(\infty)+B(\Phi f)b(\infty)\\
&= \frac{\beta_2}{\rho}\int_0^{+\infty}a(s)\Phi(s)f\left(s,u(s),u'(s)\right)ds+A(\Phi f)a(\infty)+B(\Phi f)b(\infty)\\
&\leq \max_{y_1\in[0,L],|y_2|\leq L}Q(y_1,y_2)\left(\frac{\beta_2 a(\infty)}{\rho}\int_0^{+\infty}\Phi(s)q(s)ds+A(\Phi q)a(\infty)+B(\Phi q)b(\infty)\right)\\
&< +\infty.
\end{aligned} \tag{3.10}$$

Therefore, $(Tu)(t)\subseteq C_l(R^+,R)$.

By (3.4) and (3.5), we have

$$\begin{aligned}
\|Tu\|_1 &= \sup_{t\in[0,+\infty)}\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]}\\
&\leq \max_{y_1\in[0,L],|y_2|\leq L}Q(y_1,y_2)\left(\int_0^{+\infty}\Phi(s)q(s)ds+\frac{A(\Phi q)}{\rho^{-1}\beta_2}+\frac{B(\Phi q)}{\rho^{-1}\beta_1}\right)\\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}\left\|(Tu)'\right\|_\infty &= \max_{t\in[0,+\infty)}\left|(Tu)'(t)\right| \\ &\le \sup_{t\in[0,+\infty)}\frac{1}{p(t)}\max_{y_1\in[0,L],|y_2|\le L}Q(y_1,y_2)\left(\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right) \\ &< +\infty,\end{aligned} \tag{3.11}$$

so TB is bounded.

Given $T>0$, t_1, $t_2\in[0,T]$, by (H_1) and Lemma 2.3(1), we have

$$\left|\frac{G(t_1,s)}{\rho^{-1}[1+a(t_1)b(t_1)]}-\frac{G(t_2,s)}{\rho^{-1}[1+a(t_2)b(t_2)]}\right|\Phi(s)f(s,u(s),u'(s)) \le 2\max_{y_1\in[0,L],|y_2|\le L}Q(y_1,y_2) \times\Phi(s)q(s). \tag{3.12}$$

Therefore for any $u\in B$, by (3.1), the Lebesgue dominated convergence theorem and the continuity of $G(t,s)$, $a(t)$, and $b(t)$, we have

$$\begin{aligned}&\left|\frac{(Tu)(t_1)}{\rho^{-1}[1+a(t_1)b(t_1)]}-\frac{(Tx)(t_2)}{\rho^{-1}[1+a(t_2)b(t_2)]}\right| \\ &\quad\le \int_0^{+\infty}\left|\frac{G(t_1,s)}{\rho^{-1}[1+a(t_1)b(t_1)]}-\frac{G(t_2,s)}{\rho^{-1}[1+a(t_2)b(t_2)]}\right| \\ &\quad\times\Phi(s)f(s,u(s),u'(s))ds \\ &\quad + A(\Phi f)\left|\frac{a(t_1)}{\rho^{-1}[1+a(t_1)b(t_1)]}-\frac{a(t_2)}{\rho^{-1}[1+a(t_2)b(t_2)]}\right| \\ &\quad + B(\Phi f)\left|\frac{b(t_1)}{\rho^{-1}[1+a(t_1)b(t_1)]}-\frac{b(t_2)}{\rho^{-1}[1+a(t_2)b(t_2)]}\right| \\ &\quad\longrightarrow 0,\quad \text{as } t_1\longrightarrow t_2.\end{aligned} \tag{3.13}$$

By a similar proof as (3.6), we obtain $|(Tu)'(t_1)-(Tu)'(t_2)|\to 0$, as $t_1\to t_2$. Thus, TB is equicontinuous on $[0,T]$. Since $T>0$ is arbitrary, TB is locally equicontinuous on $[0,+\infty)$.

By Lemma 2.3(2), (H_2) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{t\to+\infty}\left|\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]}\right| \\ &\quad=\frac{1}{\rho^{-1}[1+a(\infty)b(\infty)]}\left|\int_0^{+\infty}\beta_2 a(s)\Phi(s)f(s,x(s),x'(s))ds + A(\Phi f)a(\infty)+B(\Phi f)b(\infty)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\max_{y_1\in[0,L],|y_2|\leq L} Q(y_1,y_2)}{\rho^{-1}(1+\beta_1\beta_2)}\left(\beta_2 a(\infty)\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)a(\infty)+B(\Phi q)b(\infty)\right)\\
&< +\infty,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{(Tu)(t)}{\rho^{-1}[1+a(t)b(t)]}-\frac{(Tu)(\infty)}{\rho^{-1}[1+a(\infty)b(\infty)]}\right|\\
&\quad\leq \int_0^t \mathrm{a}(s)b(t)\left|\frac{1}{1+a(t)b(t)}-\frac{1}{1+a(\infty)b(\infty)}\right|\Phi(s)f(s,x(s),x'(s))ds\\
&\qquad+\int_0^t \frac{a(s)}{1+a(\infty)b(\infty)}\left|b(t)-\beta_2\right|\Phi(s)f(s,x(s),x'(s))ds\\
&\qquad+\int_t^{+\infty} b(s)\frac{|a(t)-a(s)|}{1+a(t)b(t)}\Phi(s)f(s,x(s),x'(s))ds\\
&\qquad+\int_t^{+\infty} a(s)b(s)\left|\frac{1}{1+a(t)b(t)}-\frac{1}{1+a(\infty)b(\infty)}\right|\Phi(s)f(s,x(s),x'(s))ds\\
&\qquad+\int_t^{+\infty}\frac{a(s)}{1+a(\infty)b(\infty)}\left|b(s)-\beta_2\right|\Phi(s)f(s,x(s),x'(s))ds + A(\Phi f)\frac{|a(t)-a(\infty)|}{\rho^{-1}[1+a(t)b(t)]}\\
&\qquad+B(\Phi f)\frac{|b(t)-b(\infty)|}{\rho^{-1}[1+a(t)b(t)]}+\left[A(\Phi f)a(\infty)+B(\Phi f)b(\infty)\right]\\
&\qquad\times\left|\frac{1}{\rho^{-1}[1+a(t)b(t)]}-\frac{1}{\rho^{-1}[1+a(\infty)b(\infty)]}\right|\\
&\quad\leq \max_{y_1\in[0,M],|y_2|\leq M} Q(y_1,y_2)\\
&\qquad\times\left\{b(0)a(\infty)\int_0^t\left|\frac{1}{1+a(t)b(t)}-\frac{1}{1+a(\infty)b(\infty)}\right|\Phi(s)q(s)ds\right.\\
&\qquad\quad+\frac{a(\infty)}{1+a(\infty)b(\infty)}\int_0^t\left|b(t)-\beta_2\right|\Phi(s)q(s)ds+\frac{b(0)}{1+\beta_1\beta_2}\int_t^{\infty}|a(t)-a(s)|\Phi(s)q(s)ds\\
&\qquad\quad+a(\infty)b(0)\int_t^{\infty}\left|\frac{1}{1+a(t)b(t)}-\frac{1}{1+a(\infty)b(\infty)}\right|\Phi(s)q(s)ds\\
&\qquad\quad+\frac{a(\infty)}{1+a(\infty)b(\infty)}\int_t^{+\infty}\left|b(s)-\beta_2\right|\Phi(s)q(s)ds+A(\Phi q)\frac{|a(t)-a(\infty)|}{\rho^{-1}[1+a(t)b(t)]}\\
&\qquad\quad+B(\Phi q)\frac{|b(t)-b(\infty)|}{\rho^{-1}[1+a(t)b(t)]}+\left(A(\Phi q)a(\infty)+B(\Phi q)b(\infty)\right)\\
&\qquad\quad\left.\times\left|\frac{1}{\rho^{-1}[1+a(t)b(t)]}-\frac{1}{\rho^{-1}[1+a(\infty)b(\infty)]}\right|\right\}\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty.
\end{aligned}
\tag{3.14}
$$

By (3.5), we know that $\lim_{t\to+\infty}|(Tu)'(t)| < +\infty$, then

$$\begin{aligned}
&\left|(Tu)'(t)-(Tu)'(\infty)\right| \\
&\quad = \Bigg|\frac{1}{p(t)}\int_0^t \frac{-\alpha_2 a(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds + \frac{1}{p(t)}\int_t^{+\infty}\frac{\alpha_1 b(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds \\
&\qquad + \frac{1}{p(t)}A(\Phi f)\alpha_1 - \frac{1}{p(t)}B(\Phi f)\alpha_2 + \frac{1}{p(\infty)}\int_0^t \frac{\alpha_2 a(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds \\
&\qquad + \frac{1}{p(\infty)}\int_t^{+\infty}\frac{\alpha_2 a(s)}{\rho}\Phi(s)f(s,u(s),u'(s))ds - \frac{1}{p(\infty)}A(\Phi f)\alpha_1 + \frac{1}{p(\infty)}B(\Phi f)\alpha_2\Bigg| \\
&\quad \le \max_{y_1\in[0,L],|y_2|\le L} Q(y_1,y_2)\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right| \\
&\qquad \times\Bigg\{\int_0^t \frac{\alpha_2 a(s)}{\rho}\Phi(s)q(s)ds + \frac{1}{p(t)}\int_t^{+\infty}\frac{\alpha_1 b(s)}{\rho}\Phi(s)q(s)ds + \frac{1}{p(\infty)} \\
&\qquad\qquad \int_t^{+\infty}\frac{\alpha_2 a(s)}{\rho}\Phi(s)q(s)ds + \left(A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right)\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\Bigg\} \longrightarrow 0, \\
&\quad \text{as } t \longrightarrow +\infty.
\end{aligned} \tag{3.15}$$

Therefore, TB is equiconvergent at ∞. By Lemma 2.5, TB is completely continuous.

Finally we will show that all conditions of Theorem 2.6 hold.

From the definition of α, we can get $\alpha(u) \le \|u\|$ for all $u \in P$. For all $u \in \overline{P_{r_2}}$, we have $\|u\| \le r_2$; therefore $0 \le y_1 \le r_2$, $|y_2| \le r_2$. By (3.4), (3.5), and (H_3), we have

$$\begin{aligned}
\frac{|(Tu)(t)|}{\rho^{-1}[1+a(t)b(t)]} &\le \max_{y_1\in[0,r_2],|y_2|\le r_2} Q(y_1,y_2)\left(\int_0^{+\infty}\Phi(s)q(s)ds + \frac{A(\Phi q)}{\rho^{-1}\beta_2} + \frac{B(\Phi q)}{\rho^{-1}\beta_1}\right) \\
&\le r_2, \\
\left|(Tu)'(t)\right| &\le \sup_{t\in[0,+\infty)}\frac{1}{p(t)}\max_{y_1\in[0,r_2],|y_2|\le r_2} Q(y_1,y_2) \\
&\quad \times\left(\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right) \\
&\le r_2,
\end{aligned} \tag{3.16}$$

that is, $\|Tu\| \le r_2$ for $u \in \overline{P_{r_2}}$. Thus $T : \overline{P_{r_2}} \to \overline{P_{r_2}}$.

Similarly for any $u \in \overline{P_{r_1}}$, we have $\|Tu\| < r_1$, which means that condition (c_2) of Theorem 2.6 holds.

In order to apply condition (c_1) of Theorem 2.6, we choose $u(t) = b_1\rho^{-1}[1 + a(t)b(t)]/c^*$, $t \in R_0^+$, then $\|u\| \le l_1$; this is because

$$\|u\|_1 = \frac{b_1}{c^*} \le l_1,$$
$$\|u'\|_\infty = \sup_{t\in[0,+\infty)} |u'(t)| = \sup_{t\in[0,+\infty)} \left|\frac{b_1\rho^{-1}[a'(t)b(t) + a(t)b'(t)]}{c^*}\right| \le \sup_{t\in[0,+\infty)} \frac{1}{p(t)}\frac{b_1}{c^*} \le l_1, \tag{3.17}$$

and $\alpha(u) = \min_{t\in[a^*,b^*]}(u(t)/\rho^{-1}[1 + a(t)b(t)]) = b_1/c^* > b_1$, which means that $\{u \in P(\alpha, b_1, l_1) | \alpha(u) > b_1\} \ne \phi$. For all $u \in P(\alpha, b_1, l_1)$, we have $\alpha(u) \ge b_1$ and $\|u\| \le l_1$, thus $b_1 \le u(t)/\rho^{-1}[1 + a(t)b(t)] \le l_1, |u'(t)| \le l_1$, that is, $b_1 \le y_1 \le l_1, |y_2| \le l_1$. By (H_4), we can get

$$\begin{aligned}
\alpha(Tu(t)) &= \min_{t\in[a^*,b^*]} \frac{(Tu)(t)}{\rho^{-1}[1 + a(t)b(t)]} \ge \min_{t\in[a^*,b^*]} \frac{1}{\rho^{-1}[1 + a(t)b(t)]} \\
&\quad\times \left(\int_0^{a^*} \frac{a(s)b(t)}{\rho}\Phi(s)f(s, u(s), u'(s))ds\right. \\
&\qquad + \int_{a^*}^{t} \frac{a(s)b(t)}{\rho}\Phi(s)f(s, u(s), u'(s))ds + \int_t^{b^*} \frac{a(t)b(s)}{\rho}\Phi(s)f(s, u(s), u'(s))ds \\
&\qquad \left.+ \int_{b^*}^{+\infty} \frac{a(t)b(s)}{\rho}\Phi(s)f(s, u(s), u'(s))ds\right) \\
&> \frac{a(a^*)b(b^*)}{1 + a(b^*)b(a^*)} \int_{a^*}^{b^*} \Phi(s)f(s, u(s), u'(s))ds \\
&> \frac{a(a^*)b(b^*)}{1 + a(b^*)b(a^*)} \int_{a^*}^{b^*} \Phi(s)ds\frac{b_1}{\delta} \\
&\ge b_1.
\end{aligned} \tag{3.18}$$

Consequently condition (c_1) of Theorem 2.6 holds.

We will prove that condition (c_3) of Theorem 2.6 holds. If $u \in P(\alpha, b_1, r_2)$, and $\|Tu(t)\| > l_1$, by (H_4), we have

$$\alpha(Tu(t)) = \min_{t\in[a^*,b^*]} \frac{(Tu)(t)}{\rho^{-1}[1 + a(t)b(t)]} > \frac{a(a^*)b(b^*)}{1 + a(b^*)b(a^*)} \int_{a^*}^{b^*} \Phi(s)ds\frac{b_1}{\delta} \ge b_1. \tag{3.19}$$

Therefore, condition (c_3) of Theorem 2.6 is satisfied. Then we can complete the proof of this theorem by Leggett-Williams fixed point theorem. □

Theorem 3.2. *Suppose that* (H_1), (H_2) *hold, and assume there exist* $0 < r_1 < b_1 < l_1 < r_2 < b_2 < l_2 < r_3 < \cdots < r_m$ *with* $l_i = \max\{b_i/c^*, \sup_{t\in[0,+\infty)} b_i/c^* p(t)\}$, *such that*

(H_6) $Q(y_1, y_2) < \min\{r_i / \int_0^{+\infty} \Phi(s)q(s)ds + A(\Phi q)/\rho^{-1}\beta_2 + B(\Phi q)/\rho^{-1}\beta_1, r_i / \sup_{t\in[0,+\infty)}(1/p(t))(\int_0^{+\infty} \Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2)\}$, $0 \le y_1 \le r_i$, $|y_2| \le r_i$, $1 \le i \le m$,

(H_7) $\Psi(t, y_1, y_2) > b_i/\delta$, $t \in [a^*, b^*]$, $b_i \le y_1 \le r_{i+1}$, $|y_2| \le r_{i+1}$, $1 \le i \le m-1$.

Then BVP(1.1) *has at least* $2m-1$ *positive solutions.*

Proof. When $m = 1$, it follows from (H_6) that T has at least one positive solution by the Schauder fixed point theorem. When $m = 2$, it is clear that Theorem 3.1 holds. Then we can obtain three positive solutions. In this way, we can finish the proof by the method of induction. □

4. Example

Consider the following singular Sturm-Liouville singular boundary value problems for second-order differential equation on the half-line

$$\begin{gathered}
\left((1+t)^2 u'(t)\right)' + \frac{e^{-t}(1+t)}{\sqrt{t}} f(t, u(t), u'(t)) = 0, \quad 0 < t < +\infty, \\
u(0) - \lim_{t\to 0^+} p(t)u'(t) = \sum_{i=1}^{m-2} \left(\frac{1}{3}\right)^i u(\xi_i), \quad 0 < \xi_i < +\infty, \\
\lim_{t\to+\infty} u(t) + \lim_{t\to+\infty} p(t)u'(t) = \int_0^{+\infty} \frac{1}{3} e^{-s}(1+s)u(s)ds,
\end{gathered} \tag{4.1}$$

where

$$f(t, u(t), u'(t)) = \Psi(t, y_1, y_2) = \begin{cases} y_1^4 + \dfrac{1}{550}|y_2|, & y_1 \le 1, \\ 1 + \dfrac{1}{550}|y_2|, & y_1 \ge 1, \end{cases} \tag{4.2}$$

$p(t) = (1+t)^2$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $a(t) = 2-1/(1+t)$, $b(t) = 1+1/(1+t)$, $\Phi(t) = e^{-t}(1+t)/\sqrt{t}$ which is singular at $t = 0$, $\rho = 3$, $\mathbf{T}(u) = \sum_{i=1}^{m-2}(1/3)^i u(\xi_i)$, $\mathbf{K}(u) = \int_0^{+\infty}(1/3)e^{-s}(1+s)u(s)ds$.

Set $q(t) = 1$ and

$$Q(y_1, y_2) = \begin{cases} y_1^4 + \dfrac{1}{550}|y_2|, & y_1 \le 1, \\ 1 + \dfrac{1}{550}|y_2|, & y_1 \ge 1, \end{cases} \tag{4.3}$$

then $\int_0^{+\infty} \Phi(s)q(s)ds < 3$, $a(0) = 1$, $a(\infty) = 2$, $b(0) = 2$, $b(\infty) = 1$, $1/2 < \mathbf{T}(a(\tau)) < 1$, $1/2 < \mathbf{T}(b(\tau)) < 1$, $\mathbf{K}(a(\tau)) = \mathbf{K}(b(\tau)) = 1$, $\Delta > 3$, $A(\Phi q) < 26/9$, $B(\Phi q) < 20/9$.

Choose $r_1 = 1/3$, $b_1 = 7/5$, $r_2 = 19$. When $a^* = 1$, $b^* = 2$, by the definition of δ, we may choose $\delta = 8/5$. By direct calculations, we imply that

$$\min\left\{\frac{r_1}{\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)/\rho^{-1}\beta_2 + B(\Phi q)/\rho^{-1}\beta_1},\right.$$
$$\left.\frac{r_1}{\sup_{t\in[0,+\infty)}(1/p(t))\left(\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right)}\right\} > \frac{3r_1}{55},$$
$$\min\left\{\frac{r_2}{\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)/\rho^{-1}\beta_2 + B(\Phi q)/\rho^{-1}\beta_1},\right.$$
$$\left.\frac{r_2}{\sup_{t\in[0,+\infty)}(1/p(t))\left(\int_0^{+\infty}\Phi(s)q(s)ds + A(\Phi q)\alpha_1 + B(\Phi q)\alpha_2\right)}\right\} > \frac{3r_2}{55}, \quad (4.4)$$
$$Q(y_1,y_2) \le \left(\frac{1}{3}\right)^4 + \frac{1}{550}\times\frac{1}{3} < \frac{1}{55} = \frac{3r_1}{55}, \quad \text{for } 0\le y_1\le\frac{1}{3},\ |y_2|\le\frac{1}{3},$$
$$Q(y_1,y_2) \le 1 + \frac{19}{550} < \frac{3\times 19}{55} = \frac{3r_2}{55}, \quad \text{for } 0\le y_1\le 19,\ |y_2|\le 19,$$
$$\Psi(t,y_1,y_2) \ge 1 > \frac{7/5}{8/5} = \frac{b_1}{\delta}, \quad \text{for } t\in[1,2],\ 7/5\le y_1\le 19,\ |y_2|\le 19.$$

Therefore, the conditions (H_1)–(H_5) hold. Applying Theorem 3.1 we conclude that BVP(4.1) has at least three positive solutions.

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References

[1] J. Mao, Z. Zhao, and N. Xu, "The existence and uniqueness of positive solutions for integral boundary value problems," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 34, no. 1, pp. 153–164, 2011.

[2] A. Boucherif, "Second-order boundary value problems with integral boundary conditions," *Nonlinear Analysis A*, vol. 70, no. 1, pp. 364–371, 2009.

[3] X. P. Liu, M. Jia, and W. G. Ge, "Existence of monotone positive solutions to a type of three-point boundary value problem," *Acta Mathematicae Applicatae Sinica*, vol. 30, no. 1, pp. 111–119, 2007.

[4] X. Ni and W. Ge, "Existence of multiple positive solutions for boundary value problems on the half-line," *Journal of Systems Science and Mathematical Sciences*, vol. 26, no. 1, pp. 113–120, 2006.

[5] M. H. Xing, K. M. Zhang, and H. L. Gao, "Existence of multiple positive solutions for general Sturm-Liouville boundary value problems on the half-line," *Acta Mathematica Scientia A*, vol. 29, no. 4, pp. 929–939, 2009.

[6] L. Liu, Z. Wang, and Y. Wu, "Multiple positive solutions of the singular boundary value problems for second-order differential equations on the half-line," *Nonlinear Analysis A*, vol. 71, no. 7-8, pp. 2564–2575, 2009.

[7] Z. Q. Zhao and F. S. Li, "Existence and uniqueness of positive solutions for some singular boundary value problems with linear functional boundary conditions," *Acta Mathematica Sinica*, vol. 27, no. 10, pp. 2073–2084, 2011.

[8] X. Zhang, L. Liu, and Y. Wu, "Existence of positive solutions for second-order semipositone differential equations on the half-line," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 628–635, 2007.

[9] H. Lian, H. Pang, and W. Ge, "Triple positive solutions for boundary value problems on infinite intervals," *Nonlinear Analysis A*, vol. 67, no. 7, pp. 2199–2207, 2007.

[10] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, NY, USA, 1973.

[11] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.