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Research Article

Lightlike Submanifolds of a Semi-Riemannian Manifold of Quasi-Constant Curvature

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We study the geometry of lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasiconstant curvature subject to the following conditions: (1) the curvature vector field ζ of $\widetilde{M}$ is tangent to M, (2) the screen distribution $S(TM)$ of M is totally geodesic in M, and (3) the coscreen distribution $S(TM^{\perp})$ of M is a conformal Killing distribution.

1. Introduction

In the generalization from the theory of submanifolds in Riemannian to the theory of submanifolds in semi-Riemannian manifolds, the induced metric on submanifolds may be degenerate (lightlike). Therefore, there is a natural existence of lightlike submanifolds and for which the local and global geometry is completely different than nondegenerate case. In lightlike case, the standard text book definitions do not make sense, and one fails to use the theory of nondegenerate geometry in the usual way. The primary difference between the lightlike submanifolds and nondegenerate submanifolds is that in the first case, the normal vector bundle intersects with the tangent bundle. Thus, the study of lightlike submanifolds becomes more difficult and different from the study of nondegenerate submanifolds. Moreover, the geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, and Kruskal's horizons). The universe can be represented as a four-dimensional submanifold embedded in a $(4 + n)$-dimensional spacetime manifold. Lightlike hypersurfaces are also studied in the theory of electromagnetism [1]. Thus, large number of applications but limited information available motivated us to do research on this subject matter. Kupeli [2] and Duggal and Bejancu [1] developed the general theory of degenerate (lightlike) submanifolds. They constructed a transversal vector bundle of lightlike submanifold and investigated various properties of these manifolds.

In the study of Riemannian geometry, Chen and Yano [3] introduced the notion of a *Riemannian manifold of a quasiconstant curvature* as a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with the curvature tensor $\widetilde{R}$ satisfying the condition

$$\begin{aligned} \widetilde{g}\left(\widetilde{R}(X,Y)Z,W\right) = \alpha\{&\widetilde{g}(Y,Z)\widetilde{g}(X,W) - \widetilde{g}(X,Z)\widetilde{g}(Y,W)\} \\ &+ \beta\{\widetilde{g}(X,W)\theta(Y)\theta(Z) - \widetilde{g}(X,Z)\theta(Y)\theta(W) \\ &\quad + \widetilde{g}(Y,Z)\theta(X)\theta(W) - \widetilde{g}(Y,W)\theta(X)\theta(Z)\}, \end{aligned} \tag{1.1}$$

for any vector fields $X,\ Y,\ Z$, and W on $\widetilde{M}$, where $\alpha,\ \beta$ are scalar functions and θ is a 1-form defined by

$$\theta(X) = \widetilde{g}(X, \zeta), \tag{1.2}$$

where ζ is a unit vector field on $\widetilde{M}$ which called the *curvature vector field*. It is well known that if the curvature tensor $\widetilde{R}$ is of the form (1.1), then the manifold is conformally flat. If $\beta = 0$, then the manifold reduces to a space of constant curvature.

A nonflat Riemannian manifold of dimension $n(> 2)$ is defined to be a quasi-Einstein manifold [4] if its Ricci tensor satisfies the condition

$$\widetilde{\mathrm{Ric}}(X,Y) = a\widetilde{g}(X,Y) + b\phi(X)\phi(Y), \tag{1.3}$$

where $a,\ b$ are scalar functions such that $b \neq 0$, and ϕ is a nonvanishing 1-form such that $\widetilde{g}(X,U) = \phi(X)$ for any vector field X, where U is a unit vector field. If $b = 0$, then the manifold reduces to an Einstein manifold. It can be easily seen that every Riemannian manifold of quasiconstant curvature is a quasi-Einstein manifold.

The subject of this paper is to study the geometry of lightlike submanifolds of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasiconstant curvature. We prove two characterization theorems for such a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ as follows.

Theorem 1.1. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasi-constant curvature. If the curvature vector field ζ of $\widetilde{M}$ is tangent to M and $S(TM)$ is totally geodesic in M, then one has the following results:*

(1) *if $S(TM^{\perp})$ is a Killing distribution, then the functions α and β, defined by* (1.1)*, vanish identically. Furthermore, $\widetilde{M}$, M, and the leaf M^* of $S(TM)$ are flat manifolds;*

(2) *if $S(TM^{\perp})$ is a conformal Killing distribution, then the function β vanishes identically. Furthermore, $\widetilde{M}$ and M^* are space of constant curvatures, and M is an Einstein manifold such that* $\mathrm{Ric} = (r/(m-r))g$*, where r is the induced scalar curvature of M.*

Theorem 1.2. *Let M be an irrotational r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasiconstant curvature. If ζ is tangent to M, $S(TM)$ is totally umbilical in M, and*

$S(TM^{\perp})$ is a conformal Killing distribution with a nonconstant conformal factor, then the function β vanishes identically. Moreover, $\widetilde{M}$ and M^* are space of constant curvatures, and M is a totally umbilical Einstein manifold such that $\mathrm{Ric} = (c/(m-r))g$, *where c is the scalar quantity of M.*

2. Lightlike Submanifolds

Let (M, g) be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. We follow Duggal and Bejancu [1] for notations and results used in this paper. The radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r $(1 \le r \le \min\{m, n\})$. Then, in general, there exist two complementary nondegenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $\mathrm{Rad}(TM)$ in TM and $TM^{\perp}$, respectively, called the *screen* and *coscreen distribution* on M, and we have the following decompositions:

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM); \qquad TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right), \tag{2.1}$$

where the symbol $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $M = (M, g, S(TM), S(TM^{\perp}))$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\widetilde{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_i\}$ be a lightlike basis of $\Gamma(\mathrm{ltr}(TM)_{|\mathcal{U}})$ consisting of smooth sections of $S(TM)^{\perp}_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\widetilde{g}(N_i, \xi_j) = \delta_{ij}, \qquad \widetilde{g}(N_i, N_j) = 0, \tag{2.2}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$. Then,

$$\begin{aligned} T\widetilde{M} &= TM \oplus \mathrm{tr}(TM) = \{\mathrm{Rad}(TM) \oplus \mathrm{tr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \\ &= \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{2.3}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\widetilde{M}$ is

(1) *r-lightlike submanifold* if $1 \le r < \min\{m, n\}$,

(2) *coisotropic submanifold* if $1 \le r = n < m$,

(3) *isotropic submanifold* if $1 \le r = m < n$,

(4) *totally lightlike submanifold* if $1 \le r = m = n$.

The above three classes (2)~(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$, and $S(TM) = S(TM^{\perp}) = \{0\}$, respectively.

Example 2.1. Consider in $\mathbb{R}^4_2$ the 1-lightlike submanifold M given by equations

$$x^3 = \frac{1}{\sqrt{2}}\left(x^1 + x^2\right), \qquad x^4 = \frac{1}{2}\log\left(1 + \left(x^1 - x^2\right)^2\right), \tag{2.4}$$

then we have $TM = \text{span}\{U_1, U_2\}$ and $TM^{\perp} = \{H_1, H_2\}$, where we set

$$\begin{gathered}
U_1 = \sqrt{2}\left(1 + \left(x^1 - x^2\right)^2\right)\partial x^1 + \left(1 + \left(x^1 - x^2\right)^2\right)\partial x^3 + \sqrt{2}\left(x^1 - x^2\right)\partial x^4, \\
U_2 = \sqrt{2}\left(1 + \left(x^1 - x^2\right)^2\right)\partial x^2 + \left(1 + \left(x^1 - x^2\right)^2\right)\partial x^3 + \sqrt{2}\left(x^1 - x^2\right)\partial x^4, \\
H_1 = \partial x^1 + \partial x^2 + \sqrt{2}\partial x^3, \\
H_2 = 2\left(1 + \left(x^2 - x^1\right)^2\right)\partial x^2 + \sqrt{2}\left(x^1 - x^2\right)\partial x^3 + \left(1 + \left(x^1 - x^2\right)^2\right)\partial x^4.
\end{gathered} \tag{2.5}$$

It follows that $\text{Rad}(TM)$ is a distribution on M of rank 1 spanned by $\xi = H_1$. Choose $S(TM)$ and $S(TM^{\perp})$ spanned by U_2 and H_2 where are timelike and spacelike, respectively. Finally, the lightlike transversal vector bundle

$$\text{ltr}(TM) = \text{Span}\left\{N = \frac{1}{2}\partial x^1 + \frac{1}{2}\partial x^2 + \frac{1}{\sqrt{2}}\partial x^3\right\} \tag{2.6}$$

and the transversal vector bundle

$$\text{tr}(TM) = \text{Span}\{N, H_2\} \tag{2.7}$$

are obtained.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.1). For an r-lightlike submanifold, the local Gauss-Weingartan formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y) W_{\alpha}, \tag{2.8}$$

$$\widetilde{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_{\alpha}, \tag{2.9}$$

$$\widetilde{\nabla}_X W_{\alpha} = -A_{W_{\alpha}} X + \sum_{i=1}^{r} \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X) W_{\beta}, \tag{2.10}$$

$$\nabla_X PY = \nabla_X^{*} PY + \sum_{i=1}^{r} h_i^{*}(X, PY)\xi_i, \tag{2.11}$$

$$\nabla_X \xi_i = -A_{\xi_i}^{*} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^{*} are induced linear connections on TM and $S(TM)$, respectively, the bilinear forms h_i^{ℓ} and h_{α}^{s} on M are called the *local lightlike second fundamental form* and *local screen second fundamental form* on TM, respectively, and h_i^{*} is called the *local radical second fundamental form* on $S(TM)$. A_{N_i}, $A_{\xi_i}^{*}$, and $A_{W_{\alpha}}$ are linear operators on $\Gamma(TM)$, and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$, and $\theta_{\alpha\beta}$ are 1-forms on TM.

Since $\widetilde{\nabla}$ is torsion-free, ∇ is also torsion-free and both h_i^ℓ and h_α^s are symmetric. From the fact that $h_i^\ell(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi_i)$, we know that h_i^ℓ are independent of the choice of a screen distribution. Note that h_i^ℓ, τ_{ij}, and $\rho_{i\alpha}$ depend on the section $\xi \in \Gamma(\text{Rad}(TM)|_{\mathcal{U}})$. Indeed, take $\overline{\xi}_i = \sum_{j=1}^r a_{ij}\xi_j$, then we have $d(\text{tr}(\tau_{ij})) = d(\text{tr}(\widetilde{\tau}_{ij}))$ [5].

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^r \left\{ h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y) \right\}, \tag{2.13}$$

where η_i is the 1-form such that

$$\eta_i(X) = \widetilde{g}(X,N_i), \quad \forall X \in \Gamma(TM),\ i \in \{1,\ldots,r\}. \tag{2.14}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$h_i^\ell(X,Y) = g\left(A_{\xi_i}^* X, Y\right) - \sum_{k=1}^r h_k^\ell(X,\xi_i)\eta_k(Y), \tag{2.15}$$

$$h_i^\ell(X,PY) = g\left(A_{\xi_i}^* X, PY\right), \qquad \widetilde{g}\left(A_{\xi_i}^* X, N_j\right) = 0, \tag{2.16}$$

$$\epsilon_\alpha h_\alpha^s(X,Y) = g(A_{W_\alpha}X,Y) - \sum_{i=1}^r \phi_{\alpha i}(X)\eta_i(Y), \tag{2.17}$$

$$\epsilon_\alpha h_\alpha^s(X,PY) = g(A_{W_\alpha}X,PY), \qquad \widetilde{g}(A_{W_\alpha}X,N_i) = \epsilon_\alpha\rho_{i\alpha}(X), \tag{2.18}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY), \qquad \eta_j(A_{N_i}X) + \eta_i\left(A_{N_j}X\right) = 0, \tag{2.19}$$

and $\epsilon_\beta\theta_{\alpha\beta} = -\epsilon_\alpha\theta_{\beta\alpha}$, where $X,Y \in \Gamma(TM)$. From (2.19), we know that the operators A_{N_i} are shape operators related to h_i^* for each i, called the *radical shape operators* on $S(TM)$. From (2.16), we know that the operators $A_{\xi_i}^*$ are $\Gamma(S(TM))$ valued. Replace Y by ξ_j in (2.15), then we have $h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0$ for all $X \in \Gamma(TM)$. It follows that

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0. \tag{2.20}$$

Also, replace X by ξ_j in (2.15) and use (2.20), then we have

$$h_i^\ell(X,\xi_j) = g\left(X, A_{\xi_i}^*\xi_j\right), \qquad A_{\xi_i}^*\xi_j + A_{\xi_j}^*\xi_i = 0, \qquad A_{\xi_i}^*\xi_i = 0. \tag{2.21}$$

Thus ξ_i is an eigenvector field of $A_{\xi_i}^*$ corresponding to the eigenvalue 0. For an r-lightlike submanifold, replace Y by ξ_i in (2.17), then we have

$$h_\alpha^s(X,\xi_i) = -\epsilon_\alpha\phi_{\alpha i}(X). \tag{2.22}$$

From (2.15)~(2.18), we show that the operators $A^*_{\xi_i}$ and A_{W_α} are not self-adjoint on $\Gamma(TM)$ but self-adjoint on $\Gamma(S(TM))$.

Theorem 2.2. *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, then the following assertions are equivalent:*

(i) *$A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g, for all i,*

(ii) *h^ℓ_i satisfy $h^\ell_i(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j,*

(iii) *$A^*_{\xi_i}\xi_j = 0$ for all i and j, that is, the image of* Rad(TM) *with respect to $A^*_{\xi_i}$ for each i is a trivial vector bundle,*

(iv) *$h^\ell_i(X, Y) = g(A^*_{\xi_i}X, Y)$ for all $X, Y \in \Gamma(TM)$ and i, that is, $A^*_{\xi_i}$ is a shape operator on M, related by the second fundamental form h^ℓ_i.*

Proof. From (2.15) and the fact that h^ℓ_i are symmetric, we have

$$g\left(A^*_{\xi_i}X, Y\right) - g\left(X, A^*_{\xi_i}Y\right) = \sum_{j=1}^{r}\left\{h^\ell_k(X, \xi_i)\eta_k(Y) - h^\ell_k(Y, \xi_i)\eta_k(X)\right\}. \tag{2.23}$$

(i)⇔(ii). If $h^\ell_i(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j, then we have $g(A^*_{\xi_i}X, Y) = g(A^*_{\xi_i}Y, X)$ for all $X, Y \in \Gamma(TM)$, that is, $A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g. Conversely, if $A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g, then we have

$$h^\ell_k(X, \xi_i)\eta_k(Y) = h^\ell_k(Y, \xi_i)\eta_k(X), \tag{2.24}$$

for all $X, Y \in \Gamma(TM)$. Replace Y by ξ_j in this equation and use the second equation of (2.20), then we have $h^\ell_j(X, \xi_i) = 0$ for all $X \in \Gamma(TM)$, i and j.

(ii)⇔(iii). Since $S(TM)$ is nondegenerate, from the first equation of (2.21), we have $h^\ell_i(X, \xi_j) = 0 \Leftrightarrow A^*_{\xi_i}\xi_j = 0$, for all i and j.

(ii)⇔(iv). From (2.16), we have $h^\ell_i(X, Y) = g(A^*_{\xi_i}X, Y) \Leftrightarrow h^\ell_j(X, \xi_i) = 0$ for any $X, Y \in \Gamma(TM)$ and for all i and j. □

Corollary 2.3. *Let $(M, g, S(TM), S(TM^\perp))$ be a 1-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, then the operators $A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g.*

Definition 2.4. An r-lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is said to be *irrotational* if $\widetilde{\nabla}_X\xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and i.

For an r-lightlike submanifold M of $\widetilde{M}$, the above definition is equivalent to $h^\ell_j(X, \xi_i) = 0$ and $h^s_\alpha(X, \xi_i) = 0$ for any $X \in \Gamma(TM)$. In this case, $A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g, for all i.

We need the following Gauss-Codazzi equations (for a full set of these equations see [1, chapter 5]) for M and $S(TM)$. Denote by $\widetilde{R}$, R, and R^* the curvature tensors of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{M}$, the induced connection ∇ of M, and the induced connection ∇^* on $S(TM)$, respectively:

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}(X,Y)Z,PW\right) &= g(R(X,Y)Z,PW) \\ &\quad+\sum_{i=1}^{r}\left\{h_i^{\ell}(X,Z)h_i^*(Y,PW)-h_i^{\ell}(Y,Z)h_i^*(X,PW)\right\} \\ &\quad+\sum_{\alpha=r+1}^{n}\epsilon_\alpha\{h_\alpha^s(X,Z)h_\alpha^s(Y,PW)-h_\alpha^s(Y,Z)h_\alpha^s(X,PW)\},\end{aligned}\tag{2.25}$$

$$\begin{aligned}\epsilon_\alpha\widetilde{g}\left(\widetilde{R}(X,Y)Z,W_\alpha\right) &= (\nabla_X h_\alpha^s)(Y,Z)-(\nabla_Y h_\alpha^s)(X,Z) \\ &\quad+\sum_{i=1}^{r}\left\{h_i^{\ell}(Y,Z)\rho_{i\alpha}(X)-h_i^{\ell}(X,Z)\rho_{i\alpha}(Y)\right\} \\ &\quad+\sum_{\beta=r+1}^{n}\left\{h_\beta^s(Y,Z)\theta_{\beta\alpha}(X)-h_\beta^s(X,Z)\theta_{\beta\alpha}(Y)\right\},\end{aligned}\tag{2.26}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}(X,Y)Z,N_i\right) &= \widetilde{g}(R(X,Y)Z,N_i) \\ &\quad+\sum_{j=1}^{r}\left\{h_j^{\ell}(X,Z)\eta_i\left(A_{N_j}Y\right)-h_j^{\ell}(Y,Z)\eta_i\left(A_{N_j}X\right)\right\} \\ &\quad+\sum_{\alpha=r+1}^{n}\epsilon_\alpha\{h_\alpha^s(X,Z)\rho_{i\alpha}(Y)-h_\alpha^s(Y,Z)\rho_{i\alpha}(X)\},\end{aligned}\tag{2.27}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}(X,Y)\xi_i,N_j\right) &= \widetilde{g}(R(X,Y)\xi_i,N_j) \\ &\quad+\sum_{k=1}^{r}\left\{h_k^{\ell}(X,\xi_i)\eta_j(A_{N_k}Y)-h_k^{\ell}(Y,\xi_i)\eta_j(A_{N_k}X)\right\} \\ &\quad+\sum_{\alpha=r+1}^{n}\{\rho_{j\alpha}(X)\phi_{\alpha i}(Y)-\rho_{j\alpha}(Y)\phi_{\alpha i}(X)\} \\ &= g\left(A_{\xi_i}^*X,A_{N_j}Y\right)-g\left(A_{\xi_i}^*Y,A_{N_j}X\right)-2d\tau_{ji}(X,Y) \\ &\quad+\sum_{k=1}^{r}\left\{h_k^{\ell}(X,\xi_i)\eta_j(A_{N_k}Y)-h_k^{\ell}(Y,\xi_i)\eta_j(A_{N_k}X)\right\} \\ &\quad+\sum_{k=1}^{r}\{\tau_{jk}(X)\tau_{ki}(Y)-\tau_{jk}(Y)\tau_{ki}(X)\} \\ &\quad+\sum_{\alpha=r+1}^{n}\{\rho_{j\alpha}(X)\phi_{\alpha i}(Y)-\rho_{j\alpha}(Y)\phi_{\alpha i}(X)\},\end{aligned}\tag{2.28}$$

$$\begin{aligned}\widetilde{g}(R(X,Y)PZ,PW) &= g(R^*(X,Y)PZ,PW) \\ &\quad+\sum_{i=1}^{r}\left\{h_i^*(X,PZ)h_i^{\ell}(Y,PW)-h_i^*(Y,PZ)h_i^{\ell}(X,PW)\right\},\end{aligned}\tag{2.29}$$

$$g(R(X,Y)PZ,N_i) = (\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) + \sum_{j=1}^{r}\left\{h_j^*(X,PZ)\tau_{ij}(Y) - h_j^*(Y,PZ)\tau_{ij}(X)\right\}. \tag{2.30}$$

The Ricci tensor of $\widetilde{M}$ is given by

$$\widetilde{\text{Ric}}(X,Y) = \text{trace}\left\{Z \longrightarrow \widetilde{R}(Z,X)Y\right\}, \qquad \forall X,Y \in \Gamma\left(T\widetilde{M}\right), \tag{2.31}$$

for any $X,Y \in \Gamma(T\widetilde{M})$. Let $\dim \widetilde{M} = m+n$. Locally, $\widetilde{\text{Ric}}$ is given by

$$\widetilde{\text{Ric}}(X,Y) = \sum_{i=1}^{m+n} \epsilon_i \widetilde{g}\left(\widetilde{R}(E_i,X)Y,E_i\right), \tag{2.32}$$

where $\{E_1,\dots,E_{m+n}\}$ is an orthonormal frame field of $T\widetilde{M}$. If $\dim(\widetilde{M}) > 2$ and

$$\widetilde{\text{Ric}} = \widetilde{\kappa}\widetilde{g}, \qquad \widetilde{\kappa} \text{ is a constant}, \tag{2.33}$$

then $\widetilde{M}$ is an *Einstein manifold*. If $\dim(\widetilde{M}) = 2$, any $\widetilde{M}$ is Einstein, but $\widetilde{\kappa}$ in (2.33) is not necessarily constant. The *scalar curvature* $\widetilde{r}$ is defined by

$$\widetilde{r} = \sum_{i=1}^{m+n} \epsilon_i \widetilde{\text{Ric}}(E_i,E_i). \tag{2.34}$$

Putting (2.33) in (2.34) implies that $\widetilde{M}$ is Einstein if and only if

$$\widetilde{\text{Ric}} = \frac{\widetilde{r}}{m+n}\widetilde{g}. \tag{2.35}$$

3. The Tangential Curvature Vector Field

Let $R^{(0,2)}$ denote the induced Ricci tensor of type $(0,2)$ on M, given by

$$R^{(0,2)}(X,Y) = \text{trace}\{Z \longrightarrow R(Z,X)Y\}, \qquad \forall X,Y \in \Gamma\left(T\widetilde{M}\right). \tag{3.1}$$

Consider an induced quasiorthonormal frame field

$$\{\xi_1,\dots,\xi_r; N_1,\dots,N_r; X_{r+1},\dots,X_m; W_{r+1},\dots,W_n\}, \tag{3.2}$$

where $\{N_i, W_\alpha\}$ is a basis of $\Gamma(\text{tr}(TM)|_{\mathcal{U}})$ on a coordinate neighborhood $\mathcal{U}$ of M such that $N_i \in \Gamma(\text{ltr}(TM)|_{\mathcal{U}})$ and $W_\alpha \in \Gamma(S(TM^\perp)|_{\mathcal{U}})$. By using (2.29) and (3.1), we obtain the following local expression for the Ricci tensor:

$$\begin{aligned}\widetilde{\text{Ric}}(X,Y) = &\sum_{a=r+1}^{n} \epsilon_a \widetilde{g}\Big(\widetilde{R}(W_a, X)Y, W_a\Big) + \sum_{i=1}^{r} \widetilde{g}\Big(\widetilde{R}(\xi_i, X)Y, N_i\Big) \\ &+ \sum_{b=r+1}^{m} \epsilon_b \widetilde{g}\Big(\widetilde{R}(X_b, X)Y, X_b\Big) + \sum_{i=1}^{r} \widetilde{g}\Big(\widetilde{R}(N_i, X)Y, \xi_i\Big),\end{aligned} \tag{3.3}$$

$$R^{(0,2)}(X,Y) = \sum_{a=r+1}^{m} \epsilon_a g(R(X_a, X)Y, X_a) + \sum_{i=1}^{r} \widetilde{g}(R(\xi_i, X)Y, N_i). \tag{3.4}$$

Substituting (2.25) and (2.27) in (3.3) and using (2.15)~(2.18) and (3.4), we obtain

$$\begin{aligned}R^{(0,2)}(X,Y) = \widetilde{\text{Ric}}(X,Y) &+ \sum_{i=1}^{r} h_i^\ell(X,Y) \operatorname{tr} A_{N_i} + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y) \operatorname{tr} A_{W_\alpha} \\ &- \sum_{i=1}^{r} g\Big(A_{N_i}X, A^*_{\xi_i}Y\Big) - \sum_{\alpha=r+1}^{n} \epsilon_\alpha g(A_{W_\alpha}X, A_{W_\alpha}Y) \\ &- \sum_{i,j=1}^{r} h_j^\ell(\xi_i, Y)\eta_i\Big(A_{N_j}X\Big) + \sum_{i=1}^{r}\sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)\phi_{\alpha i}(Y) \\ &- \sum_{\alpha=r+1}^{n} \epsilon_\alpha \widetilde{g}\Big(\widetilde{R}(W_\alpha, X)Y, W_\alpha\Big) - \sum_{i=1}^{r} \widetilde{g}\Big(\widetilde{R}(\xi_i, Y)X, N_i\Big),\end{aligned} \tag{3.5}$$

for any $X, Y \in \Gamma(TM)$. This shows that $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ of M, given by (3.1), is called its induced Ricci tensor if it is symmetric. From now and in the sequel, a symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

Using (2.28), (3.5), and the first Bianchi identity, we obtain

$$\begin{aligned}R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = &\sum_{i=1}^{r} \Big\{ g\Big(A^*_{\xi_i}X, A_{N_i}Y\Big) - g\Big(A^*_{\xi_i}Y, A_{N_i}X\Big)\Big\}\text{s} \\ &+ \sum_{i,\, j=1}^{r} \Big\{ h_j^\ell(X, \xi_i)\eta_i\Big(A_{N_j}Y\Big) - h_j^\ell(Y, \xi_i)\eta_i\Big(A_{N_j}Y\Big)\Big\} \\ &+ \sum_{i=1}^{r}\sum_{\alpha=r+1}^{n} \{\rho_{i\alpha}(X)\phi_{\alpha i}(Y) - \rho_{i\alpha}(Y)\phi_{\alpha i}(X)\} \\ &- \sum_{i=1}^{r} \widetilde{g}\Big(\widetilde{R}(X,Y)\xi_i, N_i\Big).\end{aligned} \tag{3.6}$$

From this equation and (2.28), we have

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = 2d\big(\operatorname{tr}(\tau_{ij})\big)(X,Y). \tag{3.7}$$

Theorem 3.1 (see[5]). *Let M be a lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, then the tensor field $R^{(0,2)}$ is a symmetric Ricci tensor* Ric *if and only if each 1-form* $\mathrm{tr}(\tau_{ij})$ *is closed, that is,* $d(\mathrm{tr}(\tau_{ij})) = 0$*, on any* $\mathcal{U} \subset M$.

Note 1. Suppose that the tensor $R^{(0,2)}$ is symmetric Ricci tensor Ric, then the 1-form $\mathrm{tr}(\tau_{ij})$ is closed by Theorem 3.1. Thus, there exist a smooth function f on $\mathcal{U}$ such that $\mathrm{tr}(\tau_{ij}) = \mathrm{df}$. Consequently, we get $\mathrm{tr}(\tau_{ij})(X) = X(f)$. If we take $\widetilde{\xi}_i = \sum_{j=1}^{r} \alpha_{ij}\xi_j$, it follows that $\mathrm{tr}(\tau_{ij})(X) = \mathrm{tr}(\widetilde{\tau}_{ij})(X) + X(\ln\Delta)$. Setting $\Delta = \exp(f)$ in this equation, we get $\mathrm{tr}(\widetilde{\tau}_{ij})(X) = 0$ for any $X \in \Gamma(TM_{|\mathcal{U}})$. We call the pair $\{\xi_i, N_i\}_i$ on $\mathcal{U}$ such that the corresponding 1-form $\mathrm{tr}(\tau_{ij})$ vanishes the *canonical null pair* of M.

For the rest of this paper, let M be a lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ of quasiconstant curvature. We may assume that the curvature vector field ζ of $\widetilde{M}$ is a unit spacelike tangent vector field of M and $\dim \widetilde{M} > 4$,

$$\widetilde{\mathrm{Ric}}(X,Y) = \{(n+m-1)\alpha + \beta\} g(X,Y) + (n+m-2)\beta\theta(X)\theta(Y), \tag{3.8}$$

$$\widetilde{g}\left(\widetilde{R}(\xi_i, Y)X, N_i\right) = \alpha g(X,Y) + \beta\theta(X)\theta(Y), \tag{3.9}$$

$$\epsilon_\alpha \widetilde{g}\left(\widetilde{R}(W_\alpha, Y)X, W_\alpha\right) = \alpha g(X,Y) + \beta\theta(X)\theta(Y), \tag{3.10}$$

for all $X, Y \in \Gamma(TM)$. Substituting (3.8)~(3.10) into (3.5), we have

$$\begin{aligned} R^{(0,2)}(X,Y) = {}& \{(m-1)\alpha + \beta\} g(X,Y) + (m-2)\beta\theta(X)\theta(Y) \\ & + \sum_{i=1}^{r} h_i^{\ell}(X,Y)\,\mathrm{tr}\, A_{N_i} + \sum_{\alpha=r+1}^{n} h_\alpha^{s}(X,Y)\,\mathrm{tr}\, A_{W_\alpha} \\ & - \sum_{i=1}^{r} g\left(A_{N_i}X, A_{\xi_i}^{*}Y\right) - \sum_{\alpha=r+1}^{n} \epsilon_\alpha g(A_{W_\alpha}X, A_{W_\alpha}Y) \\ & - \sum_{i,j=1}^{r} h_j^{\ell}(\xi_i, Y)\eta_i\left(A_{N_j}X\right) + \sum_{i=1}^{r}\sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)\phi_{\alpha i}(Y). \end{aligned} \tag{3.11}$$

Definition 3.2. We say that the screen distribution $S(TM)$ of M is *totally umbilical* [1] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ_i such that $A_{N_i}X = \gamma_i\, PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$h_i^{*}(X, PY) = \gamma_i g(X,Y), \qquad \forall X, Y \in \Gamma(TM). \tag{3.12}$$

In case $\gamma_i = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

A vector field X on $\widetilde{M}$ is said to be a *conformal Killing vector field* [6] if $\widetilde{\mathcal{L}}_X \widetilde{g} = -2\delta\widetilde{g}$ for any smooth function δ, where $\widetilde{\mathcal{L}}_X$ denotes the Lie derivative with respect to X, that is,

$$\left(\widetilde{\mathcal{L}}_X \widetilde{g}\right)(Y,Z) = X(\widetilde{g}(Y,Z)) - \widetilde{g}([X,Y],Z) - \widetilde{g}(Y,[X,Z]), \qquad \forall X, Y, Z \in \Gamma\left(T\widetilde{M}\right). \tag{3.13}$$

In particular, if $\delta = 0$, then X is called a *Killing vector field* [7]. A distribution $\mathcal{G}$ on $\widetilde{M}$ is called a *conformal Killing* (resp., *Killing*) *distribution* on $\widetilde{M}$ if each vector field belonging to $\mathcal{G}$ is a conformal Killing (resp., Killing) vector field on $\widetilde{M}$. If the coscreen distribution $S(TM^{\perp})$ is a Killing distribution, using (2.10) and (2.17), we have

$$\widetilde{g}\left(\widetilde{\nabla}_X W_\alpha, Y\right) = -g(A_{W_\alpha}X, Y) + \sum_{i=1}^{r}\phi_{\alpha i}(X)\eta_i(Y) = -\epsilon_\alpha h^s_\alpha(X,Y). \tag{3.14}$$

Therefore, since h^s_α are symmetric, we obtain

$$\left(\widetilde{\mathcal{L}}_{W_\alpha}\widetilde{g}\right)(Y,Z) = -2\epsilon_\alpha h^s_\alpha(X,Y). \tag{3.15}$$

Theorem 3.3. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$, then the coscreen distribution $S(TM^{\perp})$ is a conformal Killing (resp., Killing) distribution if and only if there exists a smooth function δ_α such that*

$$h^s_\alpha(X,Y) = \epsilon_\alpha\delta_\alpha g(X,Y), \quad \{\textit{resp. } h^s_\alpha(X,Y) = 0,\} \quad \forall X,Y \in \Gamma(TM). \tag{3.16}$$

Theorem 3.4. *Let M be an irrotational r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ of quasiconstant curvature. If the curvature vector field ζ of $\widetilde{M}$ is tangent to M, $S(TM)$ is totally umbilical in M, and $S(TM^{\perp})$ is a conformal Killing distribution, then the tensor field $R^{(0,2)}$ is an induced symmetric Ricci tensor of M.*

Proof. From (2.17)~(2.20), (2.22), (3.16), and (3.11), we have

$$h^s_\alpha(X,Y) = \epsilon_\alpha\delta_\alpha g(X,Y), \qquad \phi_{\alpha i}(X) = 0, \qquad A_{W_\alpha}X = \delta_\alpha PX + \sum_{i=1}^{r}\epsilon_\alpha\rho_{i\alpha}(X)\xi_i, \tag{3.17}$$

$$\begin{aligned} R^{(0,2)}(X,Y) = &\left\{(m-1)\alpha + \beta + (m-r-1)\sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2 + \sum_{\alpha=r+1}^{n}\sum_{i=1}^{r}\delta_\alpha\rho_{i\alpha}(\xi_i)\right\}g(X,Y) \\ &+ (m-2)\beta\,\theta(X)\theta(Y) \\ &+ (m-r-1)\sum_{i=1}^{r}\gamma_i g\left(A^*_{\xi_i}X, Y\right), \qquad \forall X,Y \in \Gamma(TM). \end{aligned} \tag{3.18}$$

Using (3.17), we show that $R^{(0,2)}$ is symmetric. □

4. Proof of Theorem 1.1

As $h^*_i = 0$, we get $\widetilde{g}(R(X,Y)PZ, N_i) = 0$ by (2.30). From (2.27) and (3.16), we have

$$\widetilde{g}\left(\widetilde{R}(X,Y)PZ, N_i\right) = \sum_{\alpha=r+1}^{n}\delta_\alpha\{g(X,PZ)\rho_{i\alpha}(Y) - g(Y,PZ)\rho_{i\alpha}(X)\}. \tag{4.1}$$

By Theorems 3.1 and 3.4, we get $d\tau = 0$ on TM. Thus, we have $\tilde{g}(\tilde{R}(X,Y)\xi_i, N_i) = 0$ due to (2.28). From the above results, we deduce the following equation:

$$\tilde{g}\left(\tilde{R}(X,Y)Z, N_i\right) = \sum_{\alpha=r+1}^{n} \delta_\alpha \{g(X,PZ)\rho_{i\alpha}(Y) - g(Y,PZ)\rho_{i\alpha}(X)\}. \tag{4.2}$$

Taking $X = \xi_i$ and $Z = X$ to (4.2) and then comparing with (3.9), we have

$$\beta\theta(X)\theta(Y) = -\left\{\alpha + \sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i)\right\} g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.3}$$

Case 1. If $S(TM^\perp)$ is a Killing distribution, that is, $\delta_\alpha = 0$, then we have

$$\beta\theta(X)\theta(Y) = -\alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.4}$$

Substituting (4.3) into (1.1) and using (2.25) and the facts $\tilde{g}(\tilde{R}(X,Y)Z, \xi_i) = 0$ and $\tilde{g}(\tilde{R}(X,Y)Z, N_i) = 0$ due to (1.1), we have

$$R(X,Y)Z = -\alpha\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.5}$$

Thus, M is a space of constant curvature $-\alpha$. Taking $X = Y = \zeta$ to (4.3), we have $\beta = -\alpha$. Substituting (4.3) into (3.18) with $\delta_\alpha = \gamma_i = 0$, we have

$$\operatorname{Ric}(X,Y) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{4.6}$$

On the other hand, substituting (4.5) and $g(R(\xi_i, Y)X, N_i) = 0$ into (3.4), we have

$$\operatorname{Ric}(X,Y) = -(m-1)\alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.7}$$

From the last two equations, we get $\alpha = 0$ as $m > 1$. Thus, $\beta = 0$, and $\widetilde{M}$ and M are flat manifolds by (1.1) and (4.5). From this result and (2.29), we show that M^* is also flat.

Case 2. If $S(TM^\perp)$ is a conformal Killing distribution, assume that $\beta \neq 0$. Taking $X = Y = \zeta$ to (4.3), we have $\beta = -\{\alpha + \sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i)\}$. From this and (4.3), we show that

$$g(X,Y) = \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.8}$$

Substituting (4.8) into (1.1) and using (2.25) with $h_i^* = 0$ and (3.16), we have

$$g(R(X,Y)Z, W) = \left(\alpha + 2\beta + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \delta_\alpha^2\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \tag{4.9}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Substituting (4.8) into (3.18) with $\gamma_i = 0$, we have

$$\mathrm{Ric}(X,Y) = (m-r-1)\left\{\alpha + \beta + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \delta_\alpha^2\right\} g(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{4.10}$$

by the fact that $\sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i) = -(\alpha+\beta)$. On the other hand, from (2.27), (3.9), and (4.3), we have $g(R(\xi_i, Y)X,\ N_i) = 0$. Substituting this result and (4.9) into (3.4), we have

$$\mathrm{Ric}(X,Y) = (m-r-1)\left\{\alpha + 2\beta + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \delta_\alpha^2\right\} g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.11}$$

The last two equations imply $\beta = 0$ as $m - r > 1$. It is a contradiction. Thus, $\beta = 0$ and $\widetilde{M}$ is a space of constant curvature α. From (2.29) and (4.9), we show that M^* is a space of constant curvature $(\alpha + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \delta_\alpha^2)$. But M is not a space of constant curvature by $(3.17)_3$. Let $\kappa = (m-r-1)(\alpha + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \delta_\alpha^2)$, then the last two equations reduce to

$$R^{(0,2)}(X,Y) = \mathrm{Ric}(X,Y) = \kappa g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.12}$$

Thus M is an Einstein manifold. The scalar quantity r of M [8], obtained from $R^{(0,2)}$ by the method of (2.34), is given by

$$r = \sum_{i=1}^{r} R^{(0,2)}(\xi_i, \xi_i) + \sum_{a=r+1}^{m} \epsilon_a R^{(0,2)}(X_a, X_a). \tag{4.13}$$

Since M is an Einstein manifold satisfying (4.12), we obtain

$$r = \kappa \sum_{i=1}^{r} g(\xi_i, \xi_i) + \kappa \sum_{a=r+1}^{m} \epsilon_a g(X_a, X_a) = \kappa(m-r). \tag{4.14}$$

Thus, we have

$$\mathrm{Ric}(X,Y) = \frac{r}{m-r} g(X,Y), \tag{4.15}$$

which provides a geometric interpretation of half lightlike Einstein submanifold (the same as in Riemannian case) as we have shown that the constant $\kappa = r/(m-r)$.

5. Proof of Theorem 1.2

Assume that ζ is tangent to M, $S(TM)$ is totally umbilical, and $S(TM^{\perp})$ is a conformal Killing vector field. Using (1.1), (2.26) reduces to

$$(\nabla_X h^s_\alpha)(Y,Z) - (\nabla_Y h^s_\alpha)(X,Z) = \sum_{i=1}^{r}\left\{h^\ell_i(X,Z)\rho_{i\alpha}(Y) - h^\ell_i(Y,Z)\rho_{i\alpha}(X)\right\} + \sum_{\beta=r+1}^{n}\left\{h^s_\beta(X,Z)\theta_{\beta\alpha}(Y) - h^s_\beta(Y,Z)\theta_{\beta\alpha}(X)\right\}, \tag{5.1}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing W by N to (1.1), we have

$$\widetilde{g}\left(\widetilde{R}(X,Y)Z, N_i\right) = \{\alpha\eta_i(X) + e_i\beta\theta(X)\}g(Y,Z) - \{\alpha\eta_i(Y) + e_i\beta\theta(Y)\}g(X,Z) + \beta\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\}\theta(Z), \tag{5.2}$$

for all $X, Y, Z \in \Gamma(TM)$ and where $e_i = \theta(N_i)$. Applying ∇_X to (3.12) and using (2.13), we have

$$(\nabla_X h^*_i)(Y,PZ) = (X[\gamma_i])g(Y,PZ) + \gamma_i\sum_{j=1}^{r}h^\ell_j(X,PZ)\eta_j(Y), \tag{5.3}$$

for all $X, Y, Z \in \Gamma(TM)$. Substituting this equation into (2.30), we obtain

$$\widetilde{g}(R(X,Y)PZ, N_i) = \left\{X[\gamma_i] - \sum_{j=1}^{r}\gamma_j\tau_{ij}(X)\right\}g(Y,PZ) - \left\{Y[\gamma_i] - \sum_{j=1}^{r}\gamma_j\tau_{ij}(Y)\right\}g(X,PZ) + \gamma_i\sum_{j=1}^{r}h^\ell_j(X,PZ)\eta_j(Y) - \gamma_i\sum_{j=1}^{r}h^\ell_j(Y,PZ)\eta_j(X), \quad \forall X,Y,Z \in \Gamma(TM). \tag{5.4}$$

Substituting this equation and (5.2) into (2.27) and using $\theta(\xi_i) = 0$, we obtain

$$\left\{X[\gamma_i] - \sum_{j=1}^{r}\gamma_j\tau_{ij}(X) - \alpha\eta_i(X) - e_i\beta\theta(X) - \sum_{\alpha=r+1}^{n}\delta_\alpha\rho_{i\alpha}(X)\right\}g(Y,Z) - \left\{Y[\gamma_i] - \sum_{j=1}^{r}\gamma_j\tau_{ij}(Y) - \alpha\eta_i(Y) - e_i\beta\theta(Y) - \sum_{\alpha=r+1}^{n}\delta_\alpha\rho_{i\alpha}(Y)\right\}g(X,Z) = \gamma_i\left\{\sum_{j=1}^{r}h^\ell_j(Y,PZ)\eta_j(X) - \sum_{j=1}^{r}h^\ell_j(X,PZ)\eta_j(Y)\right\} + \beta\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\}\theta(Z), \quad \forall X,Y,Z \in \Gamma(TM). \tag{5.5}$$

Replacing Y by ξ_i to this and using $(2.20)_1$ and the fact $\theta(\xi_i) = 0$, we have

$$\gamma_i h_i^{\ell}(X,Y) = \left\{ \xi_i[\gamma_i] - \sum_{j=1}^{r} \gamma_j \tau_{ij}(\xi_i) - \alpha - \sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i) \right\} g(X,Y) - \beta\theta(X)\theta(Y), \tag{5.6}$$

for all $X, Y \in \Gamma(TM)$. Differentiating (3.16) and using (5.1), we have

$$\begin{aligned} \sum_{i=1}^{r} \{\delta_\alpha \eta_i(X) - \epsilon_\alpha \rho_{i\alpha}(X)\} h_i^{\ell}(Y,Z) - \sum_{i=1}^{r} \{\delta_\alpha \eta_i(Y) - \epsilon_\alpha \rho_{i\alpha}(Y)\} h_i^{\ell}(X,Z) \\ = \left\{ X[\delta_\alpha] + \epsilon_\alpha \sum_{\beta=r+1}^{n} \epsilon_\beta \delta_\beta \theta_{\beta\alpha}(X) \right\} g(Y,Z) \\ - \left\{ Y[\delta_\alpha] + \epsilon_\alpha \sum_{\beta=r+1}^{n} \epsilon_\beta \delta_\beta \theta_{\beta\alpha}(Y) \right\} g(X,Z). \end{aligned} \tag{5.7}$$

Replacing Y by ξ_i in the last equation and using $(2.20)_1$, we obtain

$$\{\delta_\alpha - \epsilon_\alpha \rho_{i\alpha}(\xi_i)\} h_i^{\ell}(X,Z) = \left\{ \xi_i[\delta_\alpha] + \epsilon_\alpha \sum_{\beta=r+1}^{n} \epsilon_\beta \delta_\beta \theta_{\beta\alpha}(\xi_i) \right\} g(X,Z). \tag{5.8}$$

As the conformal factor δ_α is nonconstant, we show that $\delta_\alpha - \epsilon_\alpha \rho_{i\alpha}(\xi_i) \neq 0$. Thus, we have

$$h_i^{\ell}(X,Y) = \sigma_i g(X,Y), \qquad \forall X, Y \in \Gamma(TM), \tag{5.9}$$

where $\sigma_i = \{\xi_i[\delta_\alpha] + \epsilon_\alpha \sum_{\beta=r+1}^{n} \epsilon_\beta \delta_\beta \theta_{\beta\alpha}(\xi_i)\}(\delta_\alpha - \epsilon_\alpha \rho_{i\alpha}(\xi_i))^{-1}$. From $(3.17)_1$ and (5.9), we show that the second fundamental form tensor h, given by $h(X,Y) = \sum_{i=1}^{r} h_i^{\ell}(X,Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^{s}(X,Y) W_\alpha$, satisfies

$$h(X,Y) = \mathscr{H} g(X,Y), \qquad \forall X, Y \in \Gamma(TM). \tag{5.10}$$

Thus, M is totally umbilical [5]. Substituting (5.9) into (5.6), we have

$$\left\{ \xi_i[\gamma_i] - \sum_{j=1}^{r} \gamma_j \tau_{ij}(\xi_i) - \gamma_i \sigma_i - \alpha - \sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i) \right\} g(X,Y) = \beta\theta(X)\theta(Y), \tag{5.11}$$

for all $X, Y \in \Gamma(TM)$. Taking $X = Y = \zeta$ to this equation, we have

$$\beta = \xi_i[\gamma_i] - \sum_{j=1}^{r} \gamma_j \tau_{ij}(\xi_i) - \gamma_i \sigma_i - \alpha - \sum_{\alpha=r+1}^{n} \delta_\alpha \rho_{i\alpha}(\xi_i). \tag{5.12}$$

Assume that $\beta \neq 0$, then we have

$$g(X,Y) = \theta(X)\theta(Y), \qquad \forall X,Y \in \Gamma(TM). \tag{5.13}$$

Substituting (5.13) into (1.1) and using (2.25), (3.12), $(3.17)_1$, and (5.9), we have

$$\begin{aligned} &g(R(X,Y)Z,W) \\ &\quad = \left(\alpha + 2\beta + \sum_{i=1}^{r}\sigma_i\gamma_i + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{5.14}$$

for all $X,Y,Z,W \in \Gamma(TM)$. Substituting (5.9) and (5.13) into (3.18), we have

$$\begin{aligned} \operatorname{Ric}(X,Y) = \Bigg\{&(m-1)(\alpha+\beta) + (m-r-1)\left(\sum_{i=1}^{r}\sigma_i\gamma_i + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2\right) \\ &+ \sum_{\alpha=r+1}^{n}\sum_{i=1}^{r}\delta_\alpha\rho_{i\alpha}(\xi_i)\Bigg\}g(X,Y). \end{aligned} \tag{5.15}$$

On the other hand, substituting (5.14) and the fact that

$$\widetilde{g}(R(\xi_i,Y)X,N_i) = \left\{\alpha + \beta + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha\rho_{i\alpha}(\xi_i)\right\}g(X,Y) \tag{5.16}$$

into (3.4), we have

$$\begin{aligned} \operatorname{Ric}(X,Y) = \Bigg\{&(m-1)\alpha + 2(m-1)\beta + (m-r-1)\left(\sum_{i=1}^{r}\sigma_i\gamma_i + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2\right) \\ &+ \sum_{\alpha=r+1}^{n}\sum_{i=1}^{r}\delta_\alpha\rho_{i\alpha}(\xi_i)\Bigg\}g(X,Y). \end{aligned} \tag{5.17}$$

Comparing (5.15) and (5.17), we obtain $(m-1)\beta = 0$. As $m > 1$, we have $\beta = 0$, which is a contradiction. Thus, we have $\beta = 0$. Consequently, by (1.1), (2.29), and (5.14), we show that $\widetilde{M}$ and M^* are spaces of constant curvatures α and $(\alpha + 2\sum_{i=1}^{r}\sigma_i\gamma_i + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2)$, respectively. Let

$$\kappa = \left\{(m-1)\alpha + (m-r-1)\left(\sum_{i=1}^{r}\sigma_i\gamma_i + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\delta_\alpha^2\right) + \sum_{\alpha=r+1}^{n}\sum_{i=1}^{r}\delta_\alpha\rho_{i\alpha}(\xi_i)\right\}, \tag{5.18}$$

then (5.15) and (5.17) reduce to

$$R^{(0,2)}(X,Y) = \operatorname{Ric}(X,Y) = \kappa g(X,Y), \qquad \forall X,Y \in \Gamma(TM). \tag{5.19}$$

Thus, M is an Einstein manifold. The scalar quantity c of M is given by

$$\begin{aligned} c &= \sum_{i=1}^{r} R^{(0,2)}(\xi_i, \xi_i) + \sum_{a=r+1}^{m} \epsilon_a R^{(0,2)}(X_a, X_a) \\ &= \sum_{i=1}^{r} \kappa g(\xi_i, \xi_i) + \kappa \sum_{a=r+1}^{m} \epsilon_a \ g(X_a, X_a) = \kappa(m-r). \end{aligned} \tag{5.20}$$

Thus, we have

$$\operatorname{Ric}(X, Y) = \frac{c}{m-r} g(X, Y). \tag{5.21}$$

Example 5.1. Let (M, g) be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ equipped with a screen distribution $S(TM)$, then there exist an almost contact metric structure $(J, \zeta, \vartheta, \overline{g})$ on $\overline{M}$, where J is a $(1,1)$-type tensor field, ζ is a vector field, ϑ is a 1-form, and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ such that

$$\begin{gathered} J^2 X = -X + \vartheta(X)\zeta, \qquad J\zeta = 0, \qquad \vartheta \circ J = 0, \qquad \vartheta(\zeta) = 1, \\ \vartheta(X) = \overline{g}(\zeta, X), \qquad \overline{g}(JX, JY) = \overline{g}(X, Y) - \vartheta(X)\vartheta(Y), \\ \overline{\nabla}_X \zeta = -X + \vartheta(X)\zeta, \qquad \left(\overline{\nabla}_X J\right) Y = -\overline{g}(JX, Y)\zeta + \vartheta(Y) JX, \end{gathered} \tag{5.22}$$

for any vector fields X, Y on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$. Using the local second fundamental forms B and C of M and $S(TM)$, respectively, and the projection morphism P of M on $S(TM)$, the curvature tensors $\overline{R}$, R, and R^* of the connections $\overline{\nabla}$, ∇, and ∇^* on $\overline{M}$, M, and $S(TM)$, respectively, are given by (see [9])

$$\begin{aligned} \overline{g}\left(\overline{R}(X, Y)Z, PW\right) &= g(R(X, Y)Z, PW) \\ &\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW), \\ g(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) \\ &\quad + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW), \end{aligned} \tag{5.23}$$

for any $X, Y, Z, W \in \Gamma(TM)$. In case the ambient manifold $\overline{M}$ is a space form $\overline{M}(c)$ of constant J-holomorphic sectional curvature c, $\overline{R}$ is given by (see [10])

$$\overline{R}(X, Y)Z = g(X, Z)Y - g(Y, Z)X. \tag{5.24}$$

Assume that M is almost screen conformal, that is,

$$C(X, PY) = \varphi B(X, PY) + \eta(X)\vartheta(Y), \tag{5.25}$$

where φ is a nonvanishing function on a neighborhood $\mathcal{U}$ in M, and ζ is tangent to M, then, by the method in Section 2 of [9], we have

$$B(X,Y) = \rho\{g(X,Y) - \vartheta(X)\vartheta(Y)\}, \tag{5.26}$$

where ρ is a nonvanishing function on a neighborhood $\mathcal{U}$. Then the leaf M^* of $S(TM)$ is a semi-Riemannian manifold of quasiconstant curvature such that $\alpha = -1 + 2\varphi\rho^2$, $\beta = -2\varphi\rho^2$, and $\theta = \vartheta$ in (1.1).

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