Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 632681, 14 pages
doi:10.1155/2012/632681

Research Article

Existence and Uniqueness of Solution for a Class of Nonlinear Fractional Order Differential Equations

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Received 19 February 2012; Accepted 12 April 2012

Academic Editor: Juan J. Trujillo

We discuss the existence and uniqueness of solution to nonlinear fractional order ordinary differential equations $(\mathfrak{D}^{\alpha} - \rho t \mathfrak{D}^{\beta})x(t) = f(t, x(t), \mathfrak{D}^{\gamma}x(t))$, $t \in (0,1)$ with boundary conditions $x(0) = x_0$, $x(1) = x_1$ or satisfying the initial conditions $x(0) = 0$, $x'(0) = 1$, where $\mathfrak{D}^{\alpha}$ denotes Caputo fractional derivative, ρ is constant, $1 < \alpha < 2$, and $0 < \beta + \gamma \le \alpha$. Schauder's fixed-point theorem was used to establish the existence of the solution. Banach contraction principle was used to show the uniqueness of the solution under certain conditions on f.

1. Introduction

Fractional calculus deals with generalization of differentiation and integration to the fractional order [1, 2]. In the last few decades the fractional calculus and fractional differential equations have found applications in various disciplines [2–6]. Owing to the increasing applications, a considerable attention has been given to exact and numerical solutions of fractional differential equations [2, 6–11]. Many papers were dedicated to the existence and the uniqueness of the fractional differential equations, to the analytic methods for solving fractional differential equations, e.g., Greens function method, the Mellin transform method, and the power series (see for example references [2, 6–26] and the references therein). On this line of taught in this manuscript we proved the existence and uniqueness of a specific nonlinear fractional order ordinary differential equations within Caputo derivatives. Very recently in [27–31], the authors and other researchers studied the existence and uniqueness of solutions of some classes of fractional differential equations with delay. The paper is organized as follows: In Section 2 we introduce some necessary definitions and mathematical preliminaries of fractional calculus. In Section 3 sufficient conditions are established for the existence and uniqueness of solutions for a class fractional order differential equations satisfying the boundary conditions or satisfying the initial conditions. In order to illustrate our results several examples are presented in Section 3.

2. Fractional Integral and Derivatives

In this section, we present some notations, definitions, and preliminary facts that will be used further in this work. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations. Therefore, in this work we will use the Caputo fractional derivative $\mathfrak{D}$ proposed by Caputo in his work on the theory of viscoelasticity [32].

Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$ and $x \in C((0,\infty),\mathbb{R})$; then the Caputo fractional derivative of order α defined by

$$\mathfrak{D}^{\alpha} x(t) = \mathcal{I}^{n-\alpha}\left(\frac{d^n x(t)}{dt^n}\right), \tag{2.1}$$

where

$$\mathcal{I}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds, \tag{2.2}$$

is the Riemann-Liouville fractional integral operator of order α and Γ is the gamma function.

The fractional integral of $x(t) = (t-a)^{\beta}$, $a \geq 0$, $\beta > -1$ is given as

$$\mathcal{I}^{\alpha} x(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.3}$$

For $\alpha, \beta \geq 0$, we have the following properties of fractional integrals and derivative [33].

The fractional order integral satisfies the semigroup property

$$\mathcal{I}^{\alpha}\left(\mathcal{I}^{\beta} x(t)\right) = \mathcal{I}^{\beta}(\mathcal{I}^{\alpha} x(t)) = \mathcal{I}^{\alpha+\beta} x(t). \tag{2.4}$$

The integer order derivative operator $\mathfrak{D}^m$ commutes with fractional order $\mathfrak{D}^{\alpha}$, that is:

$$\mathfrak{D}^m(\mathfrak{D}^{\alpha} x(t)) = \mathfrak{D}^{m+\alpha} x(t) = \mathfrak{D}^{\alpha}(\mathfrak{D}^m x(t)). \tag{2.5}$$

The fractional operator and fractional derivative operator do not commute in general. Then the following result can be found in [33, 34].

Lemma 2.1 (see [33, 34]). *For $\alpha > 0$, the general solution of the fractional differential equation $\mathfrak{D}^{\alpha}x(t) = 0$ is given by*

$$x(t) = \sum_{i=0}^{r-1} c_i t^i, \quad c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, r-1,\ r = [\alpha] + 1, \tag{2.6}$$

where $[\alpha]$ denotes the integer part of the real number α.

In view of Lemma 2.1 it follows that

$$\mathcal{I}^{\alpha}(\mathfrak{D}^{\alpha}x(t)) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{r-1}t^{r-1} \quad \text{for some } c_i \in \mathbb{R},\ i = 0, 1, \ldots, r-1. \tag{2.7}$$

But in the opposite way we have

$$\mathfrak{D}^{\alpha}\left(\mathcal{I}^{\beta}(t)\right) = \mathfrak{D}^{\alpha-\beta}x(t). \tag{2.8}$$

Proposition 2.2. *Assume that $x : [0, \infty) \to \mathbb{R}$ is continuous and $0 < \beta \leq \alpha$. Then*

(i) $\mathcal{I}^{\alpha}(tx(t)) = t\mathcal{I}^{\alpha}x(t) - \alpha\mathcal{I}^{\alpha+1}x(t)$,

(ii) $\mathcal{I}^{\alpha}\{t\mathfrak{D}^{\beta}x(t)\} = t\mathcal{I}^{\alpha-\beta}x(t) - \alpha\mathcal{I}^{\alpha-\beta+1}x(t)$.

The proof of the above proposition can be found in [9, page 53].

As a pursuit of this in this paper, we discuss the existence and uniqueness of solution for nonlinear fractional order differential equations

$$\left(\mathfrak{D}^{\alpha} - \rho t\mathfrak{D}^{\beta}\right)x(t) = f(t, x(t), \mathfrak{D}^{\gamma}x(t)), \quad t \in (0, 1), \tag{2.9}$$

satisfying the boundary conditions

$$x(0) = x_0, \qquad x(1) = x_1, \tag{2.10}$$

or satisfying the initial conditions

$$x(0) = x_0, \qquad x'(0) = 1, \tag{2.11}$$

where $1 < \alpha \leq 2$ and $0 < \beta + \gamma \leq \alpha$.

In the following, we present the existence and the uniqueness results for fractional differential equation (2.9) with boundary conditions (2.10).

3. Existence and Uniqueness of Solutions

Lemma 3.1. *Assume that* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Then* $x \in C[0,1]$ *is a solution of the boundary value problem* (2.9) *and* (2.10) *if and only if* $x(t)$ *is the solution of the integral equation*

$$\begin{aligned} x(t) &= -c_0 - c_1 t + \rho t I^{\alpha-\beta} x(t) - \rho\alpha I^{\alpha-\beta+1} x(t) + I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma} x(t)) \\ &= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s)\,ds, \end{aligned} \tag{3.1}$$

for some constants c_0, c_1 *where* $\mathcal{G}(t,s)$ *given by*

$$\mathcal{G}(t,s) = \begin{cases} \mathcal{G}_1(t,s) & 0 \le s < t, \\ \mathcal{G}_2(t,s), & t \le s \le 1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mathcal{G}_1(t,s) &= \rho\left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) \\ &\quad + \left\{ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right\} f(s, x(s), \mathfrak{D}^{\gamma} x(s)), \\ \mathcal{G}_2(t,s) &= \rho t\left\{ \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s) \\ &\quad - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s)). \end{aligned} \tag{3.3}$$

Proof. Assume that $x \in C[0,1]$ is a solution of the fractional differential equation (2.9) satisfying boundary conditions (2.10). Then in view of Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned} x(t) &= \rho t I^{\alpha-\beta} x(t) - \rho\alpha I^{\alpha-\beta+1} x(t) + I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma} x(t)) - c_0 - c_1 t \\ &= \rho \int_0^t \left\{ \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)\,ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))\,ds - c_0 - c_1 t, \end{aligned} \tag{3.4}$$

for some constants c_0 and c_1. Hence using the boundary conditions (2.10) we obtain $c_0 = -x_0$ and

$$\begin{aligned} c_1 = x_0 - x_1 + \rho \int_0^1 \left\{ \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) ds \\ + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^\gamma x(s)) ds. \end{aligned} \tag{3.5}$$

Substituting $c_0 = -x_0$ and c_1 into (3.4) we get

$$x(t) = x_0 + (x_1 - x_0)t - \rho t \int_0^1 \left\{ \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) ds \tag{3.6}$$

$$- t \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^\gamma x(s)) ds \tag{3.7}$$

$$+ \int_0^t \left\{ \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) ds \tag{3.8}$$

$$\begin{aligned} &+ \rho \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^\gamma x(s)) ds, \\ &= x_0 + (x_1 - x_0)t \\ &+ \rho \int_0^t \left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) ds \\ &+ \int_0^t \left\{ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right\} f(s, x(s), \mathfrak{D}^\gamma x(s)) ds \\ &+ \rho \int_t^1 \left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s) ds \\ &- \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^\gamma x(s)) ds \\ &= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s) ds. \end{aligned} \tag{3.9}$$

We consider the space

$$\mathcal{B} = \{ x(t) : x(t) \in C[0,1], \mathfrak{D}^\gamma x(t) \in C[0,1] \}, \tag{3.10}$$

furnished with the norm

$$\|x(t)\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|\mathfrak{D}^{\gamma}x(t)|. \tag{3.11}$$

The space $\mathcal{B}$ is a Banach space [35]. □

Theorem 3.2. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous, and there exists a function $\eta : [0,1] \to [0,\infty]$, such that $f(t,x,y) \le \eta(t) + a|x| + b|y|$, $a, b \ge 0$, $2a + 2b + \alpha|\rho| \le 2\delta$ where $\delta = \min\{\Gamma(\alpha-\beta-\gamma+2), \Gamma(\alpha-\beta-\gamma+1), \Gamma(\alpha-\gamma+1), \Gamma(\alpha+1)\}$. Then, the boundary value problem* (2.9), (2.10) *has a solution.*

Proof. Define an operator $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{F}x(t) = x_0 + (x_1 - x_0)t - t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma}x(s))ds \tag{3.12}$$

$$+ \rho t\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\} x(s)ds \tag{3.13}$$

$$+ \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma}x(t)) \tag{3.14}$$

$$= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s)ds. \tag{3.15}$$

In order to show that the boundary value problem (2.9), (2.10) has a solution, it is sufficient to prove that the operator $\mathcal{F}$ has a fixed point. For $s \le t$, from (3.2), we have

$$\begin{aligned}
|\mathcal{G}(t,s)| &\le |\rho|\left\{\frac{2\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{2(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}|x(s)| \\
&\quad + \left\{\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}\right\}\left|f(s, x(s), \mathfrak{D}^{\gamma}x(s))\right| \\
&\le m_1\left\{\alpha(1-s)^{\alpha-\beta} + (1-s)^{\alpha-\beta-1}\right\}|x(s)| \\
&\quad + m_1(1-s)^{\alpha-1}\left|f(s, x(s), \mathfrak{D}^{\gamma}x(s))\right| \\
&\le m_1\left\{(\alpha+1)(1-s)^{\alpha-\beta-1}|x(s)| + (1-s)^{\alpha-1}\left|f(s, x(s), \mathfrak{D}^{\gamma}x(s))\right|\right\} \\
&\le m_1(1-s)^{\alpha-\beta-1}\left\{3|x(s)| + \left|f(s, x(s), \mathfrak{D}^{\gamma}x(s))\right|\right\} \\
&\le m_1(1-s)^{\alpha-\beta-1}\left\{(3+a)|x(s)| + \eta(s) + b|\mathfrak{D}^{\gamma}x(s)|\right\} \\
&\le m_1 m_2(1-s)^{\alpha-\beta-1},
\end{aligned} \tag{3.16}$$

where

$$m_1 = \max\left\{\frac{2|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{2|\rho|}{\Gamma(\alpha-\beta)}, \frac{2}{\Gamma(\alpha)}\right\},$$
$$m_2 = \max\left\{(3+a)|x(s)|, \eta(s), b|\mathfrak{D}^{\gamma}x(s)|,\ 0 \le s \le 1\right\}. \tag{3.17}$$

On the other hand, for $s > t$, we arrive at same conclusion. Therefore,

$$\int_0^1 |\mathcal{G}(t,s)|ds \le m_1 m_2 \int_0^1 (1-s)^{\alpha-\beta-1}ds = \frac{m_1 m_2}{\alpha-\beta}. \tag{3.18}$$

Choose $\mathcal{R} \ge \max\{\mathcal{R}_1, \mathcal{R}_2\}$, where $\mathcal{R}_1 = \max\{m_1 m_2/2(\alpha-\beta), (1/2)(2|x_0| + |x_1|)\}$ and

$$\mathcal{R}_2 = \max\left\{\frac{5|x_1 - x_0|}{2\Gamma(1-\gamma)}, \frac{5\|\eta\|}{2\Gamma(\alpha-\gamma+1)}, \frac{5\|\eta\|}{2\Gamma(\alpha+1)}, \frac{5|\rho|\alpha}{2\Gamma(\alpha-\beta+2)}, \frac{5|\rho|}{2\Gamma(\alpha-\beta+1)}\right\}. \tag{3.19}$$

Define the set $\Omega = \{x \in \mathcal{B} : \|x\| \le 8\mathcal{R}\}$. For $x \in \Omega$, using (3.15) and (3.18), we obtain

$$|\mathcal{F}x(t)| \le |x_0| + |x_1 - x_0|t + \int_0^1 |\mathcal{G}(t,s)|ds \le 2|x_0| + |x_1| + \frac{m_1 m_2}{\alpha-\beta} \le 2\mathcal{R} + 2\mathcal{R} = 4\mathcal{R}. \tag{3.20}$$

From the Caputo derivative and with using (3.12)–(3.14), we have

$$\begin{aligned}
\mathfrak{D}^{\gamma}(\mathcal{F}x(t)) &= I^{1-\gamma}\left\{\frac{d\mathcal{F}x(t)}{dt}\right\} \\
&= -I^{1-\gamma}\frac{d}{dt}\left\{t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma}x(s))ds\right\} \\
&\quad + I^{1-\gamma}\frac{d}{dt}\left\{\rho t\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}x(s)ds\right\} \\
&\quad + I^{1-\gamma}\left\{\frac{d}{dt}\left[x_0 + (x_1 - x_0)t + \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t)\right]\right\} \\
&\quad + I^{1-\gamma}\left\{\frac{d}{dt}I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma}x(t))\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -I^{1-\gamma}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),\mathfrak{D}^\gamma x(s))ds \\
&\quad + \rho I^{1-\gamma}\int_0^1 \left\{ \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s)ds \\
&\quad + I^{1-\gamma}\left\{ x_1 - x_0 + \rho(1-\alpha)I^{\alpha-\beta}x(t) + \rho t I^{\alpha-\beta-1}x(t) + I^{\alpha-1}f(t,x(t),\mathfrak{D}^\gamma x(t)) \right\}.
\end{aligned}
\tag{3.21}
$$

Then, (2.3) yields

$$
\begin{aligned}
\mathfrak{D}^\gamma(\mathcal{F}x(t)) &= -\frac{t^{1-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),\mathfrak{D}^\gamma x(s))ds \\
&\quad + \frac{\rho t^{1-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \left\{ \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s)ds \\
&\quad + \frac{(x_1-x_0)t^{1-\gamma}}{\Gamma(1-\gamma)} + \rho(1-\alpha)I^{\alpha-\beta-\gamma+1}x(t) \\
&\quad + \rho t I^{\alpha-\beta-\gamma}x(t) + I^{\alpha-\gamma}f(t,x(t),\mathfrak{D}^\gamma x(t)).
\end{aligned}
\tag{3.22}
$$

Hence,

$$
\begin{aligned}
|\mathfrak{D}^\gamma(\mathcal{F}x(t))| &\le \frac{t^{1-\gamma}}{\Gamma(\alpha+1)}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^\gamma x(t)|\} \\
&\quad + |\rho|\left\{ \frac{\alpha}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta+1)} \right\} t^{1-\gamma} \\
&\quad + \frac{x_1-x_0}{\Gamma(1-\gamma)} + \frac{|\rho(1-\alpha)|\|x\|}{\Gamma(\alpha-\beta-\gamma+1)}\int_0^t (t-s)^{\alpha-\beta-\gamma}ds \\
&\quad + \frac{|\rho|\|x\|}{\Gamma(\alpha-\beta-\gamma)}\int_0^t (t-s)^{\alpha-\beta-\gamma-1}ds + I^{\alpha-\gamma}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^\gamma x(t)|\} \\
&\le \frac{t^{1-\gamma}}{\Gamma(\alpha+1)}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^\gamma x(t)|\} \\
&\quad + |\rho|\left\{ \frac{\alpha}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta+1)} \right\} t^{1-\gamma} \\
&\quad + \frac{|x_1-x_0|}{\Gamma(1-\gamma)} + \frac{|\rho(1-\alpha)|\|x\|t^{\alpha-\beta-\gamma+1}}{\Gamma(\alpha-\beta-\gamma+2)} + \frac{|\rho|\|x\|t^{\alpha-\beta-\gamma}}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{\|\eta\| + (a+b)\|x\|}{\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma}.
\end{aligned}
\tag{3.23}
$$

Thus,

$$
\begin{aligned}
|\mathfrak{D}^{\gamma}(\mathcal{F}x(t))| \leq{}& \frac{|x_1 - x_0|}{\Gamma(1-\gamma)} + \frac{\|\eta\|}{\Gamma(\alpha-\gamma+1)} + \frac{\|\eta\|}{\Gamma(\alpha+1)} \\
&+ \mathcal{R}\left(\frac{|\rho(1-\alpha)|}{\Gamma(\alpha-\beta-\gamma+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{a+b}{\Gamma(\alpha-\gamma+1)} + \frac{a+b}{\Gamma(\alpha+1)}\right) \\
&+ \frac{\rho\alpha}{\Gamma(\alpha-\beta+2)} + \frac{|\rho|}{\Gamma(\alpha-\beta+1)} \\
\leq{}& 2\mathcal{R} + \mathcal{R}\left(\frac{|\rho|(\alpha-1)}{\delta} + \frac{|\rho|}{\delta} + \frac{2a+2b}{\delta}\right) = 2\mathcal{R} + \frac{2a+2b+\alpha|\rho|}{\delta}\mathcal{R} \leq 2\mathcal{R} + 2\mathcal{R} = 4\mathcal{R}.
\end{aligned}
\tag{3.24}
$$

Therefore, $\|\mathcal{F}x(t)\| \leq 4\mathcal{R} + 4\mathcal{R} = 8\mathcal{R}$. Thus, $\mathcal{F} : \Omega \to \Omega$. Finally, it remains to show that $\mathcal{F}$ is completely continuous. For any $x \in \Omega$, let $\ell = \max_{t\in[0,1]}|f(t, x(t), \mathfrak{D}^{\gamma}x(t))|$; then for $0 \leq t_1 \leq t_2 \leq 1$ and using (3.12)–(3.14), we have

$$
\begin{aligned}
|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| \leq{}& |x_1 - x_0||t_2 - t_1| + \ell|t_2 - t_1|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&+ |\rho||t_2 - t_1|\int_0^1 \left\{\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right\}x(s)ds \\
&+ \ell\left|\int_0^{t_2}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds\right| \\
&+ |\rho|\|x\|\left|\int_0^{t_2}\left(\frac{t_2(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_2-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)ds\right. \\
&\qquad\left. -\left(\frac{t_1(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)ds\right| \\
\leq{}& |x_1 - x_0||t_2 - t_1| + \frac{\ell|t_2 - t_1|}{\Gamma(\alpha+1)} \\
&+ |\rho|\|x\||t_2 - t_1|\left(\frac{1}{\Gamma(\alpha-\beta+1)} - \frac{\alpha}{\Gamma(\alpha-\beta+2)}\right) + \ell\frac{|t_1 - t_2|^{\alpha} + |t_1^{\alpha} - t_2^{\alpha}|}{\Gamma(\alpha+1)} \\
&+ \alpha\|x\|\frac{|t_1 - t_2|^{\alpha-\beta+1} + \left|t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}\right|}{\Gamma(\alpha-\beta+2)} + \|x\||t_1 - t_2|\frac{t_1^{\alpha-\beta} + |t_1 - t_2|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.
\end{aligned}
\tag{3.25}
$$

Hence, it follows that $\|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)\| \to 0$, as $t_2 \to t_1$. By the Arzela-Ascoli theorem, $\mathcal{F} : \Omega \to \Omega$ is completely continuous. Thus by using the Schauder fixed-point theorem, it was proved that the boundary value problem (2.9), (2.10) has a solution. □

Theorem 3.3. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. If there exists a constant* μ *such that* $|f(t,x,y) - f(t,\tilde{x},\tilde{y})| \leq \mu(|x-\tilde{x}|+|y-\tilde{y}|)$ *for each* $t \in [0,1]$ *and all* $x,\tilde{x},y,\tilde{y} \in \mathbb{R}$ *and* $4\mathcal{M}+3\mu \leq 1$, *where*

$$\mathcal{M} = \max\left\{\frac{2|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{|\rho|(1+\alpha)}{\Gamma(\alpha-\beta+2)}, \frac{|\rho(1-\alpha)|}{\Gamma(\alpha-\beta-\gamma+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)}\right\}. \tag{3.26}$$

Then the boundary value problem (2.9) *with boundary conditions* (2.10) *has a unique solution.*

Proof. Under condition on f, we have

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}\tilde{x}(t)| \leq{}& t\left|\rho\int_0^1 \left\{\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right\}[\tilde{x}(s)-x(s)]ds\right| \\
&+ t\left|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[f(s,\tilde{x}(s),\mathfrak{D}^\gamma\tilde{x}(s)) - f(s,x(s),\mathfrak{D}^\gamma x(s))\right]ds\right| \\
&+ \left|\rho\int_0^t \left\{\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right\}[x(s)-\tilde{x}(s)]ds\right| \\
&+ \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[f(s,x(s),\mathfrak{D}^\gamma x(s)) - f(s,\tilde{x}(s),\mathfrak{D}^\gamma\tilde{x}(s))\right]ds\right| \\
\leq{}& \left|\frac{\rho}{\Gamma(\alpha-\beta+1)} - \frac{\alpha\rho}{\Gamma(\alpha-\beta+2)}\right|\|x-\tilde{x}\| + \frac{2\mu\|\tilde{x}-x\|}{\Gamma(\alpha+1)} \\
&+ \left|\frac{\rho t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{\rho\alpha t}{\Gamma(\alpha-\beta+2)}\right|\|x-\tilde{x}\| + \frac{2\mu\|x-\tilde{x}\|t^\alpha}{\Gamma(\alpha+1)} \\
\leq{}& \left(\frac{|\rho|(1+t^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)} + \frac{|\rho|(1+\alpha)t}{\Gamma(\alpha-\beta+2)} + \frac{2\mu(1+t^\alpha)}{\Gamma(\alpha+1)}\right)\|x-\tilde{x}\|.
\end{aligned} \tag{3.27}$$

Using (3.22) we conclude

$$\begin{aligned}
|\mathfrak{D}^\gamma(\mathcal{F}x)(t) - \mathfrak{D}^\gamma(\mathcal{F}\tilde{x})(t)| \leq{}& |\rho(1-\alpha)|\left|I^{\alpha-\beta-\gamma+1}(x(t)-\tilde{x}(t))\right| + t\left|\rho I^{\alpha-\beta-\gamma}(x(t)-\tilde{x}(t))\right| \\
&+ \left|I^{\alpha-\gamma}\left(f(t,x(t),\mathfrak{D}^\gamma x(t)) - f(t,\tilde{x}(t),\mathfrak{D}^\gamma\tilde{x}(t))\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\rho(1-\alpha)|\|x-\tilde{x}\|}{\Gamma(\alpha-\beta-\gamma+1)}\int_0^t (t-s)^{\alpha-\beta-\gamma}ds \\
&\quad + \frac{|\rho|\|x-\tilde{x}\|}{\Gamma(\alpha-\beta-\gamma)}\int_0^t (t-s)^{\alpha-\beta-\gamma-1}ds \\
&\quad + \frac{2\mu\|x-\tilde{x}\|}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}ds \\
&\leq \left(\frac{\rho|1-\alpha|t^{\alpha-\beta-\gamma+1}}{\Gamma(\alpha-\beta-\gamma+2)} + \frac{\rho t^{\alpha-\beta-\gamma}}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{2\mu t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right)\|x-\tilde{x}\|.
\end{aligned} \tag{3.28}$$

Thus, we have

$$\|\mathcal{F}x(t)-\mathcal{F}\tilde{x}(t)\| \leq \left(4\mathcal{M}+\frac{6\mu}{\Gamma(\alpha+1)}\right) < (4\mathcal{M}+3\mu)\|x-\tilde{x}\|. \tag{3.29}$$

Therefore, by the contraction mapping theorem, the boundary value problem (2.9), (2.10) has a unique solution. □

Theorem 3.4. *Let* $f : [0,1] \to [0,\infty]$, *such that* $f(t,x,y) \leq \eta(t) + a|x| + b|y|$, $a,b \geq 0$ *with* $a+b+\alpha|\rho \leq \delta|$ *where* $\delta = \min\{\Gamma(\alpha-\beta-\gamma+1), \Gamma(\alpha-\beta-\gamma+2), \Gamma(\alpha-\beta-\gamma+3)\}$. *Then the initial value problem* (2.9), (2.10) *has a solution.*

Proof. In view of Lemma 2.1 and Proposition 2.2, we have

$$x(t) = \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha}f(t,x(t),\mathfrak{D}^{\gamma}x(t)) - c_0 - c_1 t. \tag{3.30}$$

Then,

$$x'(t) = \rho(1-\alpha)I^{\alpha-\beta}x(t) + \rho t I^{\alpha-\beta-1}x(t) + I^{\alpha=1}f(t,x(t),\mathfrak{D}^{\gamma}x(t)) - c_0 - c_1 t. \tag{3.31}$$

By initial conditions we have $c_0 = -x_0$ and $c_1 = -1$. Define an operator $\mathfrak{T} : \Omega \to \Omega$ by

$$\mathfrak{T}x(t) = x_0 + t + \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha}f(t,x(t),\mathfrak{D}^{\gamma}x(t)). \tag{3.32}$$

Can be easily to prove that $\mathfrak{T} : \Omega \to \Omega$ is completely continuous as operator $\mathcal{F}$. □

Theorem 3.5. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. If there exists a constant* μ *such that* $|f(t,x,y)-f(t,\tilde{x},\tilde{y})| \leq \mu(|x-\tilde{x}|+|y-\tilde{y}|)$ *for each* $t \in [0,1]$ *and all* $x,\tilde{x},y,\tilde{y} \in \mathbb{R}$ *and* $3(\mathcal{M}+\mu) \leq 1$, *where*

$$\mathcal{M} = \max\left\{\frac{|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{|\rho|\alpha}{\Gamma(\alpha-\beta+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)}\right\}, \tag{3.33}$$

then the initial value value problem (2.9), (2.11) *has a unique solution.*

The proof of the Theorem 3.5 is similar to the proof of Theorem 3.3. Note that

$$\frac{d\mathcal{T}x(t)}{dt} = 1 + \rho(1-\alpha)\mathcal{I}^{\alpha-\beta}x(t) + \rho t\mathcal{I}^{\alpha-\beta-1}x(t) + \mathcal{I}^{\alpha-1}f(t, x(t), \mathfrak{D}^{\gamma}x(t)). \tag{3.34}$$

Then using Proposition 2.2 we have,

$$\begin{aligned}
&\mathfrak{D}^{\gamma}(\mathcal{T}x(t)) \\
&\quad = \mathcal{I}^{1-\gamma}\left\{\frac{d\mathcal{T}x(t)}{d\mathcal{T}}\right\} \\
&\quad = \frac{t^{1-\gamma}}{\Gamma(1-\gamma)} + (\alpha(1-\rho) - \rho(1-\alpha))\mathcal{I}^{\alpha-\beta-\gamma+1}x(t) + \rho t\mathcal{I}^{\alpha-\beta-\gamma}x(t) + \mathcal{I}^{\alpha-\gamma}f(t, x(t), \mathfrak{D}^{\gamma}x(t))
\end{aligned} \tag{3.35}$$

Example 3.6. Consider the following boundary value problem for nonlinear fractional order differential equation:

$$\begin{gathered}
\left(\mathfrak{D}^{3/2} - t\mathfrak{D}^{1/2}\right)x(t) = \left(3e^{t} + \frac{1}{10}x(t) + \frac{1}{10}\mathfrak{D}^{1/2}x(t)\right)^{1/3}, \quad t \in (0,1), \\
x(0) = x_0, \qquad x(1) = x_1.
\end{gathered} \tag{3.36}$$

Then, (3.36) with assumed boundary conditions has a solution in Ω.

In Example 3.6 $f(t, x(t), \mathfrak{D}^{\gamma}x(t)) = \sqrt[3]{3e^{t} + (1/10)x(t) + (1/10)\mathfrak{D}^{1/2}x(t)}$ satisfies the conditions required in Theorem 3.2, that is

$$f\left(t, x(t), \mathfrak{D}^{1/2}x(t)\right) \leq e^{t} + \frac{1}{30}|x(t)| + \frac{1}{30}\left|\mathfrak{D}^{1/2}x(t)\right| \tag{3.37}$$

and $\delta = \min\{\Gamma(3/2), \Gamma(2), \Gamma(5/2)\} = \Gamma(3/2) = \sqrt{\pi}/2$ and $2a + 2b + \alpha\rho = 47/30 < 2\delta = \sqrt{\pi}$.

Example 3.7. Consider the following boundary value problem for nonlinear fractional order differential equation:

$$\begin{gathered}
\left(\mathfrak{D}^{3/2} - (1/8)t\mathfrak{D}^{1/2}\right)x(t) = \frac{1}{21}x(t) + \frac{1}{21}\mathfrak{D}^{1/2}x(t), \quad t \in (0,1), \\
x(0) = x_0, \qquad x(1) = x_1.
\end{gathered} \tag{3.38}$$

Then, (3.38) with assumed boundary conditions has unique solution in Ω.

In Example 3.7 $f(t, x(t), \mathfrak{D}^{\gamma}x(t)) = (1/21)x(t) + (1/21)\mathfrak{D}^{1/2}x(t)$ satisfies the conditions required in Theorem 3.3. $L = \max\{1/3\sqrt{\pi}, 1/8\sqrt{\pi}, 1/12\sqrt{\pi}, 1/4\sqrt{\pi}\} = 1/3\sqrt{\pi}$ and $4\mathcal{M} + 3\mu = 4/3\sqrt{\pi} + 1/7 < 1$.

4. Conclusion

We considered a class of nonlinear fractional order differential equations involving Caputo fractional derivative with lower terminal at 0 in order to study the existence solution satisfying the boundary conditions or satisfying the initial conditions. The unique solution under Lipschitz condition is also derived. In order to illustrate our results several examples are presented. The presented research work can be generalized to multiterm nonlinear fractional order differential equations with polynomial coefficients.

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