Research Article

Ulam-Hyers Stability for Cauchy Fractional Differential Equation in the Unit Disk

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We prove the Ulam-Hyers stability of Cauchy fractional differential equations in the unit disk for the linear and non-linear cases. The fractional operators are taken in sense of Srivastava-Owa operators.

1. Introduction

A classical problem in the theory of functional equations is that if a function f approximately satisfies functional equation $\mathcal{E}$, when does there exists an exact solution of $\mathcal{E}$ which f approximates. In 1940, Ulam [1, 2] imposed the question of the stability of the Cauchy equation, and in 1941, Hyers solved it [3]. In 1978, Rassias [4] provided a generalization of Hyers, theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [5–7]). The Ulam-Hyers stability of differential equations has been investigated by Alsina and Ger [8] and generalized by Jung [9–11]. Recently, Li and Shen [12] have investigated the Ulam-Hyers stability of the linear differential equations of second order, Abdollahpour and Najati [13] have studied the Ulam-Hyers stability of the linear differential equations of third order, and Lungu and Popa have imposed the Ulam-Hyers stability of a first-order partial differential equation [14].

The analysis on stability of fractional differential equations is more complicated than the classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Recently, Li and Zhang [15] provided an overview on the stability results of the fractional differential equations. Particularly, Li et al. [16] devoted to study the Mittag-Leffler stability and the Lyapunov's methods, Deng [17] derived sufficient conditions for

the local asymptotical stability of nonlinear fractional differential equations, and Li et al. studied the stability of fractional-order nonlinear dynamic systems using the Lyapunov direct method and generalized Mittag-Leffler stability [18]. Furthermore, there are few works on the Ulam stability of fractional differential equations, which maybe provide a new way for the researchers to investigate the stability of fractional differential equations from different perspectives. First the Ulam stability and data dependence for fractional differential equations with Caputo derivative have been posed by Wang et al. [19] and Ibrahim [20] with Riemann-Liouville derivative in complex domain. Moreover, Wang et al. [21–24] considered and established the Ulam stability for various types of fractional differential equation. Finally, the author generalized the Ulam-Hyers stability for fractional differential equation including infinite power series [25, 26].

In this work, we continue our study by imposing the Ulam-Hyers stability for the Cauchy fractional differential equations in complex domain. The operators are taken in sense of the Srivastava-Owa fractional derivative and integral.

2. Fractional Calculus

The theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruit-fully been applied in obtaining, for example, the characterization properties, coefficient estimates [27], distortion inequalities [28], and convolution structures for various subclasses of analytic functions and the works in the research monographs. In [29], Srivastava and Owa gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows.

Definition 2.1. The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta, \quad 0 \le \alpha < 1, \tag{2.1}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta, \quad \alpha > 0, \tag{2.2}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 2.3. From Definitions 2.1 and 2.2, we have

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1,\ 0 \le \alpha < 1,$$

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1,\ \alpha > 0.$$

$$(2.3)$$

We need the following preliminaries in the sequel.

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a, m]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad z \in U. \tag{2.4}$$

Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is called univalent ($\mathcal{S}$) if it is one-one in U.

Lemma 2.4 (see [28]). *Let the function $f(z)$ be in the class $\mathcal{S}$. Then*

$$\left|D_z^\alpha f(z)\right| \le \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt \quad (r = |z|,\ z \in U,\ 0 < \alpha < 1). \tag{2.5}$$

Lemma 2.5 (see [28]). *Let the function $f(z)$ be in the class $\mathcal{S}$. Then*

$$\left|D_z^{\alpha+1} f(z)\right| \le \frac{r^{-\alpha}}{\Gamma(1-\alpha)} (rF(2,1;1-\alpha;r))' \quad (r = |z|,\ z \in U \setminus \{0\},\ 0 < \alpha < 1). \tag{2.6}$$

3. Ulam-Hyers Stability for Fractional Problems

In this section, we will study the Ulam-Hyers stability for two different types of fractional Cauchy problems involving the differential operator in Definition 2.1. The first initial value problem is

$$D_z^\alpha u(z) = \rho(z)u(z), \quad (u(0) = 0,\ z \in U,\ 0 < \alpha < 1), \tag{3.1}$$

where $u(z), \rho(z) \in \mathcal{H}[U, \mathbb{C}]$ (the space of analytic function on the unit disk). While the second problem is

$$D_z^\alpha u(z) = f(z, u(z)), \quad (u(0) = 0,\ z \in U,\ 0 < \alpha < 1), \tag{3.2}$$

where $f : U \times \mathbb{C} \to \mathbb{C}$ is analytic in $z \in U$. Finally, we consider the problem

$$D_z^{1+\alpha} u(z) = f(z, u(z)), \quad (u(z_0) = c,\ z_0 \in U \setminus \{0\},\ 0 < \alpha < 1), \tag{3.3}$$

where $u(z) \in \mathcal{H}[U, \mathbb{C}]$ and $f : U \times \mathbb{C} \to \mathbb{C}$ is analytic in $z \in U$.

Definition 3.1. Problem (3.1) has the Ulam-Hyers stability if there exists a positive constant K with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^\alpha u(z) - \rho(z) u(z) \right| < \epsilon, \tag{3.4}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^\alpha v(z) = \rho(z) v(z) \tag{3.5}$$

with $v(0) = 0$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.6}$$

Definition 3.2. Problem (3.2) has the Ulam-Hyers stability if there exists a positive constant K with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^\alpha u(z) - f(z, u(z)) \right| < \epsilon, \tag{3.7}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^\alpha v(z) = f(z, v(z)) \tag{3.8}$$

with $v(0) = 0$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.9}$$

Definition 3.3. Problem (3.3) has the Ulam-Hyers stability if there exists a positive constant K with the following property: for every $\epsilon > 0$, $u \in \mathcal{H}[U, \mathbb{C}]$, if

$$\left| D_z^{1+\alpha} u(z) - f(z, u(z)) \right| < \epsilon, \tag{3.10}$$

then there exists some $v \in \mathcal{H}[U, \mathbb{C}]$ satisfying

$$D_z^{1+\alpha} v(z) = f(z, v(z)) \tag{3.11}$$

with $v(z_0) = c$, $z_0 \in U \setminus \{0\}$ such that

$$|u(z) - v(z)| < K\epsilon. \tag{3.12}$$

We start with the following result.

Theorem 3.4. *Let $u \in \mathcal{S}$, such that*

$$\max|u(z)| \leq \frac{h_\alpha}{2}, \quad \forall z \in U, \tag{3.13}$$

where

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha(1-rt)^3} dt. \tag{3.14}$$

If $\max|\rho(z)| < 1$, then problem (3.1) has the Ulam-Hyers stability.

Proof. For every $\epsilon > 0$, $u \in \mathcal{S}$, we let

$$\left|D_z^\alpha u(z) - \rho(z)u(z)\right| < \epsilon \tag{3.15}$$

with $u(0) = 0$. In view of Lemma 2.4, we obtain

$$\max|D_z^\alpha u(z)| = h_\alpha \quad \text{(sharp case)}, \tag{3.16}$$

consequently, we have

$$\begin{aligned}
\max|u(z)| &\leq \max\left|D_z^\alpha u(z) - \rho(z)u(z)\right| + \max\left|\rho(z)\right|\max|u(z)| \\
&\leq \epsilon + \max\left|\rho(z)\right|\max|u(z)|;
\end{aligned} \tag{3.17}$$

hence we impose that

$$\max|u(z)| \leq \frac{\epsilon}{1 - \max\left|\rho(z)\right|} := K\epsilon. \tag{3.18}$$

Obviously, $v(z) = 0$ is a solution of the problem (3.1) and yields

$$|u(z)| \leq K\epsilon. \tag{3.19}$$

Hence (3.1) has the Ulam-Hyers stability. $\qquad\square$

Corollary 3.5. *Let $u \in \mathcal{H}[\mathbb{D}, \mathbb{C}]$, where $\mathbb{D} \subset \mathbb{C}$ is a convex domain, satisfying one of the following conditions:*

(1) $\Re\{u'(z)\} > 0, \ z \in U$,

(2) $\Re\{zu'(z)/u(z)\} > 0, \ z \in U$,

(3) $\Re\{1 + zu''(z)/u'(z)\} > 0, \ z \in U$.

If $\max|u(z)| \leq h_\alpha/2$ and $\max|\rho(z)| < 1$, then problem (3.1) has the Ulam-Hyers stability.

Proof. Assume that $u \in \mathcal{H}[D, \mathbb{C}]$ satisfying one of the conditions (1)–(3), then u is a univalent function in the unit disk; that is, $u \in \mathcal{A}$ (see [30]). Thus, in view of Theorem 3.4, problem (3.1) has the Ulam-Hyers stability. $\qquad\square$

Remark 3.6. A function $f \in \mathcal{A}$ is called bounded turning function if it satisfies the following inequality:

$$\Re\{f'(z)\} > 0 \quad (z \in U). \tag{3.20}$$

A function $f \in \mathcal{A}$ is called star-like if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U). \tag{3.21}$$

A function $f \in \mathcal{A}$ is called convex if it satisfies the following inequality

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0 \quad (z \in U). \tag{3.22}$$

These subclasses of analytic functions in the unit disk play an important role in the theory of geometric function (see [30]).

Next, we consider the Ulam-Hyers stability for the nonlinear problems (3.2) and (3.3).

Theorem 3.7. *Let $u \in \mathcal{S}$, such that $\max |u(z)| \leq h_\alpha/2$, where*

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt. \tag{3.23}$$

If

$$\max\left|f(z, u(z))\right| \leq M \max|u(z)|, \quad M \in (0, 1), \tag{3.24}$$

then problem (3.2) *has the Ulam-Hyers stability.*

Proof. For every $\epsilon > 0$, $u \in \mathcal{S}$, we let

$$\left|D_z^\alpha u(z) - f(z, u(z))\right| < \epsilon \tag{3.25}$$

with $u(0) = 0$. In view of Lemma 2.4, it implies that

$$\max|D_z^\alpha u(z)| = h_\alpha \quad \text{(sharp case)}; \tag{3.26}$$

therefore, we pose

$$\max|u(z)| \le \max\left|D_z^\alpha u(z) - f(z, u(z))\right| + \max\left|f(z, u(z))\right|$$
$$\le \epsilon + \max\left|f(z, u(z))\right| \tag{3.27}$$
$$\le \epsilon + M \max|u(z)|;$$

that is,

$$\max|u(z)| \le \frac{\epsilon}{1 - M} := K\epsilon. \tag{3.28}$$

It is clear that

$$v(0) = I_z^\alpha f(z, v(z))\big|_{z=0} = 0 \tag{3.29}$$

yields

$$|u(z)| \le K\epsilon. \tag{3.30}$$

Hence (3.2) has the Ulam-Hyers stability. □

Now by applying Lemma 2.5, we study the Ulam-Hyers stability for the nonlinear problems (3.3).

Theorem 3.8. *Let $u \in S$, such that $\max|u(z)| \le g_\alpha/2$, where*

$$g_\alpha = \frac{r^{-\alpha}}{\Gamma(1 - \alpha)} \quad (rF(2, 1; 1 - \alpha; r))', \tag{3.31}$$
$$\left|f(z, u(z)) - f(z, v(z))\right| \le L|u(z) - v(z)|.$$

If $L \in (0, 1)$, then problem (3.3) has the Ulam-Hyers stability.

Proof. Since f is a contraction mapping, then the Banach fixed-point theorem implies that problem (3.3) has a unique solution. For every $\epsilon > 0$, $u \in S$, we let

$$\left|D_z^{1+\alpha} u(z) - f(z, u(z))\right| < \epsilon \tag{3.32}$$

with $u(z_0) = c$, $z_0 \in U \setminus \{0\}$. In view of Lemma 2.5, we impose

$$\max\left|D_z^{1+\alpha} u(z)\right| = g_\alpha \quad \text{(sharp case)}, \tag{3.33}$$

and consequently we have

$$
\begin{aligned}
\max|u(z) - v(z)| \\
&\leq \max|D_z^\alpha(u(z) - v(z))| \\
&\leq \left|D_z^\alpha u(z) - D_z^\alpha v(z) - f(z, u(z)) + f(z, v(z))\right| + \max\left|f(z, u(z)) - f(z, v(z))\right| \\
&\leq \epsilon + L \max|u(z) - v(z)|;
\end{aligned}
\tag{3.34}
$$

hence we receive

$$
\max|u(z) - v(z)| \leq \frac{\epsilon}{1 - L} := K\epsilon.
\tag{3.35}
$$

It is clear that $v(z_0) = c$ for some $z_0 \in U \setminus \{0\}$ yields

$$
|u(z) - v(z)| \leq K\epsilon.
\tag{3.36}
$$

Thus (3.3) has the Ulam-Hyers stability.   □

4. Conclusion

From above, the Ulam-Hyers stability is considered for different types of fractional Cauchy problems in the unit disk and in the puncture unit disk. We have observed that the problems (3.1) and (3.2) have the Ulam-Hyers stability when $\alpha \in (0, 1)$ and $u \in \mathcal{S}$ (univalent solution). While the Ulam-Hyers stability for higher-order fractional Cauchy problem of the form (3.3) is studied in Theorem 3.8, for $z \in U \setminus \{0\}$ and $u \in \mathcal{S}$. This leads to a set of questions:

(1) Is the fractional Cauchy problem (linear and nonlinear) has the Ulam-Hyers stability for all $u \in \mathcal{H}[U, \mathbb{C}]$? (under what conditions).

(2) Is the higher-order fractional Cauchy problem has the Ulam-Hyers stability for all $u \in \mathcal{H}[U, \mathbb{C}]$? (under what conditions). More specific,

(3) does the higher-order fractional Cauchy problem of the form

$$
D_z^{m+\alpha} u(z) = f(z, u(z)) \quad (u \in \mathcal{H}[U, \mathbb{C}], \ m = 2, 3, \ldots)
\tag{4.1}
$$

have the Ulam-Hyers stability?

(4) If we extend our study to complex Banach space, does the last problem have the Ulam-Hyers stability?

(5) If the study in complex Banach space, does the problem

$$
D^m u(z) = f(z, u(z)), \qquad D := \frac{d}{dz}
\tag{4.2}
$$

have the Ulam-Hyers stability?

More generalization

(6) If the study in complex Banach space, does the problem

$$D^m u(z) = f\left(z, u(z), D^{m-1} u(z)\right), \quad m = 2, 3, \ldots,$$ (4.3)

have the Ulam-Hyers stability?

Another special case

(7) If the study in complex Banach space, does the problem

$$D^m u(z) = f\left(z, z D^{m-1} u(z)\right), \quad m = 2, 3, \ldots,$$ (4.4)

have the Ulam-Hyers stability?

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