Research Article

A Linear Homogeneous Partial Differential Equation with Entire Solutions Represented by Laguerre Polynomials

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We study a homogeneous partial differential equation and get its entire solutions represented in convergent series of Laguerre polynomials. Moreover, the formulae of the order and type of the solutions are established.

1. Introduction and Main Results

The existence and behavior of global meromorphic solutions of homogeneous linear partial differential equations of the second order

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2a_1 \frac{\partial^2 u}{\partial t \partial z} + a_2 \frac{\partial^2 u}{\partial z^2} + a_3 \frac{\partial u}{\partial t} + a_4 \frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}$$

where $a_k = a_k(t, z)$ are polynomials for $(t, z) \in \mathbb{C}^2$, have been studied by Hu and Yang [1]. Specially, in [1, 2], they have studied the following cases of (1.1)

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + (2t + 2) \frac{\partial u}{\partial t} - 2z \frac{\partial u}{\partial z} = 0, \tag{1.2}$$

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + t \frac{\partial u}{\partial t} - z \frac{\partial u}{\partial z} + t^2 u = 0 \tag{1.3}$$

and showed that the solutions of (1.2) and (1.3) are closely related to Bessel functions and Bessel polynomials, respectively. Hu and Li [3] studied meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables:

$$\left(1 - t^2\right)\frac{\partial^2 u}{\partial t^2} + z^2\frac{\partial^2 u}{\partial z^2} - \{\alpha - \beta + (\alpha + \beta + 2)t\}\frac{\partial u}{\partial t} + (\alpha + \beta + 2)z\frac{\partial u}{\partial z} = 0, \tag{1.4}$$

where $\alpha, \beta \in \mathbb{C}$. Equation (1.4) has a lot of entire solutions on $\mathbb{C}^2$ represented by Jacobian polynomials. Global solutions of some first-order partial differential equations (or system) were studied by Berenstein and Li [4], Hu and Yang [5], Hu and Li [6], Li [7], Li and Saleeby [8], and so on.

In this paper, we concentrate on the following partial differential equation (PDE)

$$t\frac{\partial^2 u}{\partial t^2} + (\alpha + 1 - t)\frac{\partial u}{\partial t} + z\frac{\partial u}{\partial z} = 0 \tag{1.5}$$

for a real $\alpha > 0$. We will characterize the entire solutions of (1.5), which are related to Laguerre polynomials. Further, the formulae of the order and type of the solutions are obtained.

It is well known that the Laguerre polynomials are defined by

$$L_n(\alpha, t) = \sum_{k=0}^{n}\binom{n + \alpha}{n - k}\frac{(-t)^k}{k!}, \tag{1.6}$$

which are solutions of the following ordinary differential equations (ODE):

$$t\frac{d^2\omega}{dt^2} + (\alpha + 1 - t)\frac{d\omega}{dt} + n\omega = 0. \tag{1.7}$$

Moreover, Hu [9] pointed out that the generating function of $L_n(\alpha, t)$

$$F(\alpha, t, z) = (1 - z)^{-\alpha - 1}e^{-tz/(1-z)} = \sum_{n=0}^{\infty}L_n(\alpha, t)z^n \tag{1.8}$$

is a solution of the PDE (1.5). Based on the methods from Hu and Yang [2], we get the following results.

Theorem 1.1. *The partial differential equation* (1.5) *has an entire solution* $u = f(t, z)$ *on* $\mathbb{C}^2$, *if and only if* $u = f(t, z)$ *has a series expansion*

$$f(t, z) = \sum_{n=0}^{\infty}c_n L_n(\alpha, t)z^n \tag{1.9}$$

such that

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{1.10}$$

If $f(t, z)$ is an entire function on $\mathbb{C}^2$, set

$$M(r, f) = \max_{|t|\leq r, |z|\leq r} |f(t, z)|, \tag{1.11}$$

we define its order by

$$\mathrm{ord}(f) = \limsup_{r\to\infty} \frac{\log^+\log^+ M(r, f)}{\log r}, \tag{1.12}$$

where

$$\log^+ x = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{if } x < 1. \end{cases} \tag{1.13}$$

Theorem 1.2. *If $f(t, z)$ is defined by (1.9) and (1.10), then*

$$\rho \leq \mathrm{ord}(f) \leq \max(1, \rho), \tag{1.14}$$

where

$$\rho = \limsup_{n\to\infty} \frac{\log n}{\log\left(1/\sqrt[n]{|c_n|}\right)}. \tag{1.15}$$

Valiron [10] showed that each entire solution of a homogeneous linear ODE with polynomial coefficients was of finite order. By studying (1.2) and (1.3), Hu and Yang showed that Valiron's theorem was not true for general partial differential equations. Here by using Theorems 1.1 and 1.2, we can construct entire solution of (1.5) with arbitrary order ρ ($\rho \geq 1$).

If $0 < \lambda = \mathrm{ord}(f) < \infty$, we define the type of f by

$$\mathrm{typ}(f) = \limsup_{r\to\infty} \frac{\log^+ M(r, f)}{r^\lambda}. \tag{1.16}$$

Theorem 1.3. *If $f(t, z)$ is defined by (1.9) and (1.10), and $1 < \lambda = \mathrm{ord}(f) < \infty$, then the type $\sigma = \mathrm{typ}(f)$ satisfies*

$$e\sigma\lambda = \limsup_{n\to\infty} n\sqrt[n]{|c_n|^\lambda}. \tag{1.17}$$

Lindelöf-Pringsheim theorem [11] gave the expression of order and type for one complex variable entire function, and for two variable entire function the formulae of order and type were obtained by Bose and Sharma in [12]. Hu and Yang [2] established an analogue of Lindelöf-Pringsheim theorem for the entire solution of PDE (1.2). But from Theorems 1.2 and 1.3, we find that the analogue theorem for the entire solution of (1.5) is different from the results due to Hu and Yang.

2. An Estimate of Laguerre Polynomials

Before we prove our theorems, we give an upper bound of $L_n(\alpha, t)$, which will play an important role in this paper. The following asymptotic properties of $L_n(\alpha, t)$ can be found in [13]:

(a)

$$L_n(\alpha, t) = \frac{1}{2\sqrt{\pi}} e^{t/2} (-t)^{-\alpha/2 - 1/4} n^{\alpha/2 - 1/4} e^{2\sqrt{-nt}} \left(1 + O\left(n^{-1/2}\right)\right) \quad (n \longrightarrow \infty) \tag{2.1}$$

holds for t in the complex plane cut along the positive real semiaxis; thus, for $|t| \leq r$, we obtain that

$$|L_n(\alpha, t)| \leq n^{\alpha/2 - 1/4} e^{r/2} r^{-\alpha/2 - 1/4} e^{2\sqrt{nr}} \tag{2.2}$$

holds when n is large enough.

(b)

$$\frac{L_n(\alpha, t)}{n^{\alpha/2}} = e^{t/2} t^{-\alpha/2} J_\alpha\left(2\sqrt{nt}\right) + O\left(n^{-3/4}\right) \quad (n \longrightarrow \infty) \tag{2.3}$$

holds uniformly on compact subsets of $(0, +\infty)$, where J_α is the Bessel function and

$$J_\alpha\left(2\sqrt{nt}\right) = \frac{2}{\sqrt{\pi}\Gamma(\alpha + 1/2)} \left(\sqrt{nt}\right)^\alpha \int_0^{\pi/2} \cos\left(\sqrt{nt}\cos x\right) \sin^{2\alpha} x \, dx, \tag{2.4}$$

combining with (2.3), for $|t| \leq r$ we can deduce that

$$|L_n(\alpha, t)| \leq \frac{\pi n^\alpha}{\sqrt{\pi}\Gamma(\alpha + 1/2)} e^{t/2} \leq \frac{\pi n^\alpha}{\sqrt{\pi}\Gamma(\alpha + 1/2)} e^{r/2} \leq n^{\alpha/2 - 1/4} e^{r/2} r^{-\alpha/2 - 1/4} e^{2\sqrt{nr}} \tag{2.5}$$

holds when n is large enough. Then (2.2) and (2.5) imply

$$M(r, L_n) \leq n^{\alpha/2 - 1/4} e^{r/2} r^{-\alpha/2 - 1/4} e^{2\sqrt{nr}}, \tag{2.6}$$

where

$$M(r, L_n) = \max_{|t| \leq r} |L_n(\alpha, t)|. \tag{2.7}$$

3. Proof of Theorem 1.1

Assuming that $u = f(t,z)$ is an entire solution on $\mathbb{C}^2$ satisfying (1.5), we have Taylor expansion

$$f(t,z) = \sum_{n=0}^{\infty} \frac{w_n(t)}{n!} z^n, \tag{3.1}$$

where

$$w_n(t) = \frac{\partial^n f}{\partial z^n}(t,0). \tag{3.2}$$

Hence $w_n(t)$ is an entire solution of (1.7).

By the method of Frobenius (see [14]), we can get a second independent solution $X_n(\alpha,t)$ of (1.7) which is

$$X_n(\alpha,t) = qL_n(\alpha,t)\log t + \sum_{i=0}^{\infty} p_i t^i, \tag{3.3}$$

where $q(\neq 0)$, p_i are constants.

So there exist c_n and b_n satisfying

$$w_n(t) = n! c_n L_n(\alpha,t) + b_n X_n(\alpha,t). \tag{3.4}$$

Because of the singularity of $X_n(\alpha,t)$ at $t = 0$, we obtain $b_n = 0$. That shows

$$f(t,z) = \sum_{n=0}^{\infty} c_n L_n(\alpha,t) z^n. \tag{3.5}$$

Now we need to estimate the terms of c_n. Since

$$f(0,z) = \sum_{n=0}^{\infty} c_n L_n(\alpha,0) z^n \tag{3.6}$$

is an entire function, we have

$$\limsup_{n \to \infty} \sqrt[n]{|c_n| L_n(\alpha,0)} = 0. \tag{3.7}$$

Since

$$L_n(\alpha,0) = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}, \tag{3.8}$$

we easily get

$$\limsup_{n \to \infty} \sqrt[n]{|c_n|} = 0. \tag{3.9}$$

Conversely, the relations (1.7), (1.9), and (1.10) imply that

$$t\frac{\partial^2 f}{\partial t^2} + (\alpha + 1 - t)\frac{\partial f}{\partial t} + z\frac{\partial f}{\partial z} = \sum_{n=0}^{\infty} c_n \left\{ t\frac{d^2 L_n(\alpha, t)}{dt^2} + (\alpha + 1 - t)\frac{dL_n(\alpha, t)}{dt} + nL_n(\alpha, t) \right\} z^n = 0$$

(3.10)

holds for all $(t, z) \in \mathbb{C}^2$. Since (2.6) implies

$$\limsup_{n \to \infty} \sqrt[n]{M(r, L_n)} \leq 1,$$

(3.11)

we have

$$\limsup_{n \to \infty} \sqrt[n]{|c_n L_n(\alpha, t)|} \leq \limsup_{n \to \infty} \sqrt[n]{|c_n| M(r, L_n)} \leq \limsup_{n \to \infty} \sqrt[n]{|c_n|}.$$

(3.12)

Combining (1.10), (3.10) with (3.12), we can get that $u = f(t, z)$ is obviously an entire solution of (1.5) on $\mathbb{C}^2$.

4. Proof of Theorem 1.2

Firstly, we prove $\rho \leq \text{ord}(f)$. If $\rho = 0$, the result is trivial. Now we assume $0 < \rho \leq \infty$ and prove $\text{ord}(f) \geq k_1$ for any $0 < k_1 < \rho$. The relation (1.15) implies that there exists a sequence $n_j \to \infty$ such that

$$n_j \log n_j \geq k_1 \log \frac{1}{|c_{n_j}|}.$$

(4.1)

By using Cauchy's inequality of holomorphic functions, we have

$$\left| \frac{\partial^n f}{\partial z^n}(0, 0) \right| \leq n! r^{-n} M(r, f),$$

(4.2)

together with the formula of the coefficients of the Taylor expansion

$$\frac{\partial^n f}{\partial z^n}(0, 0) = c_n L_n(\alpha, 0) n!,$$

(4.3)

we obtain $M(r, f) \geq |c_n L_n(\alpha, 0)| r^n$. Since $|L_n(\alpha, 0)| \geq 1$, we have

$$|c_n| r^n \leq M(r, f),$$

(4.4)

then

$$\log M(r, f) \geq \log|c_n| + n \log r \geq n_j \left(\log r - \frac{1}{k_1} \log n_j \right). \tag{4.5}$$

Putting $r_j = (en_j)^{1/k_1}$, we have

$$\log M(r_j, f) \geq \frac{r_j^{k_1}}{ek_1}, \tag{4.6}$$

which means $\mathrm{ord}(f) \geq k_1$. Then we can get $\mathrm{ord}(f) \geq \rho$.

Next, we will prove $\mathrm{ord}(f) \leq \max(1, \rho)$. Set $\rho' = \max(1, \rho)$. The result is easy for $\rho' = \infty$; then we assume $\rho' < \infty$. For any $\varepsilon > 0$, (1.15) implies that there exists $n_0 > 0$, when $n > n_0$, we have

$$|c_n| < n^{-n(1+2\varepsilon)/\rho'(1+3\varepsilon)} = n^{-n(1+2\varepsilon)/k_2}, \tag{4.7}$$

where $k_2 = \rho'(1 + 3\varepsilon) > 1$. For any $\alpha \geq 0$, there exists $n_1 > n_0$ such that when $n > n_1$,

$$n^{(\alpha/2 - 1/4)} < n^{n(\varepsilon/k_2)}, \tag{4.8}$$

combining with (2.6) and (4.7), we get

$$\begin{aligned}
M(r, f) &\leq \sum_{n=0}^{\infty} |c_n| M(r, L_n) r^n \\
&\leq Cr^{2n_1} + C \sum_{n>n_1}^{\infty} |c_n| n^{\alpha/2 - 1/4} e^{r/2} e^{2\sqrt{nr}} r^n \\
&\leq Cr^{2n_1} + Ce^{r/2} \sum_{n>n_1}^{\infty} n^{-(1+\varepsilon)n/k_2} e^{2\sqrt{nr}} r^n,
\end{aligned} \tag{4.9}$$

where C is a constant but not necessary to be the same every time.

Set $m_1(r) = ((1/\varepsilon)(2\sqrt{r}/\log r))^2$, which means that $e^{2\sqrt{nr}} < r^{\varepsilon n}$ for $n > m_1(r)$. Further set $m_2(r) = (2r)^{k_2}$, which yields that when $n \geq m_2(r)$,

$$\left(n^{-1/k_2} r \right)^{(1+\varepsilon)n} \leq \left(\frac{1}{2} \right)^{(1+\varepsilon)n} < \left(\frac{1}{2} \right)^n. \tag{4.10}$$

Obviously, we can choose $r_0 > 0$ such that $m_2(r) > m_1(r)$ for $r > r_0$. Then

$$\sum_{n>m_2(r)} n^{-(1+\varepsilon)n/k_2} e^{2\sqrt{nr}} r^n \leq \sum_{n>m_2(r)}^{\infty} n^{-(1+\varepsilon)n/k_2} r^{(1+\varepsilon)n} \leq \sum_{n>m_2(r)}^{\infty} \left(\frac{1}{2} \right)^n \leq 1. \tag{4.11}$$

We also have

$$
\sum_{n_1 < n \le m_2(r)} n^{-(1+\varepsilon)n/k_2} e^{2\sqrt{nr}} r^n \le \sum_{n_1 < n \le m_2(r)} n^{-(1+\varepsilon)n/k_2} e^{2\sqrt{m_2(r)r}} r^{m_2(r)}
$$

$$
\le \sum_{n_1 < n \le m_2(r)} n^{-(1+\varepsilon)n/k_2} r^{m_2(r)} r^{\varepsilon m_2(r)}
$$

$$
\le r^{(1+\varepsilon)m_2(r)} \sum_{n_1 < n \le m_2(r)} n^{-(1+\varepsilon)n/k_2} \tag{4.12}
$$

$$
\le C r^{(1+\varepsilon)m_2(r)}
$$

$$
= C r^{(1+\varepsilon)(2r)^{k_2}}.
$$

Therefore, when $|t| = r \ge r_0$, we have

$$
M(r, f) \le C r^{2n_1} + C e^{r/2} + C e^{r/2} r^{(1+\varepsilon)(2r)^{k_2}}, \tag{4.13}
$$

which means $\operatorname{ord}(f) \le k_2$. Hence $\operatorname{ord}(f) \le \rho' = \max(1, \rho)$ follows by letting $\varepsilon \to 0$.

5. Proof of Theorem 1.3

Set

$$
\kappa = \limsup_{n \to \infty} n \sqrt[n]{|c_n|^\lambda}. \tag{5.1}
$$

At first, we prove $e\lambda\sigma \ge \kappa$. The result is trivial for $\kappa = 0$, we assume $0 < \kappa \le \infty$ and take ε with $0 < \varepsilon < \kappa$, set

$$
k_3 = \begin{cases} \kappa - \varepsilon, & \text{if } \kappa < \infty, \\ \dfrac{1}{\varepsilon}, & \text{if } \kappa = \infty. \end{cases} \tag{5.2}
$$

Equation (5.1) implies that there exists a sequence $n_j \to \infty$ satisfying

$$
\left| c_{n_j} \right| > \left(\frac{k_3}{n_j} \right)^{n_j/\lambda}, \tag{5.3}
$$

combining with (4.4), we can deduce that

$$
M(r, f) \ge \left| c_{n_j} \right| r^{n_j} \ge \left(\frac{k_3}{n_j} \right)^{n_j/\lambda} r^{n_j} = \left(\frac{k_3}{n_j} r^\lambda \right)^{n_j/\lambda}. \tag{5.4}
$$

Taking $r_j^\lambda = e n_j / k_3$, we get $M(r_j, f) > e^{k_3 r_j^\lambda/e\lambda}$, which yields $\sigma \ge k_3/e\lambda$, so $e\lambda\sigma \ge \kappa$.

Next, we prove $e\lambda\sigma \leq \kappa$. We may assume $\kappa < \infty$. Equation (5.1) implies that for any $\varepsilon > 0$, there exists $n_0 > 0$, such that when $n > n_0$,

$$|c_n| < \left(\frac{\kappa + (\varepsilon/2)}{n}\right)^{n/\lambda}. \tag{5.5}$$

For any $\alpha \geq 0$, we choose $n_1 (> n_0)$ such that when $n > n_1$,

$$n^{\alpha/2 - 1/4} < \left(\frac{\kappa + \varepsilon}{\kappa + (\varepsilon/2)}\right)^{n/\lambda}, \tag{5.6}$$

combining with (2.6), we have

$$\begin{aligned}
M(r, f) &\leq \sum_{n=0}^{\infty} |c_n| M(r, L_n) r^n \\
&\leq Cr^{2n_1} + C \sum_{n > n_1}^{\infty} |c_n| n^{\alpha/2 - 1/4} e^{r/2} e^{2\sqrt{nr}} r^n \\
&\leq Cr^{2n_1} + Ce^{r/2} \sum_{n > n_1}^{\infty} \left(\frac{\kappa + \varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n.
\end{aligned} \tag{5.7}$$

Set $m_3(r) = 16r\lambda^2$, when $n > m_3(r)$, we deduce $e^{2\sqrt{nr}} < e^{n/2\lambda}$. Set $m_4(r) = 2(\kappa + \varepsilon)r^\lambda$, it is obvious that $(\kappa + \varepsilon)r^\lambda/n < 1/2$ for $n > m_4(r)$. Since $\lambda > 1$, there exists r_1 such that when $r > r_1$,

$$m_4(r) = 2(\kappa + \varepsilon)r^\lambda > 16r\lambda^2 = m_3(r). \tag{5.8}$$

Then

$$\sum_{n > m_4(r)}^{\infty} \left(\frac{\kappa + \varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n = \sum_{n > m_4(r)}^{\infty} \left(\frac{(\kappa + \varepsilon)r^\lambda}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} \leq \sum_{n > m_4(r)}^{\infty} \left(\frac{e}{4}\right)^{n/2\lambda} \leq C. \tag{5.9}$$

We note that for $a > 0$, $b > 0$, $\max_{x > 0}(a/x)^{x/b} = e^{a/eb}$, then we have

$$\left(\frac{\kappa + \varepsilon}{n}\right)^{n/\lambda} r^n = \left(\frac{(\kappa + \varepsilon)r^\lambda}{n}\right)^{n/\lambda} \leq e^{(\kappa+\varepsilon)r^\lambda/e\lambda}. \tag{5.10}$$

This shows

$$\begin{aligned}
\sum_{n_1 < n \leq m_4(r)} \left(\frac{\kappa + \varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n &\leq m_4(r) e^{2\sqrt{m_4(r)r}} e^{(\kappa+\varepsilon)r^\lambda/e\lambda} \\
&\leq 2(\kappa + \varepsilon)r^\lambda e^{2\sqrt{2(\kappa+\varepsilon)r^{\lambda+1}}} e^{(\kappa+\varepsilon)r^\lambda/e\lambda}.
\end{aligned} \tag{5.11}$$

Therefore when $|t| = r \geq r_1$,

$$M(r, f) \leq C e^{r/2} (\kappa + \varepsilon) r^\lambda e^{2\sqrt{2(\kappa+\varepsilon)r^{\lambda+1}}} e^{(\kappa+\varepsilon)r^\lambda/e\lambda} + C r^{2n_1} + C e^{r/2}. \tag{5.12}$$

Together with $\lambda > 1$ and the definition of type, we can get $\sigma \leq (\kappa+\varepsilon)/e\lambda$, which yields $e\lambda\sigma \leq \kappa$ by letting $\varepsilon \to 0$.

Acknowledgments

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