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Research Article

Comparison Functions and Fixed Point Results in Partial Metric Spaces

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The aim of this article is to clearly formulate various possible assumptions for a comparison function in contractive conditions and to deduce respective (common) fixed point results in partial metric spaces. Since standard metric spaces are special cases, these results also apply for them. We will show by examples that there exist situations when a partial metric result can be applied, while the standard metric one cannot.

1. Introduction

In recent years many authors have worked on domain theory in order to equip semantics domain with a notion of distance. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks and obtained, among other results, a nice relationship between partial metric spaces and so-called weightable quasimetric spaces. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [2–11]) proved fixed point theorems in partial metric spaces.

Contractive conditions with the so-called comparison function φ of the form

$$d(Tx, Ty) \le \varphi(M(x, y)) \tag{1.1}$$

have been used for obtaining (common) fixed point results of mappings in metric spaces since the celebrated result of Boyd and Wong [12]. In various articles, different assumptions for function φ were made. Sometimes these assumptions were not clearly stated, and sometimes the assumptions were stronger than needed. This includes some recent fixed point results in partial metric spaces.

The aim of this paper is to clearly formulate various possible conditions for a comparison function and to deduce respective (common) fixed point results in partial metric spaces. Since standard metric spaces are special cases, these results also apply for them. We will show by examples that there exist situations when a partial metric result can be applied, while the standard metric one cannot.

2. Preliminaries

The following definitions and details can be seen, for example, in [1, 3, 6, 9, 11].

Definition 2.1. A partial metric space is a pair $(X,\ p)$ of a nonempty set X and a partial metric p on X, that is, a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \le p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

It is clear that if $p(x,y) = 0$, then, from (p_1) and (p_2), it follows that $x = y$. But $p(x,x)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x)+\varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$, with respect to τ_p, if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$. This will be denoted as $x_n \to x$, $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. If $T : X \to X$ is continuous at $x_0 \in X$ (in τ_p), then, for each sequence $\{x_n\}$ in X,

$$x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0. \tag{2.1}$$

Remark 2.2. Clearly, a limit of a sequence in a partial metric space needs not to be unique. For example, if $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$for $x,y \in X$, then for $\{x_n\} = \{1\}, p(x_n,x) = x = p(x,x)$ for each $x \ge 1$ and so, for example, $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$. Moreover, the function $p(\cdot,\cdot)$ needs not to be continuous in the sense that $x_n \to x$ and $y_n \to y$ imply $p(x_n,y_n) \to p(x,y)$.

Definition 2.3. Let (X,p) be a partial metric space. Then we have the following.

(1) A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).

(2) The space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

It is easy to see that every closed subset of a complete partial metric space is complete. If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.2}$$

is a metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.3}$$

Lemma 2.4. *Let (X, p) be a partial metric space:*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s),*

(b) *the space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

3. Auxiliary Results

We will consider the following properties of functions $\varphi : [0,+\infty) \to [0,+\infty)$. φ^n will denote the nth iteration of φ:

(I) $\varphi(t) < t$ for each $t > 0$ and $\varphi^n(t) \to 0$, $n \to \infty$ for each $t \geq 0$,

(II) φ is nondecreasing and $\varphi^n(t) \to 0$, $n \to \infty$ for each $t \geq 0$,

(III) φ is right-continuous, and $\varphi(t) < t$ for each $t > 0$,

(IV) φ is nondecreasing and $\sum_{n\geq 1} \varphi^n(t) < +\infty$ for each $t \geq 0$.

Lemma 3.1. (1) *(II) $\Rightarrow$ (I).*

(2) *(III) $+\varphi$ is nondecreasing $\Rightarrow$ (II).*

(3) *(IV) $\Rightarrow$ (II).*

(4) *(III) and (IV) are not comparable (even if φ is nondecreasing).*

Proof. (1) (see [13]). Suppose that (II) holds and that there is some $t_0 \in (0,+\infty)$ such that $\varphi(t_0) \geq t_0$. Then monotonicity of φ implies that $\varphi^2(t_0) \geq \varphi(t_0) \geq t_0$. Continuing by induction we get that $\varphi^n(t_0) \geq t_0$ and so $\varphi^n(t_0) \to 0$, $n \to \infty$ is impossible.

(2) Let (III) hold and let φ be nondecreasing. Monotonicity of φ implies that, for each fixed $t \geq 0$, the sequence $\{\varphi^n(t)\}$ is nonincreasing (and nonnegative); hence, there exists $\lim_{n\to\infty} \varphi^n(t) = \alpha \geq 0$. Suppose that $\alpha > 0$. Then it follows by (III) that

$$0 < \alpha \leq \lim_{n\to\infty} \varphi^{n+1}(t) = \lim_{\varphi^n(t)\to\alpha+} \varphi(\varphi^n(t)) = \varphi\left(\lim_{\varphi^n(t)\to\alpha+} \varphi^n(t)\right) = \varphi(\alpha), \tag{3.1}$$

which is a contradiction with $\varphi(t) < t$.

(3) Obvious.

(4) It is demonstrated in the following example. □

Example 3.2. (1) The function

$$\varphi(t) = \begin{cases} \dfrac{1}{3}t, & 0 \leq t < 1, \\[2ex] \dfrac{1}{2}, & t = 1, \\[2ex] \dfrac{2}{3}t, & t > 1, \end{cases} \tag{3.2}$$

satisfies (IV) but not (III).

(2) The function $\varphi(t) = t/(1+t)$ is nondecreasing and satisfies (III) but not (IV), since $\varphi^n(t) = t/(1+nt)$.

Assertions similar to the following lemma (see, e.g., [14]) were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 3.3. *Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that $\{d(y_{n+1},y_n)\}$ is nonincreasing and*

$$\lim_{n\to\infty} d(y_{n+1},y_n) = 0. \tag{3.3}$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to $\varepsilon + 0$ when $k \to \infty$:

$$d(y_{2m_k},y_{2n_k}), \qquad d(y_{2m_k},y_{2n_k+1}), \qquad d(y_{2m_k-1},y_{2n_k}), \qquad d(y_{2m_k-1},y_{2n_k+1}). \tag{3.4}$$

As a consequence we obtain the following.

Lemma 3.4. *Let (X,p) be a partial metric space and let $\{y_n\}$ be a sequence in X such that $\{p(y_{n+1}, y_n)\}$ is nonincreasing and*

$$\lim_{n\to\infty} p(y_{n+1},y_n) = 0. \tag{3.5}$$

If $\{y_{2n}\}$ is not a Cauchy sequence in (X,p), then there exist $\varepsilon > 0$ and two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to $\varepsilon + 0$ when $k \to \infty$:

$$p(y_{2m_k},y_{2n_k}), \qquad p(y_{2m_k},y_{2n_k+1}), \qquad p(y_{2m_k-1},y_{2n_k}), \qquad p(y_{2m_k-1},y_{2n_k+1}). \tag{3.6}$$

Proof. Suppose that $\{y_n\}$ is a sequence in (X,p) satisfying (3.5) such that $\{y_{2n}\}$ is not Cauchy. According to Lemma 2.4, it is not a Cauchy sequence in the metric space (X,p^s), either. Applying Lemma 3.3 we get the sequences

$$p^s(y_{2m_k},y_{2n_k}), \qquad p^s(y_{2m_k},y_{2n_k+1}), \qquad p^s(y_{2m_k-1},y_{2n_k}), \qquad p^s(y_{2m_k-1},y_{2n_k+1}) \tag{3.7}$$

tending (from above) to some $2\varepsilon > 0$ when $k \to \infty$. Using definition (2.2) of the associated metric and (3.5) (which implies that also $\lim_{n\to\infty} p(y_n,y_n) = 0$), we get that the sequences (3.6) tend to $\varepsilon + 0$ when $k \to \infty$. □

4. Common Fixed Point Results for Four Mappings

In this section we prove two common fixed point results for four mappings in partial metric spaces, using two distinct properties of a comparison function mentioned in Section 3.

Theorem 4.1. *Let (X,p) be a complete partial metric space, and let $A,B,S,T : X \to X$. Suppose that $AX \subset TX$, $BX \subset SX$ and one of these four subsets of X is closed. Let further*

$$p(Ax,By) \le \varphi(M(x,y)) \tag{4.1}$$

hold for all $x, y \in X$, where

$$M(x,y) = \max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \ \frac{1}{2}(p(Sx,By) + p(Ax,Ty))\right\}, \tag{4.2}$$

and $\varphi : [0,+\infty) \to [0,+\infty)$ satisfies property (III), that is, φ is right-continuous, and $\varphi(t) < t$ for each $t > 0$. Then A and S, as well as B and T, have a coincidence point. If, moreover, pairs (A,S) and (B,T) are weakly compatible, then A, B, S, T have a unique common fixed point.

Recall that a point $y \in X$ is called a point of coincidence for mappings $f, g : X \to X$ if there exists $x \in X$ such that $f(x) = g(x) = y$. Then, x is called a coincidence point. Mappings f and g are called weakly compatible if they commute at their coincidence points.

Proof. Starting from arbitrary $x_0 \in X$ and using that $AX \subset TX$, $BX \subset SX$ construct a Jungck sequence $\{y_n\}$ by

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \quad n = 0, 1, 2, \dots. \tag{4.3}$$

Consider two possible cases.

(1) $p(y_n, y_{n+1}) = 0$ (and so $y_n = y_{n+1}$) for some $n \in \mathbb{N}$. Let, for example, $n = 2k - 1$, $k \in \mathbb{N}$. Then

$$p(y_{2k}, y_{2k+1}) = p(Ax_{2k}, Bx_{2k+1}) \leq \varphi(M(x_{2k}, x_{2k+1})), \tag{4.4}$$

where

$$\begin{aligned} &M(x_{2k}, x_{2k+1}) \\ &\quad = \max\left\{p(y_{2k-1}, y_{2k}), p(y_{2k}, y_{2k-1}), p(y_{2k+1}, y_{2k}), \ \frac{1}{2}(p(y_{2k-1}, y_{2k+1}) + p(y_{2k}, y_{2k}))\right\} \\ &\quad = p(y_{2k}, y_{2k+1}) \quad (\text{by } (\mathrm{p}_4)). \end{aligned} \tag{4.5}$$

It follows that $p(y_{2k}, y_{2k+1}) \leq \varphi(p(y_{2k}, y_{2k+1})) < p(y_{2k}, y_{2k+1})$, which is impossible, unless $p(y_{2k}, y_{2k+1}) = 0$ and $y_{2k} = y_{2k+1}$. In a similar way, if $n = 2k$, $k \in \mathbb{N}$, it follows that also $y_{2k+1} = y_{2k+2}$. Hence, in both cases we obtain that the sequence $\{y_n\}$ is eventually constant, and so a Cauchy one.

(2) Suppose that $p(y_n, y_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then, as above,

$$p(y_{2n}, y_{2n+1}) = p(Ax_{2n}, Bx_{2n+1}) \leq \varphi(M(x_{2n}, x_{2n+1})), \tag{4.6}$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &\quad = \max\left\{p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n-1}), p(y_{2n+1}, y_{2n}), \frac{1}{2}(p(y_{2n-1}, y_{2n+1}) + p(y_{2n}, y_{2n}))\right\} \\ &\quad = \max\{p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1})\} \ (\text{by}(\mathrm{p}_4)). \end{aligned} \tag{4.7}$$

Similarly, $p(y_{2n+1}, y_{2n+2}) \leq \varphi(M(x_{2n+1}, x_{2n+2}))$, $M(x_{2n+1}, x_{2n+2}) = \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}$, that is

$$p(y_n, y_{n+1}) \leq \varphi(\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}), \quad n = 1, 2, \ldots. \tag{4.8}$$

If $p(y_n, y_{n+1}) \geq p(y_{n-1}, y_n)$, then $p(y_n, y_{n+1}) \leq \varphi(p(y_n, y_{n+1})) < p(y_n, y_{n+1})$, a contradiction. It follows that

$$p(y_n, y_{n+1}) < p(y_{n-1}, y_n), \quad n = 1, 2, \ldots. \tag{4.9}$$

Thus, in this case $\{p(y_n, y_{n+1})\}$ is a decreasing sequence of positive numbers. Denote $\lim_{n\to\infty} p(y_n, y_{n+1}) = r \geq 0$. In fact, $r = 0$. Indeed, if $r > 0$, then passing to the limit when $n \to \infty$ in $p(y_n, y_{n+1}) \leq \varphi(p(y_{n-1}, y_n))$, and using properties (III) of φ, we get that

$$r \leq \varphi(r) < r, \tag{4.10}$$

a contradiction.

We have proved that $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, Lemma 3.4 implies that there exist $\varepsilon > 0$ and two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences (3.6) tend to ε (from above) when $k \to \infty$. Now, using (4.1) with $x = x_{2m_k}$ and $y = x_{2n_k+1}$, we get that

$$p(y_{2m_k}, y_{2n_k+1}) \leq \varphi(M(x_{2m_k}, x_{2n_k+1})), \tag{4.11}$$

where

$$\begin{aligned} &M(x_{2m_k}, x_{2n_k+1}) \\ &\quad= \max\left\{p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k}, y_{2m_k-1}), p(y_{2n_k+1}, y_{2n_k}), \frac{1}{2}(p(y_{2m_k-1}, y_{2n_k+1}) + p(y_{2m_k}, y_{2n_k}))\right\} \\ &\quad\longrightarrow \varepsilon + 0, \quad k \longrightarrow \infty. \end{aligned} \tag{4.12}$$

Using properties (III) of φ, we obtain a contradiction $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, since $\varepsilon > 0$.

Thus $\{y_{2n}\}$ and so also $\{y_n\}$ is a Cauchy sequence, both in (X, p) and in (X, p^s). Suppose that, for example, SX is closed in X, and hence complete. It follows that sequence $\{y_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(y_n, z) = 0$, where $z = Su$ for some $u \in X$. Again from Lemma 2.4, we have

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n,m\to\infty} p(y_n, y_m). \tag{4.13}$$

Moreover, since $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim_{n,m\to\infty} p^s(y_n, y_m) = 0$ and so, by the definition of p^s, we have $\lim_{n,m\to\infty} p(y_n, y_m) = 0$. Then (4.13) implies that $p(z, z) = 0$ and

$$\lim_{n\to\infty} p(y_n, z) = p(z, z) = 0. \tag{4.14}$$

We will prove that z is a point of coincidence of A and S, that is, $Au = z$.

Suppose that $p(Au,z)>0$. Then, using (p_4),

$$\begin{aligned}p(Au,z)&\le p(Au,Bx_{2n+1})+p(Bx_{2n+1},z)-p(Bx_{2n+1},Bx_{2n+1})\\&\le\varphi\left(\max\left\{p(z,y_{2n}),p(Au,z),p(y_{2n+1},y_{2n}),\frac{1}{2}(p(z,y_{2n+1})\ +p(Au,y_{2n}))\right\}\right)\\&\quad+p(y_{2n+1},z)\\&\le\varphi\left(\max\left\{p(z,y_{2n}),p(Au,z),p(y_{2n+1},y_{2n}),\frac{1}{2}(p(z,y_{2n+1})+p(Au,z)+p(z,y_{2n}))\right\}\right)\\&\quad+p(y_{2n+1},z).\end{aligned}\tag{4.15}$$

All terms in the previous set $\{\cdots\}$, which depend on n, tend to 0 when $n\to\infty$ and they are smaller, for n large enough, than, say, $(1/2)p(Au,z)$. It follows that

$$p(Au,z)\le\varphi(p(Au,z))+p(y_{2n+1},z).\tag{4.16}$$

Letting $n\to\infty$, and using that $\varphi(t)<t$, we get that $p(Au,z)\le\varphi(p(Au,z))<p(Au,z)$, a contradiction. Hence, $p(Au,z)=0$ and $Au=z$.

Now $AX\subset TX$ implies that $z=Au\in TX$ and so there exists $v\in X$ such that $Tv=z$. If $p(z,Bv)>0$, then

$$\begin{aligned}p(z,Bv)&=p(Au,Bv)\\&\le\varphi\left(\max\left\{p(Su,Tv),p(Au,Su),p(Bv,Tv),\frac{1}{2}(p(Su,Bv)+p(Au,Tv))\right\}\right)\\&\le\varphi(p(z,Bv))<p(z,Bv),\end{aligned}\tag{4.17}$$

a contradiction. Hence, $p(z,Bv)=0$ and $Bv=z$, so z is a point of coincidence, both for (A,S) and for (B,T). It order to show that this point of coincidence is unique, one has only to use that $\varphi(t)<t$ for $t>0$ and property (p_2) of partial metric. Hence, if these pairs of mappings are weakly compatible, it follows by a well-known result that z is a unique common fixed point of A,B,S,T. The theorem is proved. □

Theorem 4.2. *Let (X,p) be a complete partial metric space, and let $A,B,S,T:X\to X$. Suppose that $AX\subset TX$, $BX\subset SX$ and one of these four subsets of X is closed. Let further*

$$p(Ax,By)\le\varphi(M(x,y))\tag{4.18}$$

hold for all $x,y\in X$, where

$$M(x,y)=\max\left\{p(Sx,Ty),p(Ax,Sx),p(By,Ty),\frac{1}{2}(p(Sx,By)+p(Ax,Ty))\right\},\tag{4.19}$$

and $\varphi:[0,+\infty)\to[0,+\infty)$ satisfies property (IV), that is, φ is nondecreasing and $\sum_{n\ge1}\varphi^n(t)<+\infty$ for each $t\ge0$. Then A and S, as well as B and T, have a coincidence point. If, moreover, pairs (A,S) and (B,T) are weakly compatible, then A,B,S,T have a unique common fixed point.

Proof. Construct a Jungck sequence $\{y_n\}$ as in the proof of Theorem 4.1. By Lemma 3.1, conditions (IV) imply that $\varphi^n(t) \to 0$, $n \to \infty$ for each $t \in [0, +\infty)$. Since φ is nondecreasing, it further implies that $\varphi(t) < t$ for each $t > 0$. Condition (4.9) follows as in the proof of Theorem 4.1. It further implies that

$$p(y_n, y_{n+1}) < \varphi^n(p(y_0, y_1)), \quad n \in \mathbb{N}. \tag{4.20}$$

By the definition of associated metric (2.2), we get that

$$p^s(y_n, y_{n+1}) < 4\varphi^n(p(y_0, y_1)), \quad n \in \mathbb{N}. \tag{4.21}$$

Now, for arbitrary $k \in \mathbb{N}$,

$$\begin{aligned} p^s(y_n, y_{n+k}) &\leq p^s(y_n, y_{n+1}) + p^s(y_{n+1}, y_{n+2}) + \cdots + p^s(y_{n+k-1}, y_{n+k}) \\ &\leq 4\sum_{i=n}^{n+k-1} \varphi^i(p(y_0, y_1)) \leq 4\sum_{i=n}^{\infty} \varphi^i(p(y_0, y_1)) \longrightarrow 0 \end{aligned} \tag{4.22}$$

as $n \to \infty$, by conditions (IV). Hence $\{y_n\}$ is a Cauchy sequence in X.

The rest of the proof is the same as in Theorem 4.1, since it uses only the contractive condition and that $\varphi(t) < t$ for $t > 0$. □

Remark 4.3. (1) It follows from the proofs of previous theorems that, under the respective hypotheses, an arbitrary Jungck sequence $\{y_n\}$ converges to the (unique) common fixed point z of A, B, S, T.

(2) Taking $\varphi(t) = kt$, $k \in [0, 1)$ in Theorems 4.1 and 4.2, we obtain an extension of Fisher's theorem for four mappings [15] to the setting of partial metric spaces.

(3) Taking $A = B = S = T$, Theorems 3 and 4 from [8] are obtained, and so also extensions of [3, Theorem 1] and [2, Theorem 1]. For [4, Theorem 2.1], precise assumptions for the function φ are stated.

(4) Our results are far extensions of the classical Boyd-Wong result [12] to the setting of partial metric spaces.

(5) A related result for so-called Geraghty-type mappings was recently obtained in [5].

We illustrate the results of this section with an example.

Example 4.4. Let $X = [0, 1]$ be equipped with the partial metric $p(x, y) = \max\{x, y\}$ and let $\varphi(t) = (t/(t + 1))e^{-t}$ for $t \geq 0$ (φ satisfies condition (III)). The mappings

$$Ax = \frac{1}{6}x^2, \qquad Bx = \frac{1}{8}x^2, \qquad Tx = \frac{1}{2}x^2, \qquad Sx = x^2 \tag{4.23}$$

are such that $AX \subset TX$, $BX \subset SX$, and (A, S) as well as (B, T) are weakly compatible. In order to check condition (4.1), take arbitrary $x, y \in X$ with, first, $x \geq y$. Then,

$$\begin{aligned} p(Ax, By) &= p\left(\frac{1}{6}x^2, \frac{1}{8}y^2\right) = \frac{1}{6}x^2, \\ M(x, y) &= \max\left\{p\left(x^2, \frac{1}{2}y^2\right), p\left(\frac{1}{6}x^2, x^2\right), p\left(\frac{1}{8}y^2, \frac{1}{2}y^2\right), \frac{1}{2}\left[p\left(x^2, \frac{1}{8}y^2\right) + p\left(\frac{1}{6}x^2, \frac{1}{2}y^2\right)\right]\right\} \end{aligned}$$

$$= \max\left\{x^2, x^2, \frac{1}{2}x^2, \frac{1}{2}\left[x^2 + \max\left\{\frac{1}{6}x^2, \frac{1}{2}y^2\right\}\right]\right\} = x^2,$$
$$\varphi(M(x,y)) = \frac{x^2}{x^2+1}e^{-x^2} \geq \frac{1}{6}x^2, \tag{4.24}$$

since $e^{-x^2}/(x^2+1) \geq 1/2e > 1/6$ for $x \in [0,1]$. In the case $x < y$, the same can be checked after careful calculations. Hence, A, B, S, T satisfy all the conditions of Theorem 4.1 and they have a unique common fixed point ($z = 0$).

5. Common Fixed Points for Two Mappings under Weaker Condition for the Comparison Function

In the next theorem we consider weaker condition (II) for the comparison function φ. As a compensation, we assume a bit stronger contractive condition.

Theorem 5.1. *Let (X, p) be a complete partial metric space and let $A, T : X \to X$ be two self-maps. Let $AX \subset TX$ and at least one of these subsets of X is closed. Suppose that there is a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying property (II), such that*

$$p(Ax, Ay) \leq \varphi\left(\max\left\{p(Tx,Ty), p(Ax,Tx), p(Ay,Ty), \frac{1}{2}p(Tx,Ay)\right\}\right) \tag{5.1}$$

holds for all $x, y \in X$. Then A and T have a unique point of coincidence. If A and T are weakly compatible, then they have a unique common fixed point $u \in X$.

Proof. We will construct a Jungck sequence in the usual way. Take arbitrary $x_0 \in X$ and using that $AX \subset TX$, choose $x_n \in X$ such that $y_n = Ax_n = Tx_{n+1}$, $n = 0, 1, 2, \ldots$. First, suppose that there exists an n such that $p(y_n, y_{n-1}) = 0$. Then the sequence $\{y_n\}$ is eventually constant. Indeed, from (5.1) it follows that

$$\begin{aligned} p(y_{n+1}, y_n) &= p(Ax_{n+1}, Ax_n) \\ &\leq \varphi\left(\max\left\{p(Tx_{n+1}, Tx_n), p(Ax_{n+1}, Tx_{n+1}), p(Ax_n, Tx_n), \frac{1}{2}p(Tx_{n+1}, Ax_n)\right\}\right) \\ &= \varphi\left(\max\left\{p(y_n, y_{n-1}), p(y_n, y_{n+1}), p(y_{n-1}, y_n), \frac{1}{2}p(y_n, y_n)\right\}\right) \\ &= \varphi(p(y_{n+1}, y_n)), \end{aligned} \tag{5.2}$$

since $(1/2)p(y_n, y_n) \leq p(y_n, y_n) \leq p(y_{n+1}, y_n)$ by (p_2). This is a contradiction with $\varphi(t) < t$ (which follows from assumption (I)), unless $p(y_{n+1}, y_n) = 0$. Hence, $y_{n+1} = y_n = y_{n-1}$. Continuing this process, we obtain that $\{y_n\}$ is an eventually constant sequence and hence a Cauchy one. Moreover, from $y_n = Ax_n = Tx_{n+1} = y_{n+1} = Ax_{n+1} = Tx_{n+1}$ it follows that $Ax_{n+1} = Tx_{n+1}$ and A and T have a coincidence point.

Suppose now that $p(y_n, y_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then, similarly as above, we get

$$p(y_{n+1}, y_n) \leq \varphi(\max\{p(y_n, y_{n-1}), p(y_{n+1}, y_n)\}). \tag{5.3}$$

Using property (I) of function φ it follows that

$$p(y_{n+1}, y_n) \leq \varphi(p(y_n, y_{n-1})) < p(y_n, y_{n-1}). \tag{5.4}$$

Hence, $p(y_{n+1}, y_n) < \varphi^n(p(y_0, y_1)) \to 0$ when $n \to \infty$ and $\{p(y_{n+1}, y_n)\}$ is a decreasing sequence, tending to 0.

Now, using mathematical induction, we prove that $\{y_n\}$ is a Cauchy sequence. Since $p(y_n, y_{n+1}) \to 0$, for each $\varepsilon > 0$, there exists $n(\varepsilon)$ such that $p(y_n, y_{n+1}) < \varepsilon - \varphi(\varepsilon)$ for $n > n(\varepsilon)$. Let, for some $k \in \mathbb{N}$, $p(y_n, y_{n+k}) < \varepsilon$. Then we have

$$p(y_n, y_{n+k+1}) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+k+1}), \tag{5.5}$$

where

$$\begin{aligned}
&p(y_{n+1}, y_{n+k+1}) \\
&\quad = p(Ax_{n+1}, Ax_{n+k+1}) \\
&\quad \leq \varphi\left(\max\left\{p(Tx_{n+1}, Tx_{n+k+1}), p(Ax_{n+1}, Tx_{n+1}), p(Ax_{n+k+1}, Tx_{n+k+1}), \frac{1}{2}p(Tx_{n+1}, Ax_{n+k+1})\right\}\right) \\
&\quad = \varphi\left(\max\left\{p(y_n, y_{n+k}), p(y_{n+1}, y_n), p(y_{n+k+1}, y_{n+k}), \frac{1}{2}p(y_n, y_{n+k+1})\right\}\right).
\end{aligned} \tag{5.6}$$

The first three members of the last set are smaller than ε. Concerning the fourth one, we have that

$$\frac{1}{2}p(y_n, y_{n+k+1}) \leq \frac{1}{2}p(y_n, y_{n+k}) + \frac{1}{2}p(y_{n+k}, y_{n+k+1}) < \frac{1}{2}\varepsilon + \frac{1}{2}(\varepsilon - \varphi(\varepsilon)) < \varepsilon, \tag{5.7}$$

and it follows that

$$p(y_{n+1}, y_{n+k+1}) < \varphi(\varepsilon). \tag{5.8}$$

Hence,

$$p(y_n, y_{n+k+1}) < \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon, \tag{5.9}$$

and the inductive proof is over.

Using the definition of the associated metric p^s, we get that

$$p^s(y_n, y_{n+k}) \leq 4p(y_n, y_{n+k}) < 4\varepsilon, \tag{5.10}$$

for each $n > n(\varepsilon)$ and $k \in \mathbb{N}$. Thus, $\{y_n\}$ is also a p^s-Cauchy sequence, and so there exists $y \in X$ such that

$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{m,n\to\infty} p(y_n, y_m). \tag{5.11}$$

Suppose that, for example, TX is closed. Then $y = Tu$ for some $u \in X$. We will prove that also $Au = y$. Suppose that $p(Au, y) > 0$. Then, using (5.1), we get

$$p(y, Au) \leq p(y, Ax_n) + p(Ax_n, Au)$$

$$\leq p(y, Ax_n) + \varphi\left(\max\left\{p(Tx_n, Tu), p(Ax_n, Tx_n), p(Au, Tu), \frac{1}{2}p(Tu, Ax_n)\right\}\right)$$
$$= p(y, Ax_n) + \varphi\left(\max\left\{p(y_{n-1}, Tu), p(y_n, y_{n-1}), p(Au, y), \frac{1}{2}p(Tu, y_n)\right\}\right). \tag{5.12}$$

Since $y_n \to y = Tu$, there exists n_0 such that $p(y_{n-1}, Tu) < p(Au, y)$ and $p(y_n, y_{n-1}) < p(Au, y)$ whenever $n > n_0$. It follows that

$$p(y, Au) \leq p(y, Ax_n) + \varphi(p(Au, y)) \tag{5.13}$$

and, passing to the limit when $n \to \infty$, we obtain that

$$p(y, Au) \leq 0 + \varphi(p(Au, y)) < p(Au, y), \tag{5.14}$$

which is a contradiction. Hence, $p(Au, y) = 0$ and $Au = y = Tu$ is a point of coincidence of A and T.

Note that continuity of φ was not needed in the previous conclusions.

In order to prove that the point of coincidence is unique, suppose that there exists $y_1 \neq y$ (and so $p(y_1, y) > 0$) such that $Au_1 = Tu_1 = y_1$ for some $u_1 \in X$. Then (5.1) implies that

$$p(y_1, y) = p(Au_1, Au) \leq \varphi\left(\max\left\{p(Tu_1, Tu), p(Au_1, Tu_1), p(Au, Tu), \frac{1}{2}p(Tu_1, Au)\right\}\right)$$
$$= \varphi\left(\max\left\{p(y_1, y), p(y_1, y_1), p(y, y), \frac{1}{2}p(y_1, y)\right\}\right) \leq \varphi(p(y_1, y)) < p(y_1, y), \tag{5.15}$$

since $p(y, y) \leq p(y, y_1)$ and $p(y_1, y_1) \leq p(y, y_1)$. This contradiction shows that $p(y_1, y) = 0$ and $y_1 = y$.

The last assertion of the theorem follows easily from the definition of weak compatibility and the uniqueness of the coincidence point. □

We present an example where the existence of a common fixed point can be proved using Theorem 5.1 (and conditions formulated in terms of a partial metric) but cannot be obtained using respective conditions in the associated (standard) metric.

Example 5.2. Consider on $X = \{0, 1, 2\}$ both the partial metric $p(x, y) = \max\{x, y\}$ and the associated metric $d(x, y) = |x - y|$. Let $A : X \to X$ be given by $A0 = A1 = 0$, $A2 = 1$, let $T = i_X$ (identity map), and let $\varphi(t) = (3/4)t$, $t \in [0, +\infty)$. Then

$$\begin{aligned} &p(A0, A1) = p(0, 0) = 0, \\ &p(A0, A2) = p(0, 1) = 1 < 3/2 = \varphi(2) = \varphi(\max\{p(0, 2), p(0, 0), p(2, 1), 1/2p(0, 1)\}), \\ &p(A1, A2) = p(0, 1) = 1 < 3/2 = \varphi(2) = \varphi(\max\{p(1, 2), p(1, 0), p(2, 1), 1/2p(1, 1)\}), \end{aligned} \tag{5.16}$$

and condition (5.1) is satisfied. The respective condition in the standard metric does not hold since

$$d(A1,A2)=d(0,1)=1>3/4=\varphi(1)=\varphi(\max\{d(1,2),d(1,0),d(2,1),1/2d(1,1)\}). \quad (5.17)$$

6. Meir Keeler-Type Result in Partial Metric Spaces

It is well-known that the celebrated Meir-Keeler fixed point result [16] can also be formulated in a form with a comparison function. We present a partial metric version of this result. The function $\varphi:[0,+\infty)\to[0,+\infty)$ will be called a Meir-Keeler function if

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall t\ \varepsilon\le t<\varepsilon+\delta \implies \varphi(t)<\varepsilon. \quad (6.1)$$

Theorem 6.1. *Let (X,p) be a complete partial metric space and let a mapping $f:X\to X$ satisfies the following condition*

$$\forall x,y\in X \quad p(fx,fy)\le\varphi(p(x,y)), \quad (6.2)$$

for a Meir-Keeler function φ. Then f has a unique fixed point, say u, and for each $x\in X$, the Picard sequence $\{f^nx\}$ converges to u, satisfying $p(u,u)=0$.

Proof. We prove first the uniqueness. Let u,v be two fixed points of f and let $p(u,v)>0$. Then, taking $\varepsilon=p(u,v)$ in (9), it follows that there exists $\delta>0$ such that $p(u,v)\le t<p(u,v)+\delta$ implies that $\varphi(t)<p(u,v)$. In particular, $\varphi(p(u,v))<p(u,v)$. But then, by (6.2), $p(u,v)=p(fu,fv)\le\varphi(p(u,v))<p(u,v)$, a contradiction. Hence, $p(u,v)=0$ and $u=v$.

Take now arbitrary $x\in X$ and form the Picard sequence $\{f^nx\}$. If, for some n, $p(f^nx,f^{n+1}x)=0$, then $ff^nx=f^nx$ and f^nx is a (unique) fixed point of f. Suppose that $p(f^nx,f^{n+1}x)>0$ for each $n=0,1,2,\ldots$. Taking (for fixed n) $\varepsilon=p(f^nx,f^{n+1}x)$ in (6.1), there exists $\delta>0$ such that

$$p\left(f^nx,f^{n+1}x\right)\le t<p\left(f^nx,f^{n+1}x\right)+\delta \implies \varphi(t)<p\left(f^nx,f^{n+1}x\right). \quad (6.3)$$

Then, by (6.2),

$$p\left(f^{n+1}x,f^{n+2}x\right)=p\left(ff^nx,ff^{n+1}x\right)\le\varphi\left(p\left(f^nx,f^{n+1}x\right)\right)<p\left(f^nx,f^{n+1}x\right). \quad (6.4)$$

It follows that $\{p(f^nx,f^{n+1}x)\}$ is a decreasing sequence of positive numbers, tending to some $r\ge0$. If $r>0$ then, again by (6.1) and (6.2), we can find δ such that

$$r\le p\left(f^nx,f^{n+1}x\right)<r+\delta \implies p\left(f^{n+1}x,f^{n+2}x\right)<r. \quad (6.5)$$

This is a contradiction since $p(f^{n+1}x,f^{n+2}x)\ge r$. We conclude that $p(f^nx,f^{n+1}x)\to0$ when $n\to\infty$.

In order to prove that $\{f^n x\}$ is a Cauchy sequence, we use again Lemma 3.4. If we suppose the contrary, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $p(f^{n_k}x, f^{m_k}x) \to \varepsilon^+$ and also $p(f^{n_k+1}x, f^{m_k+1}x) \to \varepsilon^+$. But, (6.1) and (6.2) imply that there exists $\delta > 0$ such that $\varepsilon \le p(f^{n_k}x, f^{m_k}x) < \varepsilon + \delta$ implies $p(f^{n_k+1}x, f^{m_k+1}x) < \varepsilon$. This is a contradiction since $p(f^{n_k+1}x, f^{m_k+1}x) \ge \varepsilon$ for large k.

Thus, $\{f^n x\}$ is a Cauchy sequence, both in (X, p) and in (X, p^s) (Lemma 2.4) and there exists an element u in the complete (partial) metric space X such that $\lim_{n\to\infty} p^s(f^n x, u) = 0$, wherefrom

$$p(u,u) = \lim_{n\to\infty} p(f^n x, u) = \lim_{m,n\to\infty} p(f^n x, f^m x). \tag{6.6}$$

By the definition (2.2) of metric p^s, and

$$\lim_{n\to\infty} p(f^n x, f^n x) = \lim_{m\to\infty} p(f^m x, f^m x) = 0, \tag{6.7}$$

it follows that

$$p(u,u) = \lim_{n\to\infty} p(f^n x, u) = \lim_{m,n\to\infty} p(f^n x, f^m x) = 0. \tag{6.8}$$

We have proved that

$$0 = p(u,u) = \lim_{n\to\infty} p(f^n x, u). \tag{6.9}$$

It remains to show that $fu = u$. Take $f^n x$ instead of x and u instead of y in (6.2). Then, using (6.1) we get that

$$\begin{aligned} p\left(f^{n+1}x, fu\right) &< p(f^n x, u), \\ p(u, fu) &\le p\left(u, f^{n+1}x\right) + p\left(f^{n+1}x, fu\right) - p\left(f^{n+1}x, f^{n+1}x\right) \\ &\le p\left(u, f^{n+1}x\right) + p\left(f^{n+1}x, fu\right) \\ &< p\left(u, f^{n+1}x\right) + p(f^n x, u) \\ &\longrightarrow p(u,u) + p(u,u) = 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{6.10}$$

If follows that $p(u, fu) = 0$ and $fu = u$. □

Example 6.2. Let X, p and d be as in Example 5.2. Consider mapping $f : X \to X$ and function $\varphi : [0, +\infty) \to [0, +\infty)$ given by

$$f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varphi(t) = \frac{3}{4}t. \tag{6.11}$$

Then φ is a Meir-Keeler function. Indeed, for arbitrary $\varepsilon > 0$ choose $\delta = (1/3)\varepsilon$ and $\varepsilon \le t < \varepsilon + \delta = (4/3)\varepsilon$ implies that $\varphi(t) < \varepsilon$. We will check that f satisfies condition (6.2) of Theorem 6.1.

In the cases $x = y = 0$; $x = y = 1$; and $x = 0, y = 1$, the left-hand side of (6.2) is equal to zero. In all other cases ($x = y = 2$; $x = 0, y = 2$; and $x = 1, y = 2$), it is $p(fx, fy) = 1$ and $\varphi(p(x, y)) = \varphi(2) = 3/2$. Hence, condition (6.2) always holds true, and mapping f has a unique fixed point ($u = 0$).

Note again that in the case when standard metric d is used instead of partial metric p, this conclusion cannot be obtained. Indeed, for $x = 1$, $y = 2$ we have that

$$d(f1, f2) = d(0,1) = 1 > \frac{3}{4} = \varphi(1) = \varphi(d(1,2)). \tag{6.12}$$

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References

[1] S. G. Matthews, "Partial metric topology," in *Proceedings of the 8th Summer Conference on General Topology and Applications*, pp. 183–197, The New York Academy of Sciences, 1994.

[2] I. Altun and K. Sadarangani, "Corrigendum to "Generalized contractions on partial metric spaces" [Topology and Its Applications vol. 157, pp. 2778–2785, 2010] [MR2729337]," *Topology and Its Applications*, vol. 158, no. 13, pp. 1738–1740, 2011.

[3] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology and Its Applications*, vol. 157, no. 18, pp. 2778–2785, 2010.

[4] L. Ćirić, B. Samet, H. Aydi, and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2398–2406, 2011.

[5] D. Đukić, Z. Kadelburg, and S. Radenović, "Fixed points of geraghty-type mappings in various generalized metric spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 561245, 13 pages, 2011.

[6] R. Heckmann, "Approximation of metric spaces by partial metric spaces," *Applied Categorical Structures*, vol. 7, no. 1-2, pp. 71–83, 1999.

[7] D. Ilić, V. Pavlović, and V. Rakočević, "Some new extensions of Banach's contraction principle to partial metric space," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1326–1330, 2011.

[8] S. Romaguera, "Fixed point theorems for generalized contractions on partial metric spaces," *Topology and Its Applications*, vol. 159, no. 1, pp. 194–199, 2012.

[9] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, vol. 36, no. 1-2, pp. 17–26, 2004.

[10] W. Shatanawi, B. Samet, and M. Abbas, "Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces," *Mathematical and Computer Modelling*, vol. 55, pp. 680–687, 2012.

[11] O. Valero, "On Banach fixed point theorems for partial metric spaces," *Applied General Topology*, vol. 6, no. 2, pp. 229–240, 2005.

[12] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.

[13] J. Matkowski, "Fixed point theorems for mappings with a contractive iterate at a point," *Proceedings of the American Mathematical Society*, vol. 62, no. 2, pp. 344–348, 1977.

[14] S. Radenović, Z. Kadelburg, D. Jandrlić, and A. Jandrlić, "Some results on weak contraction maps," *Bulletin of the Iranian Mathematical Society*. In press.

[15] B. Fisher, “Four mappings with a common fixed point,” *The Journal of the University of Kuwait*, vol. 8, pp. 131–139, 1981.

[16] A. Meir and E. Keeler, “A theorem on contraction mappings,” *Journal of Mathematical Analysis and Applications*, vol. 28, pp. 326–329, 1969.