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Research Article

On Global Solutions for the Cauchy Problem of a Boussinesq-Type Equation

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We will give conditions which will guarantee the existence of global weak solutions of the Boussinesq-type equation with power-type nonlinearity $\gamma|u|^p$ and supercritical initial energy. By defining new functionals and using potential well method, we readdressed the initial value problem of the Boussinesq-type equation for the supercritical initial energy case.

1. Introduction

This paper is devoted to the initial value problem of a Boussinesq-type equation:

$$u_{tt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = \left(f(u)\right)_{xx}, \quad x \in R,\ t > 0, \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in R, \tag{1.2}$$

where $f(u) = \gamma|u|^p, \gamma > 0$. Equation (1.1) is of fundamental physical interest because it arises in the study of shallow water theory, nonlinear lattice theory, and some other physical phenomena. In the absence of the sixth-order term, (1.1) becomes the good Boussinesq equation and has been intensively studied from a mathematical viewpoint with various additional terms [1–9].

Concerning the initial value problem (1.1) and (1.2), it is important to cite the works of Xu et al. [10], Y. Z. Wang and Y. X. Wang, [11] and Wang and Xu [12].

Wang and Xu [12] investigated the Cauchy problem (1.1) and (1.2). They first proved the existence and nonexistence of global solutions of the problem for a general nonlinear function $f(u)$ provided that the antiderivative F of f satisfies $F(u) \geq 0$ or $f'(u)$ is bounded below. Then by the potential well method, they proved the global existence of solutions for a special case of the nonlinear term $f(u) = -\beta|u|^p u$ with $0 < E(0) \leq d$.

In [11], the initial value problem for a class of nonlinear wave equations of higher order:

$$u_{tt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = \left(\varphi(u)_x\right)_x \tag{1.3}$$

was considered. The global well-posedness of the initial value problem of (1.3) with $\varphi(s) = \alpha|s|^p, \alpha \neq 0$ was studied making use of the potential well method.

Very recently, Xu et al. [10] studied the multidimensional Boussinesq-type equation:

$$u_{tt} - \Delta u + \Delta^2 u + \Delta^2 u_{tt} = \Delta\left(f(u)\right), \tag{1.4}$$

with the nonlinear terms $\pm a|u|^p,\ -a|u|^{p-1}u,\ \pm au^{2k}$, and $-au^{2k+1}$. When $E(0) < d$, the global existence and finite-time blowup of solutions were proved by the aid of the potential well method.

All of the above-mentioned three studies considered the cases $E(0) < d$ or $0 < E(0) \leq d$. The case $E(0) > 0$ was also investigated in [12], but the conditions imposed on the nonlinear term are only valid for the odd-degree nonlinearities, that is, positivity condition is not enough for the nonlinear term $f(u) = \gamma|u|^p$. In the present paper, we again investigate problem (1.1) and (1.2) and give the global existence of solutions for $f(u) = \gamma|u|^p$ with supercritical initial energy. We emphasize that our main results would seem to be the first contribution to global well-posedness of the sixth-order Boussinesq equation with supercritical initial energy and this type of nonlinearity.

The plan of the paper is as follows. Section 2 contains some definitions and an abbreviated description of the global existence theory given in [10, 12]. Also it contains our first new functional and some theorems about sign preserving property of this functional. The global existence theory for supercritical initial energy is presented in Section 3, following the lines laid down by Kutev et al. [13]. For this purpose, we introduce the second new functional and give the invariance of this functional under the flow of (1.1) and (1.2). A few concluding remarks and an example are to be found in the last section.

The notation employed is standard and may be found explained in detail in [10].

2. Preliminaries

In this section, we give some definitions and some theorems from the papers mentioned in the first section. We also introduce a new functional, and we try to prove global well-posedness for supercritical energy by the aid of this functional.

Let $s \geq 1$. For the Cauchy problem (1.1) and (1.2), we define

$$E(t) = E(u(t), u_t(t)) = \frac{1}{2}\left\|\left(-\partial_x^2\right)^{-1/2} u_t\right\|^2 + \|u_{xt}\|^2 + \|u\|_{H^1}^2 + \frac{\gamma}{p+1}\int_R |u|^p u dx = E(0), \tag{2.1}$$

$$J(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{\gamma}{p+1}\int_R |u|^p u dx, \tag{2.2}$$

$$I(u) = \|u\|_{H^1}^2 + \gamma\int_R |u|^p u dx, \tag{2.3}$$

$$d = \inf_{u\in N} J(u), \qquad N = \left\{u \in H^1 : I(u) = 0,\ \|u\|_{H^1} \neq 0\right\}, \tag{2.4}$$

which are all well defined.

In the following, we show one more characterization of d used before in the literature

$$d = \frac{p-1}{2(p+1)}\left(\gamma S_p^{p+1}\right)^{-2/(p-1)}, \tag{2.5}$$

where S_p is the imbedding constant from $H^1(R)$ into $L^{p+1}(R)$ given by

$$S_p = \sup_{u\in H^1}\frac{\|u\|_{p+1}}{\|u\|_{H^1}}. \tag{2.6}$$

By the use of (2.1) and definition of S_p, (2.5) can be easily obtained as follows:

$$\frac{1}{2}\|u\|_{H^1}^2 - \frac{\gamma}{p+1}S_p^{p+1}\|u\|_{H^1}^{(p+1)/2} \leq J(u) \leq E(0). \tag{2.7}$$

The function

$$h(k) = \frac{1}{2}k - \frac{\gamma}{p+1}S_p^{p+1}k^{(p+1)/2} \tag{2.8}$$

is strictly increasing in $[0, k_1)$ and strictly decreasing in (k_1, ∞), where $k_1 = \gamma^{-2/(p+1)} S_p^{-2(p+1)/(p-1)}$. By (2.4), we get

$$d = \max_{k\in[0,\infty)} h(k) = h(k_1) = \frac{p-1}{2(p+1)}\left(\gamma S_p^{p+1}\right)^{-2/(p-1)}. \tag{2.9}$$

The following theorem is a generalization of Theorem 3.4 in [12] and Theorem 6.2 in [10].

Theorem 2.1. *Assume that* $\varphi \in H^1$, $\psi \in H^1$ *and* $(-\partial_x^2)^{-1/2}\varphi$, $(-\partial_x^2)^{-1/2}\psi \in L^2(R)$.

(i) *If* $E(0) < 0$, *then all weak solutions of* (1.1) *and* (1.2) *blow up in finite time;*

(ii) *if* $E(0) = 0$, *then all weak solutions of* (1.1) *and* (1.2), *except the trivial one blow up in finite time;*

(iii) *let* $0 < E(0) < d$. *If* $I(\varphi) > 0$, *then the weak solution of* (1.1) *and* (1.2) *is globally defined for every* $t \in [0,\infty)$, *if* $I(\varphi) < 0$, *then the weak solution of* (1.1) *and* (1.2) *blows up in finite time.*

Remark 2.2. For the critical and subcritical initial energy case, the functional $I(\varphi)$ determines the behavior of solutions of (1.1) and (1.2). But, the numerical results in [13] showed that in the supercritical case the sign of the functional $I(\varphi)$ cannot guarantee the global well-posedness of the problem. Due to the choice of initial data, for $I(\varphi) > 0$, there are some solutions that blow up in finite time, and for $I(\varphi) < 0$, there are some solutions that exist globally. A new functional I_δ was considered in [6]. In the rest of this section, we prove that even if we take a more general functional than I_δ, it will be insufficient in determining the behavior of problem (1.1) and (1.2) in supercritical case. For a satisfactory result, we have to take into account the initial velocity in a new functional. This will be done in the next section.

Now, we define the first new functional:

$$I_\sigma(u) = (1-\sigma)\|u\|_{H^1}^2 + \gamma \int_R |u|^p u dx = I(u) - \sigma\|u\|_{H^1}^2, \tag{2.10}$$

for $\sigma > -(p-1)/2$. The depth D_σ and N_σ are as follows:

$$D_\sigma = \inf_{u\in N_\sigma} J(u), \qquad N_\sigma = \left\{u \in H^1 : I_\sigma(u) = 0,\ \|u\|_{H^1} \neq 0\right\}. \tag{2.11}$$

Obviously, taking $\sigma = 0$ corresponds to the functional $I(u)$. Moreover in the case of $\sigma < -(p-1)/2$ we have $D_\sigma < 0$. From Theorem 2.1, we know that in this case all weak solutions of (1.1) and (1.2) blow up in a finite time.

For $\sigma > -(p-1)/2$, we have the following lemmas.

Lemma 2.3. *Assume that $u \in H^1(R)$. If $I_\sigma(u) < 0$, then $\|u\|_{H^1} > r(\sigma)$, and if $I_\sigma(u) = 0$, then $\|u\|_{H^1} \geq r(\sigma)$ or $\|u\|_{H^1} = 0$, where $r(\sigma) = ((1-\sigma)/\gamma S_p^{p+1})^{1/(p-1)}$.*

Proof. First, from $I_\sigma(u) < 0$, we have $\|u\|_{H^1} \neq 0$. Hence, by

$$(1-\sigma)\|u\|_{H^1}^2 < -\gamma \int_R |u|^p u dx \leq \gamma S_p^{p+1} \|u\|_{H^1}^{p+1}, \tag{2.12}$$

we have $\|u\|_{H^1} > r(\sigma)$.

If $\|u\|_{H^1} = 0$, then $I_\sigma(u) = 0$, if $I_\sigma(u) = 0$ and $\|u\|_{H^1} \neq 0$, then from

$$(1-\sigma)\|u\|_{H^1}^2 = -\gamma \int_R |u|^p u dx \leq \gamma S_p^{p+1} \|u\|_{H^1}^{p+1}, \tag{2.13}$$

we have $\|u\|_{H^1} \geq r(\sigma)$. □

Lemma 2.4. *If $\|u\|_{H^1} < r(\sigma)$, then $I_\sigma(u) \geq 0$.*

Proof. From $\|u\|_{H^1} < r(\sigma)$, we obtain

$$-\gamma \int_R |u|^p u dx \leq \gamma S_p^{p+1} \|u\|_{H^1}^{p+1} < (1-\sigma)\|u\|_{H^1}^2 \tag{2.14}$$

from which follows $I_\sigma(u) \geq 0$. □

The properties of $I(u)$ have been studied in detail. Particularly, analogous results to above lemmas are obtained in [10] (Lemma 3.3), for $\sigma = 0$.

Theorem 2.5. *Let D_σ be defined as above. Then for $\sigma > -(p-1)/2$, one has*

$$D_\sigma = b(\sigma) r^2(\sigma), \tag{2.15}$$

where $b(\sigma) = (1/2) - ((1-\sigma)/(p+1))$. If one writes D_σ in terms of d, one obtains the following statement:

$$D_\sigma = b(\sigma)(1-\sigma)^{2/(p-1)}\frac{2(p+1)}{p-1}d. \tag{2.16}$$

Proof. If $u \in N_\sigma$, we have by Lemma 2.3 that $\|u\|_{H^1} \geq r(\sigma)$. In the proof of Lemma 2.3, the inequality (2.12) is an equality if and only if u is a minimizer of the imbedding H^1 into L^{p+1}. Since $\|u\|_{p+1} = S_p\|u\|_{H^1}$ is attained only for $\tilde{u} = (\cosh((p-1)/2)x)^{-2/(p-1)}$ [14], and it has a constant sign, we have

$$\inf_{u\in N_\sigma}\|u\|_{H^1} = r(\sigma). \tag{2.17}$$

Hence from

$$\begin{aligned}\inf_{u\in N_\sigma} J(u) &= \inf_{u\in N_\sigma}\left(\frac{1}{2}\|u\|_{H^1}^2 + \frac{\gamma}{p+1}\int_R |u|^p u dx\right)\\ &= \inf_{u\in N_\sigma}\left[\left(\frac{1}{2} - \frac{(1-\sigma)}{p+1}\right)\|u\|_{H^1}^2 + \frac{1}{p+1}I_\sigma(u)\right]\\ &= \left(\frac{1}{2} - \frac{(1-\sigma)}{p+1}\right)\inf_{u\in N_\sigma}\|u\|_{H^1}^2,\end{aligned} \tag{2.18}$$

and by definition of D_σ, we obtain $D_\sigma = b(\sigma)r^2(\sigma)$. □

As properties of D_σ, the following corollary can be obtained by a simple computation.

Corollary 2.6. *D_σ is strictly increasing on $\sigma \in (-(p-1)/2, 0) \cup (1,\infty)$ and strictly decreasing on $(0,1)$. Moreover $\lim_{\sigma\to 1} D_\sigma = 0$, and $D_{\sigma_0} = 0$, where $\sigma_0 = -(p-1)/2$.*

The following theorems show the invariance of I_σ under the flow of (1.1) and (1.2) in the framework of weak solutions for $0 < E(0) < d$ and $E(0) = d$, respectively.

Theorem 2.7. *Assume that $\varphi,\psi \in H^1(R)$, $(-\partial_x^2)^{-1/2}\psi \in L^2(R)$. Let $0 < E(0) < d$. Then the sign of I_σ is invariant under the flow of (1.1) and (1.2) for $\sigma \in (\sigma_1, \sigma_2]$, where $(\sigma_1, \sigma_2]$ is the maximal interval such that $D_\sigma = E(0)$.*

Proof. In consequence of assumption $E(0) > 0$, we have $\|u\|_{H^1} > 0$. If for $\sigma \in (\sigma_1, \sigma_2]$, the sign of I_σ is changeable, then we must have a $\overline{\sigma} \in (\sigma_1, \sigma_2]$ such that $I_{\overline{\sigma}} = 0$. First, we prove the theorem for $\overline{\sigma} \in (\sigma_1, \sigma_2)$. Notice from (2.16) that $\sigma_1 < 0$ and $\sigma_2 \in (0,1)$. Thus, we have $D_{\sigma_2} \geq E(0) \geq J(u) \geq D_{\overline{\sigma}}$. Since D_σ is strictly decreasing on $(0,1)$, from Corollary 2.6 we have $D_{\overline{\sigma}} > D_{\sigma_2}$, which contradicts with the previous inequality. So, the theorem is proved for $\overline{\sigma} \in (\sigma_1, \sigma_2)$. For $\overline{\sigma} = \sigma_2$, assume that $I_{\sigma_2} = 0$ and there exists some $\underline{\sigma} \in (0, \sigma_2)$ such that $I_{\underline{\sigma}} = I_{\sigma_2} + (\sigma_2 - \underline{\sigma})\|u\|_{H^1}^2$. Hence $I_{\underline{\sigma}} = (\sigma_2 - \underline{\sigma})\|u\|_{H^1}^2 > 0$ and for some $t' > 0$, we get $I_{\underline{\sigma}}(u(t')) = 0$. Then, by the similar argument used for $\overline{\sigma} \in (\sigma_1, \sigma_2)$, we obtain a contradiction. Thus, the proof is finished. □

Theorem 2.8. *If on top of all the assumptions of Theorem 2.7, we suppose that* $E(0) = d$*. Then the sign of* I_0 *(recall that when* $E(0) = d$*, we have* $\sigma_1 = \sigma_2 = 0$*) is invariant with respect to* (1.1) *and* (1.2) *for every* $t \in [0, \infty)$*.*

Proof. The theorem states that if $I_0(\varphi) \geq 0$, then $I_0(u(t)) \geq 0$, contrary if $I_0(\varphi) < 0$, then $I_0(u(t)) < 0$. We only give the proof of the first statement, the second is similar. To check that I_0 does not change sign, we proceed as follows. Let $u(t)$ be any weak solution of problem (1.1),(1.2) with $E(0) = d$. If the first statement is false, then there must exist a $t' > 0$ such that $I_0(u(t')) < 0$. It follows from Lemma 2.3 that $\|u(t')\|_{H^1} > 0$. From the energy identity, we get

$$\begin{aligned} d = E(0) &= \frac{1}{2}\left\| \left(-\partial_x^2\right)^{-1/2} u_t(t') \right\|^2 + \|u_{xt}(t')\|^2 + J(u(t')) \\ &\geq J(u(t')) \geq \inf_{u \in N_0} J(u) = d. \end{aligned} \tag{2.19}$$

If $u(t') \in N_0$, then $u(t')$ must be a minimizer of $J(u)$ for $u \in N_0$, and we have $I_0(u(t')) = 0$. This, however, is impossible, since it violates $I_0(u(t')) < 0$. Thus the lemma is proved. □

Now, we give a lemma for $\sigma > 1$, which states similar results to Lemmas 2.3 and 2.4, and can be proved similarly.

Lemma 2.9. *Assume that* $u \in H^1(R)$*. For* $\sigma > 1$*, if* $I_\sigma(u) > 0$*, then* $\|u\|_{H^1} > s(\sigma)$*, and if* $I_\sigma(u) = 0$*, then* $\|u\|_{H^1} \geq s(\sigma)$ *or* $\|u\|_{H^1} = 0$*, where* $s(\sigma) = ((\sigma - 1)/\gamma S_p^{p+1})^{1/(p-1)}$*. Moreover, if* $\|u\|_{H^1} < s(\sigma)$*, then* $I_\sigma(u) \leq 0$ *and* $I_\sigma(u) = 0$ *if and only if* $\|u\|_{H^1} = 0$*.*

Theorem 2.10. *Assume that* $\varphi, \psi \in H^1(R)$*,* $(-\partial_x^2)^{-1/2}\psi \in L^2(R)$*. If* $E(0) > 0$*, then* $I_\sigma(u(t)) \leq 0$ *for every* $t > 0$ *and* $\sigma \geq \sigma_m$*, where* σ_m *is the maximal positive root of* $D_\sigma = E(0)$*.*

Proof. We give the proof of the theorem for $\sigma = \sigma_m$ and $\sigma > \sigma_m$ separately. First we prove the theorem for $\sigma = \sigma_m$. Proceeding by contradiction, assume that there exists some $t' > 0$ such that $I_{\sigma_m}(u(t')) > 0$. By Lemma 2.3, we have $\|u\|_{H^1} > 0$, and there exists a value σ, $\sigma > \sigma_m$ such that $I_\sigma(u(t')) = 0$. Then by (2.1), $D_{\sigma_m} = E(0) \geq J(u(t')) \geq \inf_{u \in N_\sigma} J(u) = D_\sigma$. From definition of D_σ for $\sigma > \sigma_m > 1$, we have $D_\sigma > D_{\sigma_m}$. A contradiction occurs, which proves the theorem for $\sigma = \sigma_m$. For $\sigma \geq \sigma_m$, $I_{\sigma_m}(u(t)) \geq I_\sigma(u(t))$ implies that the theorem is true for every $\sigma \geq \sigma_m$. □

The following corollary is a direct consequence of Theorems 2.1 and 2.10.

Corollary 2.11. *Suppose* $\varphi, \psi \in H^1$*,* $(-\partial_x^2)^{-1/2}\psi \in L^2(R)$*. Let* $0 < E(0) < d$ *and* $I_0(\varphi) > 0$*. Then*

$$0 < I_0(u(t)) < \sigma_m \|u\|_{H^1}^2, \tag{2.20}$$

for every $t > 0$*.*

Remark 2.12. We tried to characterize the behavior of solutions for $E(0) > 0$ in terms of initial displacement. We constituted a new functional $I_\sigma(u)$ and proved the sign invariance of $I_\sigma(u)$ for $0 < E(0) < d$ and $E(0) = d$. But the case $E(0) > 0$ is still an open question, because from Theorem 2.10, we concluded that in this case $I_\sigma(u)$ is always nonpositive. Due to numerical results of [13], we know that such a functional to prove global existence must include the initial velocity too. We will introduce this new functional in the next section.

3. Global Existence for Supercritical Initial Energy

In this section, we state the main result of the paper. The functional we introduce here, which was used before in [13] in a similar form, permits us to establish global existence of solutions for (1.1) and (1.2) in the supercritical initial energy case

$$\begin{aligned} \widetilde{H}(v,\omega) &= \|v\|_{H^1}^2 + \gamma \int_R |v|^p v\,dx - \left\| \left(-\partial_x^2\right)^{-1/2} \omega \right\|^2 - \|\omega_x\|^2 \\ &= I_0(v) - \left\| \left(-\partial_x^2\right)^{-1/2} \omega \right\|^2 - \|\omega_x\|^2. \end{aligned} \tag{3.1}$$

In order to simplify notation, we rewrite $\widetilde{H}(u(\cdot,t), u_t(\cdot,t))$ as

$$H(u,t) = \widetilde{H}(u(\cdot,t), u_t(\cdot,t)). \tag{3.2}$$

Once we have proved the invariance of the above functional with respect to (1.1) and (1.2), then global existence can be proved by the aid of invariance of this functional.

Theorem 3.1. *Assume that φ, $\psi \in H^1(R)$, $(-\partial_x^2)^{-1/2}\varphi$, $(-\partial_x^2)^{-1/2}\psi \in L^2(R)$ and $E(0) > 0$. For some $\sigma > \sigma_m, \sigma_m$ defined as above, one have*

$$\begin{aligned} &\left(\left(-\partial_x^2\right)^{-1/2}\psi, \left(-\partial_x^2\right)^{-1/2}\varphi\right) + (\psi_x, \varphi_x) + \frac{1}{2}\left\| \left(-\partial_x^2\right)^{-1/2}\varphi \right\|^2 \\ &\quad + \frac{1}{2}\|\varphi_x\|^2 + \frac{(p+1)\sigma}{p-1+(p+3)\sigma} E(0) \le 0. \end{aligned} \tag{3.3}$$

Moreover, $H(u,t)$ is positive provided that $H(u,0)$ is positive, for every $t \in [0,\infty)$.

Proof. Looking for the global solution is equivalent to showing that there is no blow up. So, we modify a blow up technique for the proof [15]. To this end, we define

$$\theta(t) = \left\| \left(-\partial_x^2\right)^{-1/2} u \right\|^2 + \|u_x\|^2. \tag{3.4}$$

Direct computations yield

$$\begin{aligned}
\theta'(t) &= 2\left(\left(-\partial_x^2\right)^{-1/2}u_t,\left(-\partial_x^2\right)^{-1/2}u\right)+2(u_{xt},u_x),\\
\theta''(t) &= 2\left\|\left(-\partial_x^2\right)^{-1/2}u_t\right\|^2+2\left(\left(-\partial_x^2\right)^{-1/2}u_{tt},\left(-\partial_x^2\right)^{-1/2}u\right)+2(u_{xtt},u_x)+2\|u_{xt}(t)\|^2\\
&= 2\left\|\left(-\partial_x^2\right)^{-1/2}u_t\right\|^2+2\|u_{xt}(t)\|^2+2\left(\left(-\partial_x^2\right)^{-1}u_{tt},u\right)-2(u_{xxtt},u)\\
&= 2\left\|\left(-\partial_x^2\right)^{-1/2}u_t\right\|^2+2\|u_{xt}(t)\|^2-2I_0(u)\\
&= -2H(u,t).
\end{aligned} \tag{3.5}$$

For contradiction, assume that there exists some $t'>0$ such that $H(u,t')=0$. Since $\theta''(t)<0$, we conclude that $\theta'(t)$ is strictly decreasing on $[0,t')$. Moreover, (3.3) implies $\theta'(0)<0$ and therefore $\theta'(t)<0$ in $[0,t']$, from which follows that $\theta(t)$ is strictly decreasing on $[0,t']$. By the energy identity and $H(u,t')=0$, we have

$$\begin{aligned}
E(0) &= \frac{1}{2}\left(\left\|\left(-\partial_x^2\right)^{-1/2}u_t(t')\right\|^2+\|u_{xt}(t')\|^2\right)+\frac{p-1}{2(p+1)}\|u(t')\|_{H^1}^2+\frac{1}{p+1}I(u(t'))\\
&= \left(\frac{1}{2}+\frac{1}{p+1}\right)\left(\left\|\left(-\partial_x^2\right)^{-1/2}u_t(t')\right\|^2+\|u_{xt}(t')\|^2\right)+\frac{p-1}{2(p+1)}\|u(t')\|_{H^1}^2.
\end{aligned} \tag{3.6}$$

Theorem 2.10, Corollary 2.11, and $H(u,t')=0$ yield

$$\|u\|_{H^1}^2 \geq \sigma_m^{-1}I_0(u(t')) \geq \sigma^{-1}\left(\left\|\left(-\partial_x^2\right)^{-1/2}u_t(t')\right\|^2+\|u_{xt}(t')\|^2\right). \tag{3.7}$$

The use of the above inequality in (3.6) shows that

$$E(0) \geq \left(\frac{1}{2}+\frac{1}{p+1}+\frac{p-1}{2(p+1)\sigma}\right)\left(\left\|\left(-\partial_x^2\right)^{-1/2}u_t(t')\right\|^2+\|u_{xt}(t')\|^2\right). \tag{3.8}$$

This can be rephrased in terms of $\theta(t)$ and $\theta'(t)$ as

$$\begin{aligned}
E(0) \geq \frac{(p+3)\sigma+p-1}{2(p+1)\sigma}\Bigg[&\left\|\left(-\partial_x^2\right)^{-1/2}(u_t(t')+u(t'))\right\|^2+\|u_{xt}(t')+u(t')\|^2\\
&-2\left(\left(-\partial_x^2\right)^{-1/2}u_t(t'),\left(-\partial_x^2\right)^{-1/2}u(t')\right)-2(u_{xt}(t'),u_x(t'))\\
&-\left\|\left(-\partial_x^2\right)^{-\frac{1}{2}}u(t')\right\|^2-\|u_x(t')\|^2\Bigg].
\end{aligned} \tag{3.9}$$

From the monotonicity of $\theta(t)$ and $\theta'(t)$, we get

$$E(0) > \frac{(p+3)\sigma+p-1}{(p+1)\sigma}\left[-\left(\left(-\partial_x^2\right)^{-1/2}\psi,\left(-\partial_x^2\right)^{-1/2}\varphi\right)-(\psi_x,\varphi_x)\right.$$
$$\left.-\frac{1}{2}\left\|\left(-\partial_x^2\right)^{-1/2}\varphi\right\|^2-\frac{1}{2}\|\varphi_x\|^2\right], \tag{3.10}$$

which contradicts (3.3). Thus, the theorem is proved. □

Theorem 3.2. *Assume that* $\varphi,\psi\in H^1(R)$, $(-\partial_x^2)^{-1/2}\varphi$, $(-\partial_x^2)^{-1/2}\psi\in L^2(R)$. *Suppose that* $E(0)>0$, $H(u,0)>0$ *and* (3.3) *holds for some* $\sigma>\sigma_m$. *Then, the weak solution of* (1.1) *and* (1.2) *is globally defined for every* $t\in[0,\infty)$.

Proof. The proof of this theorem follows from adding some arguments to the local existence result of Corollary 2.10 of [10]. If $(-\partial_x^2)^{-1/2}\psi\in L^2$, then $(-\partial_x^2)^{-1/2}u_t\in L^2$ (Lemma 2.8 of [10]). $H(u,0)>0$ implies from the sign preserving property of $H(u,t)$ that $H(u,t)>0$, thereby $I_0(u)>0$ for every $t>0$. From the energy identity we have

$$E(0)=\frac{1}{2}\left(\left\|\left(-\partial_x^2\right)^{-1/2}u_t\right\|^2+\|u_{xt}\|^2\right)+\frac{p-1}{2(p+1)}\|u\|_{H^1}^2+\frac{1}{p+1}I(u)$$
$$\geq\frac{1}{2}\left\|\left(-\partial_x^2\right)^{-1/2}u_t\right\|^2+\frac{1}{2}\|u_{xt}\|^2+\frac{p-1}{2(p+1)}\|u\|_{H^1}^2. \tag{3.11}$$

Therefore $\|u\|_{H^1}$ and $\|u_t\|_{H^1}$ are bounded for every $t>0$. The previously mentioned local existence theory completes the proof. □

4. Final Remarks

Remark 4.1. In a section of the paper of Y. Z. Wang and Y. X. Wang [11], problem (1.1) and (1.2) was studied with the supercritical initial energy. A global existence result was obtained under the assumption $F(u)\geq 0$ or $f'(u)$ is bounded below, that is, $f'(u)\geq A_0$. As the authors have mentioned in an example, this condition is valid only for odd-degree nonlinearities, namely for $f(u)=\beta u^{2p+1}, u\in R, \beta>0, p$ is a nonnegative integer. For $f(u)=\beta|u|^p$, this condition cannot guarantee the global existence. For example, if we take $f(u)=(3/2)u^2$, then the antiderivative and derivative of f are $F(u)=(1/2)u^3$ and $f'(u)=3u$, respectively. For $f(u)=\beta|u|^p$, we have $F(u)=(\beta/(p+1))|u|^p u$ and $f'(u)=ap|u|^{p-2}u$, which do not always satisfy the positivity condition. To remedy this, we generate a new functional for potential well method, which also contains the initial velocity different from the previous ones, and use the invariance of this functional with respect to problem (1.1) and (1.2).

Remark 4.2. In Section 2, we introduce the first new functional $I_\sigma(u)$, which is more general than the ones introduced before in some papers for the fourth-order Boussinesq equation. However, in the case of $E(0)>0$ we are not able to prove the sign invariance of $I_\sigma(u)$, because we see that for $E(0)>0, I_\sigma(u)$ is always nonpositive. Eventually, a satisfactory result comes from $H(u,t)$ which includes not only the initial displacement φ, but also the initial velocity ψ.

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