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Review Article

Generalized Multiparameters Fractional Variational Calculus

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This paper builds upon our recent paper on generalized fractional variational calculus (FVC). Here, we briefly review some of the fractional derivatives (FDs) that we considered in the past to develop FVC. We first introduce new one parameter generalized fractional derivatives (GFDs) which depend on two functions, and show that many of the one-parameter FDs considered in the past are special cases of the proposed GFDs. We develop several parts of FVC in terms of one parameter GFDs. We point out how many other parts could be developed using the properties of the one-parameter GFDs. Subsequently, we introduce two new two- and three-parameter GFDs. We introduce some of their properties, and discuss how they can be used to develop FVC. In addition, we indicate how these formulations could be used in various fields, and how the generalizations presented here can be further extended.

1. Introduction

For over a century, many researchers have been in search for a fundamental law that can be used to describe the behavior of the nature. One law that comes very close to it is the universal law of extremum which states that the nature always behaves in a way such that some quantity is an extremum. A catenary takes a shape so that the total potential energy is minimum, light travels from a point to another so that the travel time is minimum, a particle in a flow takes a path of least resistance, and even in social settings, we behave so that our conflict with others within our conviction is minimum. Related laws, principles, and theories have been developed in almost every field of science, engineering, mathematics, biology, economics and social science. For example, applications of such laws, principles, and theories in continuum mechanics, classical and quantum mechanics, relativistic quantum mechanics, and electromagnetics could be found in [1–5] and many other textbooks, monographs, and papers. Opponent of the universal principle may argue that nature behaves in its own way, and the extremum principles are our creations where we design a functional that is extremum for the nature's trajectory. Whatever may be the reality, the underlying theories have advanced our understanding of the nature tremendously.

The field that deals with the mathematical theories of the extremum principles is known as the variational calculus. Excellent books have been written in this field, see for example [6, 7]. These books provide not only the foundations for theoretical work in the field, but they have also been a basis for many numerical techniques (see, [8, 9]). However, the traditional variational calculus subject has one major drawback; it deals with functionals containing integer-order derivative terms only. Recent progress in last two decades have demonstrated that many phenomena in various fields of science, mathematics, engineering, bioengineering, and economics are more accurately described using fractional derivatives. As a result, many books, monographs, and papers have been written recently on this subject (see, e.g. [10–21]). We assume that many fractional models would follow the universal law of extremum. If this is true, then it is very likely that a variational calculus that deals with fractional derivatives would be necessary. In other words, we need fractional variational calculus.

The subject of fractional variational calculus was initiated by Riewe [22, 23] in 1996. Riewe was interested in developing a variational formulation for a linear damper. He observed that a quadratic term of type $(Dy(x))^2$ in a functional leads to a second-order derivative term of type D^2y in the resulting differential equation. Here D is the derivative operator. Therefore, he argued that a first-order derivative term of type Dy in a differential equation will come from a quadratic term of type $(D^{1/2}y)^2$ in the functional. Here $D^{1/2}$ is a half-order derivative operator (Precise definitions of fractional derivative would be discussed later in the paper.) Using this hypothesis, he proceeded to develop a variational formulation in terms of fractional derivatives. Subsequently, he developed fractional Lagrangian, fractional Hamiltonion, and fractional mechanics.

Klimek [24, 25] and Agrawal [26] brought this subject to the main stream and initiated the field of fractional variational calculus. These authors identified the key integration-by-parts formulas for fractional derivatives, and showed that using these formulas a fractional variational formulation can be obtained in the same way as it is done for integer variational formulation. Note that variational calculus has been applied to an extensively large number of problems, theories, and formulations most of which could be reexamined in the light of fractional variational calculus. Thus, the above work has opened significant opportunities for many new research.

Recently, the field of fractional variational calculus has indeed grown very rapidly. A citation search of [26] and some related papers suggests that in the last 10 years over 300 papers have been published which are directly related to fractional variational calculus; here we cite a few of them [27–36]. These papers further (1) develop fractional variational calculus and Fractional Euler-Lagrange Equations in terms of fractional derivatives not considered earlier, (2) derive trasnversality conditions for fractional problems, (3) propose new fractional Lagrangians and fractional Hamiltonians and fractional mechanics, and (4) develop applications of fractional derivatives in optimal control and fractional inverse problems. Recently, Klimek has written a book dedicated to fractional variational calculus and analytical techniques to solve problems resulting from fractional variational formulations [37]. Reference [38] presents a two- and a three-parameter generalizations of fractional variational calculus. It is demonstrated that by setting these parameters to different values we obtain several different fractional variational formulations presented previously. These citations are clear indications that significant progress has been made in the area of fractional variational calculus. However, compared to the progress that has already been made in ordinary variational calculus, this progress in fractional variational calculus is very small.

In this paper, we provide further generalization of fractional variational calculus. Specifically we introduce Hadamard type, Erdélyi-Kober type fractional integral and fractional derivatives, and fractional integrals and derivatives of a function (say $f(x)$) with respect to another function (say $\phi(x)$). We also introduce some new fractional integrals and derivatives Caputo, Hilfer, and Riesz types. This leads to two-, three-, and four-parameter generalized fractional derivatives of a function with respect to another function. We develop integration-by-parts formulas and Euler-Lagrange equations in terms of these new two-, three- and four-parameters fractional derivatives. It is demonstrated that by taking different values for different parameters and different function $\phi(x)$, we obtain many old and many new fractional derivatives and fractional variational formulations. We also introduce fractional Lagrangians and Hamiltonians in terms of these derivatives, and develop a more general fractional mechanics. Since, function could be selected from a large set, it provides a large number of fractional variational formulations for modeling purpose. Finally, we discuss how this work can be extended further.

At this point, we would like to emphasize that a comprehensive treatment and an excellent review of many generalized fractional operators proposed in the field could be found in [13, 39] (In this regards, please also see [19] and many references cited in [13, 39]). Generalizations of integral operators with specific weights could be found in [12]. In contrast to these references, some of the fractional operators proposed here are more general. In addition, our focus here is to develop some of the theories for generalized fractional variational calculus in terms of these operators, and provide an outline for other formulations in the field. We consider here functions dependent on one parameter. However, the theories developed here could easily be extended to field variable and distributed order systems ([21]).

2. Preliminaries

In this paper, we will introduce several general multiparameter fractional integrals and derivatives, and show that many specific integrals and derivatives can be obtained from these general derivatives. For ease in the discussion to follow and to make this paper self-contained, we first introduce several symbols and notations, and provide some preliminaries. A large part of these symbols and definitions could be found in [12, 19, 38]. We shall denote the order of the fractional integrals and derivatives as α. In [12, 19], in general, α is taken as a complex number, and restrictions are imposed on it as necessary. Same approach can be taken here. However, in our discussion to follow, we shall implicitly consider that α is a positive real number. We shall assume that a and b are real such that $a < b$, and consider the domain of the functions and operators as $[a, b]$, although in some cases, the domain of the functions and operators may not contain some isolated points of $[a, b]$. Further, we shall consider $[a, b]$ to be a finite domain. But, in many cases, a could be $-\infty$ and b could be ∞, and some special cases could be derived by setting $a = 0$. However, these would be left as an exercise. Exception to these would be noted as necessary. We shall further assume that our functions are "sufficiently good" so that the operations considered on them are valid.

In this paper, we consider several fractional integrals and derivatives. We begin with the Riemann-Liouville and the Caputo fractional integrals and derivatives.

2.1. Riemann-Liouville and Caputo Fractional Integrals and Derivatives

Many fractional derivatives are defined using the left/forward and the right/backward Riemann-Liouville fractional integrals (RLFIs). These integrals are defined as follows.

Left/Forward Riemann-Liouville fractional integral of order α

$$\left(I_{a+}^{\alpha}y\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}y(\tau)d\tau,\quad (t>a). \tag{2.1}$$

Right/Backward Riemann-Liouville fractional integral of order α

$$\left(I_{b-}^{\alpha}y\right)(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(\tau-t)^{\alpha-1}y(\tau)d\tau,\quad (t<b), \tag{2.2}$$

where $\Gamma(*)$ is the Gamma function, and α is the order of integration. Here $\alpha>0$, however, we shall restrict our attention to $(0<\alpha<1)$. The fractional integral operators I_{a+}^{α} and I_{b-}^{α} satisfy the semigroup property, namely,

$$I_{a+}^{\alpha}I_{a+}^{\beta}y=I_{a+}^{\alpha+\beta}y=I_{a+}^{\beta}I_{a+}^{\alpha}y,\quad I_{b-}^{\alpha}I_{b-}^{\beta}y=I_{b-}^{\alpha+\beta}y=I_{b-}^{\beta}I_{b-}^{\alpha}y,\quad \alpha>0,\ \beta>0. \tag{2.3}$$

We now consider the *reflection operator* Q which is defined as

$$(Qx)(t)=x(a+b-t). \tag{2.4}$$

Operator Q satisfies the following identities,

$$Q\left(I_{a+}^{\alpha}\right)=\left(I_{b-}^{\alpha}\right)Q,\qquad Q\left(I_{b-}^{\alpha}\right)=\left(I_{a+}^{\alpha}\right)Q. \tag{2.5}$$

An advantage of operator Q is that one needs to examine the properties of I_{a+}^{α} only, and obtain the properties of I_{b-}^{α} by using the properties of I_{a+}^{α} and Q.

Using (2.1) and (2.2), the left/forward and the right/backward Riemann-Liouville fractional derivatives (RLFDs) of order α, $(n-1<\alpha<n)$, n an integer, are defined as follows.

Left/Forward Riemann-Liouville fractional derivative of order α

$$\left(D_{a+}^{\alpha}y\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t(t-\tau)^{n-\alpha-1}y(\tau)d\tau=D^n\left(I_{a+}^{n-\alpha}y\right)(t) \tag{2.6}$$

Right/Backward Riemann-Liouville fractional derivative of order α

$$\left(D_{b-}^{\alpha}y\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n\int_t^b(\tau-t)^{n-\alpha-1}y(\tau)d\tau=(-D)^n\left(I_{b-}^{n-\alpha}y\right)(t). \tag{2.7}$$

where $D = d/dt$ represents the ordinary differential operator. Operators D^{α}_{a+} and D^{α}_{b-} are the left inverse of operators I^{α}_{a+} and I^{α}_{b-}, that is, they satisfy the following identities

$$D^{\alpha}_{a+} I^{\alpha}_{a+} = E = D^{\alpha}_{b-} I^{\alpha}_{b-}, \tag{2.8}$$

where E is the identity operator. However, in general

$$I^{\alpha}_{a+} D^{\alpha}_{a+} \neq E \neq I^{\alpha}_{b-} D^{\alpha}_{b-}. \tag{2.9}$$

The left/forward and the right/backward Caputo fractional derivatives (CFDs) of order $\alpha (n-1 < \alpha < n)$ are defined as

Left/Forward Caputo fractional derivative of order α

$$\left({}^{C}D^{\alpha}_{a+} y\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n y(\tau) d\tau = \left(I^{n-\alpha}_{a+} D^n y\right)(t). \tag{2.10}$$

Right/Backward Caputo fractional derivative of order α

$$\left({}^{C}D^{\alpha}_{b-} y\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{d}{dt}\right)^n y(\tau) d\tau = \left(I^{n-\alpha}_{b-} (-D)^n y\right)(t). \tag{2.11}$$

Note that the ordinary derivative operators are applied in the RLFDs after the fractional integrals whereas in the CFDs before the fractional integrals. Therefore, the differentiability required of $y(t)$ by the CFDs is higher than those by the RLFDs.

Operators D^{α}_{a+}, D^{α}_{b-}, ${}^{C}D^{\alpha}_{a+}$, ${}^{C}D^{\alpha}_{b-}$, and Q satisfy the following identities:

$$\begin{aligned} Q D^{\alpha}_{a+} &= D^{\alpha}_{b-} Q, \qquad Q D^{\alpha}_{b-} = D^{\alpha}_{a+} Q, \\ Q\,{}^{C}D^{\alpha}_{a+} &= {}^{C}D^{\alpha}_{b-} Q, \qquad Q\,{}^{C}D^{\alpha}_{b-} = {}^{C}D^{\alpha}_{a+} Q. \end{aligned} \tag{2.12}$$

The RLFDs and the CFDs are related by the following formulas [12, 19]:

$$\left(D^{\alpha}_{a+} y\right)(t) = \left({}^{C}D^{\alpha}_{a+} y\right)(t) + \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \tag{2.13}$$

$$\left(D^{\alpha}_{b-} y\right)(t) = \left({}^{C}D^{\alpha}_{b-} y\right)(t) + \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \tag{2.14}$$

where $y^{(k)}$ represents the kth derivative of $y(t)$ with respect to t. Equation (2.13) can be obtained using the identity

$$y(t) = I^{n}_{a+} D^n y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t-a)^k \tag{2.15}$$

and the semigroup property of the integral operators, and (2.14) can be obtained using the property of the Q operator.

The integral operators (I_{a+}^{α}) and (I_{b-}^{α}) and the derivative operators (D_{a+}^{α}), and (D_{b-}^{α}) satisfy the following fractional integration by parts formulas [12]

$$\int_a^b f(t)(I_{a+}^{\alpha})g(t)dt = \int_a^b g(t)(I_{b-}^{\alpha})f(t)dt. \tag{2.16}$$

Similarly, operators (D_{a+}^{α}) and (D_{b-}^{α}) satisfy and

$$\int_a^b f(t)(D_{a+}^{\alpha}g)(t)dt = \int_a^b g(t)(D_{b-}^{\alpha}f)(t)dt. \tag{2.17}$$

The conditions under which (2.16) and (2.17) are valid can be found in [12]. It will be implicitly assumed that these conditions are satisfied. In a more general setting, operators (D_{a+}^{α}), (D_{b-}^{α}), $({}^C D_{a+}^{\alpha})$, and $({}^C D_{b-}^{\alpha})$ satisfy the following fractional integration by parts formula,

$$\begin{aligned}\int_a^b f(t)(D_{a+}^{\alpha}g)(t)dt &= \int_a^b g(t)\left({}^C D_{b-}^{\alpha}f\right)(t)dt \\ &\quad + \sum_{j=0}^{n-1}(-D)^j f(t)\left(D_{a+}^{\alpha-1-j}\right)g(t)\Bigg|_a^b,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\int_a^b f(t)(D_{b-}^{\alpha}g)(t)dt &= \int_a^b g(t)\left({}^C D_{a+}^{\alpha}\right)f(t)dt \\ &\quad - \sum_{j=0}^{n-1}D^j f(t)\left(D_{b-}^{\alpha-1-j}g\right)(t)\Bigg|_a^b.\end{aligned} \tag{2.19}$$

Here, $D_{a+}^{-\alpha}$ and $D_{b-}^{-\alpha}$ must be interpreted as I_{a+}^{α} and I_{b-}^{α}, respectively. Equations (2.18) and (2.19) can be obtained by considering the Riemann-Liouville and the Caputo fractional derivatives in terms of fractional integrals and ordinary derivatives and (2.16). Conditions under which (2.18) and (2.19) are true can be found in [37]. Using the properties of functions $f(t)$ and $g(t)$, these equalities can be further specialized. Equations (2.17) to (2.19) have played key roles in developing fractional variational calculus.

2.2. Hadamard Fractional Integrals and Derivatives

The left/forward and the right/backward Hadamard fractional integrals (HFIs) of order α are defined as follows [12, 19].

Left/Forward Hadamard fractional integral of order α

$$\left({}^H I_{a+}^{\alpha}y\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{y(\tau)d\tau}{\tau}, \quad (a < t < b). \tag{2.20}$$

Right/Backward Hadamard fractional integral of order α

$$\left({}^{H}I_{b-}^{\alpha}y\right)(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{b}\left(\log\frac{\tau}{t}\right)^{\alpha-1}y(\tau)d\tau, \quad (a < t < b). \tag{2.21}$$

Like the RLFIs, the HFIs also satisfy the semigroup properties, that is,

$${}^{H}I_{a+}^{\alpha}\,{}^{H}I_{a+}^{\beta} = {}^{H}I_{a+}^{\alpha+\beta} = {}^{H}I_{a+}^{\beta}\,{}^{H}I_{a+}^{\alpha}, \qquad {}^{H}I_{b-}^{\alpha} = {}^{H}I_{b-}^{\alpha+\beta} = {}^{H}I_{b-}^{\beta}\,{}^{H}I_{b-}^{\alpha}. \tag{2.22}$$

The left/forward and the right/backward Hadamard fractional derivatives (HFDs) of order α are defined as follows [12, 19].

Left/Forward Hadamard fractional derivative of order α

$$\begin{aligned}\left({}^{H}D_{a+}^{\alpha}y\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_{a}^{t}\left(\log\frac{t}{\tau}\right)^{n-\alpha+1}\frac{y(\tau)d\tau}{\tau}\\ &= \left(t\frac{d}{dt}\right)^{n}\left({}^{H}I_{a+}^{\alpha}y\right)(t), \quad (a < t < b),\end{aligned} \tag{2.23}$$

Right/Backward Hadamard fractional derivative of order α

$$\begin{aligned}\left({}^{H}D_{b-}^{\alpha}y\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(-t\frac{d}{dt}\right)^{n}\int_{t}^{b}\left(\log\frac{\tau}{t}\right)^{n-\alpha+1}y(\tau)d\tau\\ &= \left(-t\frac{d}{dt}\right)^{n}\left({}^{H}I_{b-}^{\alpha}y\right)(t) \quad (a < t < b),\end{aligned} \tag{2.24}$$

where, as before, $n-1 < \alpha < n$. The left and the right Hadamard fractional derivative operators (HFDOs) ${}^{H}D_{a+}^{\alpha}$ and ${}^{H}D_{b-}^{\alpha}$ are the left inverse of the left and the right Hadamard Fractional Integral Operator (HFIOs) ${}^{H}I_{a+}^{\alpha}$ and ${}^{H}I_{b-}^{\alpha}$, respectively, that is, they satisfy the following identities

$${}^{H}D_{a+}^{\alpha}\,{}^{H}I_{a+}^{\alpha} = E = {}^{H}D_{b-}^{\alpha}\,{}^{H}I_{b-}^{\alpha}. \tag{2.25}$$

Note that the left and the right HFDOs are obtained by applying $(t(d/dt))^{n}$ and $(-t(d/dt))^{n}$ to the left- and the right HFIOs, respectively. We shall define another set of fractional derivatives by changing the order of the derivative and the integral operators in (2.23) and (2.24). Due to their similarity with Caputo derivatives, we call them Hadamard-Caputo derivatives. Thus, the left/forward and the right/backward Hadamard-Caputo fractional derivatives (HCFDs) of order α are defined as follows.

Left/Forward Hadamard-Caputo fractional derivative of order α

$$\begin{aligned}\left({}^{\mathrm{HC}}D_{a+}^{\alpha}y\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\left(\log\frac{t}{\tau}\right)^{n-\alpha+1}\left(\tau\frac{d}{d\tau}\right)^{n}y(\tau)\frac{d\tau}{\tau}\\ &= {}^{H}I_{a+}^{\alpha}\left(\left(t\frac{d}{dt}\right)^{n}y\right)(t) \quad (a < t < b),\end{aligned} \tag{2.26}$$

Right/Backward Hadamard-Caputo fractional derivative of order α

$$\begin{aligned}\left({}^{\mathrm{HC}}D_{b-}^{\alpha}y\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\int_t^b \left(\log\frac{\tau}{t}\right)^{n-\alpha+1}\left(-\tau\frac{d}{d\tau}\right)^n y(\tau)d\tau \\ &= {}^{H}I_{b-}^{\alpha}\left(\left(-t\frac{d}{dt}\right)^n y\right)(t), \quad (a < t < b).\end{aligned} \tag{2.27}$$

The HFDOs ${}^{H}D_{a+}^{\alpha}$ and ${}^{H}D_{b-}^{\alpha}$ are related to the HCFDOs ${}^{\mathrm{HC}}D_{a+}^{\alpha}$ and ${}^{\mathrm{HC}}D_{b-}^{\alpha}$ by the following relations

$$\begin{aligned}\left({}^{H}D_{a+}^{\alpha}y\right)(t) &= \left({}^{\mathrm{HC}}D_{a+}^{\alpha}y\right)(t) + \sum_{k=0}^{n-1}\frac{(tD)^{(k)}y(a)}{\Gamma(k-\alpha+1)}\left(\log\frac{t}{a}\right)^{k-\alpha}, \\ \left({}^{H}D_{b-}^{\alpha}y\right)(t) &= \left({}^{\mathrm{HC}}D_{b-}^{\alpha}y\right)(t) + \sum_{k=0}^{n-1}\frac{(-tD)^{(k)}y(b)}{\Gamma(k-\alpha+1)}\left(\log\frac{b}{t}\right)^{k-\alpha},\end{aligned} \tag{2.28}$$

where $D = d/dt$ is the time derivative operator. The proofs of (2.28) are the same as those for (2.13) and (2.14).

We can develop integration by parts formula for Hadamard operators also. It can be demonstrated that the HFIOs ${}^{H}I_{a+}^{\alpha}$ and ${}^{H}I_{b-}^{\alpha}$ satisfy the following integration by parts formula:

$$\int_a^b \frac{1}{t}f(t)\left({}^{H}I_{a+}^{\alpha}g\right)(t)dt = \int_a^b \frac{1}{t}g(t)\left({}^{H}I_{b-}^{\alpha}f\right)(t)dt. \tag{2.29}$$

One can prove (2.29) using the definitions of the HFIOs and the Dirichlet formula.

Similar to (2.17) to (2.19), the HFDOs and the HCFDOs satisfy the following identity

$$\int_a^b \frac{1}{t}f(t)\left({}^{H}D_{a+}^{\alpha}g\right)(t)dt = \int_a^b \frac{1}{t}g(t)\left({}^{H}D_{b-}^{\alpha}f\right)(t)dt, \tag{2.30}$$

and in a more general setting, the HFDOs and the HCFDOs satisfy the following fractional integration by parts formula

$$\begin{aligned}\int_a^b \frac{1}{t}f(t)\left({}^{H}D_{a+}^{\alpha}g\right)(t)dt &= \int_a^b \frac{1}{t}g(t)\left({}^{\mathrm{HC}}D_{b-}^{\alpha}f\right)(t)dt \\ &\quad + \sum_{j=0}^{n-1}(-tD)^j f(t)\left({}^{H}D_{a+}^{\alpha-1-j}\right)g(t)\Bigg|_a^b,\end{aligned} \tag{2.31}$$

$$\int_a^b \frac{1}{t} f(t)\left({}^{H}D^{\alpha}_{b-}g\right)(t)dt = \int_a^b \frac{1}{t} g(t)\left({}^{\mathrm{HC}}D^{\alpha}_{a+}\right)f(t)dt - \sum_{j=0}^{n-1}(tD)^j f(t)\left({}^{H}D^{\alpha-1-j}_{b-}g\right)(t)\Bigg|_a^b. \tag{2.32}$$

As stated earlier, one can take $a = 0$ or $-\infty$ and $b = \infty$, and include some weight functions in the integral to obtain some other types of HFIs and HFDs. However, these will be considered later.

Equations (2.20) to (2.32) provide sufficient number of formulas to develop Euler-Lagrange formulations in terms of Hadamard fractional derivatives. For the time being, we proceed to define the Erdélyi-Kober type fractional integrals and derivatives, and develop some of their properties pertinent to fractional variational calculus.

2.3. Erdélyi-Kober Fractional Integrals and Derivatives

The left/forward and right/backward Erdélyi-Kober fractional integrals (EKFIs) are defined as [12, 19] as follows.

Left/Forward Erdélyi-Kober fractional integral of order α

$$\left({}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\sigma\eta+\sigma-1}y(\tau)d\tau}{\left(t^{\sigma}-\tau^{\sigma}\right)^{1-\alpha}}, \quad (a < t < b) \tag{2.33}$$

and *Right/Backward Erdélyi-Kober fractional integral of order α*

$$\left({}^{\mathrm{EK}}I^{\alpha}_{b-;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{\sigma\eta}}{\Gamma(\alpha)}\int_t^b \frac{\tau^{\sigma(1-\alpha-\eta)-1}y(\tau)d\tau}{\left(\tau^{\sigma}-t^{\sigma}\right)^{1-\alpha}}, \quad (a < t < b). \tag{2.34}$$

As pointed out earlier, by setting a to $-\infty$ or 0 and b to ∞, one can obtain several other types of EKFIs. Indeed, many of such integrals are defined and discussed in [12, 19]. Many of the formulations discussed here can directly be applied to these other EKFIs.

These integrals satisfy the following semigroup properties:

$${}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]}\,{}^{\mathrm{EK}}I^{\beta}_{a+;[\sigma,\eta+\alpha]} = {}^{\mathrm{EK}}I^{\alpha+\beta}_{a+;[\sigma,\eta]}, \qquad {}^{\mathrm{EK}}I^{\alpha}_{b-;[\sigma,\eta]}\,{}^{\mathrm{EK}}I^{\beta}_{b-;[\sigma,\eta+\alpha]} = {}^{\mathrm{EK}}I^{\alpha+\beta}_{b-;[\sigma,\eta]}. \tag{2.35}$$

The Erdélyi-Kober fractional derivatives (EKFDs) of order α corresponding to the EKFIs are defined as.

Left/Forward Erdélyi-Kober fractional derivative of order α

$$\left({}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) = t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}D_t\right)^m t^{\sigma(m+\eta)}\left({}^{\mathrm{EK}}I^{m-\alpha}_{a+;[\sigma,\eta+\alpha]}y\right)(t) \tag{2.36}$$

and *Right/Backward Erdélyi-Kober fractional derivative of order* α

$$\left({}^{\mathrm{EK}}D^{\alpha}_{b-;[\sigma,\eta]}y\right)(t) = t^{\sigma(\alpha+\eta)}\left(\frac{-1}{\sigma t^{\sigma-1}}D_t\right)^{m} t^{\sigma(m-\eta-\alpha)}\left({}^{\mathrm{EK}}I^{m-\alpha}_{b-;[\sigma,\eta+\alpha-m]}y\right)(t). \tag{2.37}$$

For sufficiently good function $y(t)$, the Erdélyi-Kober fractional differential operators (EKFDOs) ${}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}$ and ${}^{\mathrm{EK}}D^{\alpha}_{b-;[\sigma,\eta]}$ are left inverse of the Erdélyi-Kober fractional integral operators (EKFIOs) ${}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]}$ and ${}^{\mathrm{EK}}I^{\alpha}_{b-;[\sigma,\eta]}$, respectively, that is, they satisfy the following identities,

$${}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}\,{}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]} = E = {}^{\mathrm{EK}}D^{\alpha}_{b-;[\sigma,\eta]}\,{}^{\mathrm{EK}}I^{\alpha}_{b-;[\sigma,\eta]}. \tag{2.38}$$

The EKFIs and the EKFDs satisfy the following integration by parts formulas:

$$\int_a^b t^{(\sigma-1)}f(t)\left({}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]}g\right)(t)dt = \int_a^b \sigma t^{(\sigma-1)}g(t)\left({}^{\mathrm{EK}}I^{\alpha}_{b-;[\sigma,\eta]}f\right)(t)dt, \tag{2.39}$$

$$\int_a^b t^{(\sigma-1)}f(t)\left({}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}g\right)(t)dt = \int_a^b \sigma t^{(\sigma-1)}g(t)\left({}^{\mathrm{EK}}D^{\alpha}_{b-;[\sigma,\eta]}f\right)(t)dt. \tag{2.40}$$

Equation (2.39) could be found in [12, 19], and (2.40) could be derived using the definitions of EKFIs and EKFDs and the Dirichlet formula.

Equations (2.33) to (2.40) are sufficient to develop some fractional variational formulations in terms of EKFIs and EKFDs. Note that in the EKFDs, the EKFIOs are applied first and some derivative operators are applied next. One can also take some derivative operators first and the EKFIOs next to define some Caputo type EKFIs and EKFDs. Indeed, one can show that the following relation is valid

$$\begin{aligned}\left({}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}f\right)(t) &= \left({}^{\mathrm{EKC}}D^{\alpha}_{a+;[\sigma,\eta]}f\right)(t) + \sum_{k=0}^{m-1}\frac{(t^{\sigma}-a^{\sigma})^{k-\alpha}}{\Gamma(m-\alpha)}\\ &\quad * \left[\left(\frac{1}{\sigma x^{\sigma-1}}D_x\right)^{k}\left(x^{\sigma(\eta+\alpha)}f(x)\right)\right]\Bigg|_{x=a},\end{aligned} \tag{2.41}$$

where

$$\left({}^{\mathrm{EKC}}D^{\alpha}_{a+;[\sigma,\eta]}f\right)(t) = t^{-\sigma\eta}\left[\frac{1}{\Gamma(m-\alpha)}\int_t^b \frac{\sigma\tau^{\sigma-1}d\tau}{(\tau^{\sigma}-t^{\sigma})^{1-m+\alpha}}\left(\frac{1}{\sigma\tau^{\sigma-1}}D_{\tau}\right)^{m}\left(\tau^{\sigma(\eta+\alpha)}f(\tau)\right)\right] \tag{2.42}$$

is the left/forward Erdélyi-Kober-Caputo type *fractional derivative of order* α. Following above discussion, a right/backward Erdélyi-Kober-Caputo type Fractional Derivative (EKC-FD) of order α and related integration by parts formula could be developed. However, note that Erdélyi-Kober operators may lead to some nonstandard cases. For example, in (2.35), one cannot simply interchange the first two operator, as it can be done in (2.3). Therefore, the EKC-FDs and related identities and their generalizations will be considered in the future. Note that references [12, 19] also introduce operators M_σ and N_σ defined as $(M_\sigma\phi)(x) = x^\sigma\phi(x)$ and $(N_\sigma\phi)(x) = \phi(x^\sigma)$. These operators link the Erdélyi-Kober operators to the Riemann-Liouville operators, which could simplify some of the formulations. Properties of these formulations in the context of fractional variational calculus will be examined in the future.

2.4. Modified Erdélyi-Kober Fractional Integrals and Derivatives

It was pointed out above that the Erdélyi-Kober operators defined by (2.33), (2.34), (2.36), and (2.37) lead to some nonstandard cases. For example, in (2.35) the first two operators cannot be interchanged. Furthermore, in order for the relations in (2.35) to be valid, some of the parameters of the operators must be related (e.g., notice the presence of α in the first two operators in (2.35)). To overcome this difficulty, we define modified Erdélyi-Kober fractional integrals and derivatives in the following way.

Left/Forward modified Erdélyi-Kober fractional integral (MEKFI) of order α

$$\left({}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{-\sigma(\eta)}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\sigma\eta+\sigma-1}y(\tau)d\tau}{(t^\sigma-\tau^\sigma)^{1-\alpha}}, \quad (a<t<b) \tag{2.43}$$

and *right/backward modified Erdélyi-Kober fractional integral (MEKFI) of order* α

$$\left({}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{\sigma\eta}}{\Gamma(\alpha)}\int_t^b \frac{\tau^{\sigma(1-\eta)-1}y(\tau)d\tau}{(\tau^\sigma-t^\sigma)^{1-\alpha}}, \quad (a<t<b). \tag{2.44}$$

It should be pointed out that like in the case of EKFIs, one can obtain several other types of MEKFIs by setting a to $-\infty$ or 0 and b to ∞. Such substitutions may lead to specialized cases, and may result in simplified formulations. These cases will be considered in the future. In any case, the formulation here will also be applicable to these specialized MEKFIs.

These integrals satisfy the following semigroup properties

$$\begin{aligned}
{}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\beta}_{a+;[\sigma,\eta]} &= {}^{\mathrm{MEK}}I^{\alpha+\beta}_{a+;[\sigma,\eta]} = {}^{\mathrm{MEK}}I^{\beta}_{a+;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]},\\
{}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\beta}_{b-;[\sigma,\eta]} &= {}^{\mathrm{MEK}}I^{\alpha+\beta}_{b-;[\sigma,\eta]} = {}^{\mathrm{MEK}}I^{\beta}_{b-;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}.
\end{aligned} \tag{2.45}$$

The proof of these identities follows the same steps as those for the proof of (2.35). Note that in the modified definition, the operators ${}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]}$ $({}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]})$ and ${}^{\mathrm{MEK}}I^{\beta}_{a+;[\sigma,\eta]}$ $({}^{\mathrm{MEK}}I^{\beta}_{b-;[\sigma,\eta]})$ commute.

We define the modified EKFDs of order α corresponding to the modified EKFIs as follows.

Left/Forward modified Erdélyi-Kober fractional derivative of order α

$$\left({}^{\mathrm{MEK}}D^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) = t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}D_t\right)^m t^{\sigma\eta}\left({}^{\mathrm{MEK}}I^{m-\alpha}_{a+;[\sigma,\eta]}y\right)(t), \tag{2.46}$$

and *right/backward modified Erdélyi-Kober fractional derivative of order* α

$$\left({}^{\mathrm{MEK}}D^{\alpha}_{b-;[\sigma,\eta]}y\right)(t) = t^{\sigma(\eta)}\left(\frac{-1}{\sigma t^{\sigma-1}}D_t\right)^m t^{\sigma(-\eta)}\left({}^{\mathrm{MEK}}I^{m-\alpha}_{b-;[\sigma,\eta]}y\right)(t). \tag{2.47}$$

Note that in the case of the modified EKFIOs and the modified EKFDOs, the value of the parameters σ and η are not modified.

For sufficiently good function $y(t)$, the modified EKFDOs ${}^{\mathrm{EK}}D^{\alpha}_{a+;[\sigma,\eta]}$ and ${}^{\mathrm{MEK}}D^{\alpha}_{b-;[\sigma,\eta]}$ are left inverse of the modified EKFIOs ${}^{\mathrm{EK}}I^{\alpha}_{a+;[\sigma,\eta]}$ and ${}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}$, respectively, that is, they satisfy the following identities:

$${}^{\mathrm{MEK}}D^{\alpha}_{a+;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]} = E = {}^{\mathrm{MEK}}D^{\alpha}_{b-;[\sigma,\eta]}\,{}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}. \tag{2.48}$$

The modified EKFIs and the modified EKFDs satisfy the following integration by parts formulas

$$\int_a^b t^{(\sigma-1)}f(t)\left({}^{\mathrm{MEK}}I^{\alpha}_{a+;[\sigma,\eta]}g\right)(t)dt = \int_a^b \sigma t^{(\sigma-1)}g(t)\left({}^{\mathrm{MEK}}I^{\alpha}_{b-;[\sigma,\eta]}f\right)(t)dt, \tag{2.49}$$

$$\int_a^b t^{(\sigma-1)}f(t)\left({}^{\mathrm{MEK}}D^{\alpha}_{a+;[\sigma,\eta]}g\right)(t)dt = \int_a^b \sigma t^{(\sigma-1)}g(t)\left({}^{\mathrm{MEK}}D^{\alpha}_{b-;[\sigma,\eta]}f\right)(t)dt. \tag{2.50}$$

The method of proof for (2.49) and (2.50) are the same as that for (2.39) and (2.40).

Equation (2.50) can be used to obtain a fractional variational formulation in terms of modified EKFDs. However, such formulations limit the terminal conditions and obscure the posibility of other terminal and transversality conditions. Further, note that in (2.46) and (2.47), the integral operators are applied first and the derivative operators are applied next. We can interchange the operation, and these leads to Caputo type derivatives. Accordingly, we define the modified Caputo type Erdélyi-Kober fractional derivatives as follows.

Left/Forward modified Caputo type Erdélyi-Kober fractional derivative (MCEKFD) of order α

$$\left({}^{\mathrm{MCEK}}D^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{-\sigma\eta}}{\Gamma(m-\alpha)}\int_a^x \frac{\tau^{\sigma-1}d\tau}{\left[x^{\sigma}-\tau^{\sigma}\right]^{1+\alpha-m}}\left(\frac{1}{\sigma\tau^{\sigma-1}}D_{\tau}\right)^m\left(\tau^{\sigma\eta}y(\tau)\right) \tag{2.51}$$

and *right/backward modified Caputo type Erdélyi-Kober fractional derivative (MCEKFD) of order* α

$$\left({}^{\mathrm{MCEK}}D^{\alpha}_{b-;[\sigma,\eta]}y\right)(t) = \frac{\sigma t^{\sigma\eta}}{\Gamma(m-\alpha)}\int_x^b \frac{\tau^{\sigma-1}d\tau}{\left[\tau^{\sigma}-x^{\sigma}\right]^{1+\alpha-m}}\left(\frac{-1}{\sigma\tau^{\sigma-1}}D_{\tau}\right)^m\left(\tau^{-\sigma\eta}y(\tau)\right). \tag{2.52}$$

The left MEKFD and the left MCEKFD are related in the following way:

$$\begin{aligned}\left({}^{\mathrm{MEK}}D^{\alpha}_{a+;[\sigma,\eta]}y\right)(t) &= \left({}^{\mathrm{MCEK}}D^{\alpha}_{a+;[\sigma,\eta]}y\right)(t)\\ &\quad + x^{-\sigma\eta}\sum_{k=0}^{m-1}\frac{\left(x^{\sigma}-a^{\sigma}\right)}{\Gamma(k-\alpha+1)}\left(\frac{1}{\sigma\tau^{\sigma-1}}D_{\tau}\right)^{(k)}\left(\tau^{\sigma\eta}y(\tau)\right)\big|_{\tau=a}.\end{aligned} \tag{2.53}$$

A similar relation exits between the right MEKFD and the right MCEKFD. The MEKFDs and MCEKFDs satisfy the following integration by parts formula:

$$\begin{aligned}&\int_a^b \sigma x^{\sigma-1} f(x)\left({}^{\mathrm{MEK}}D^{\alpha}_{a+;[\sigma,\eta]} g\right)(x)dx\\&\quad=\int_a^b \sigma x^{\sigma-1} g(x)\left({}^{\mathrm{MCEK}}D^{\alpha}_{b-;[\sigma,\eta]} f\right)(x)dx\\&\qquad+\sum_{k=0}^{m-1}\left[x^{\sigma\eta}\left(\frac{-1}{\sigma x^{\sigma-1}}D_x\right)^k\left(x^{-\sigma\eta}f(x)\right)\right]\left({}^{\mathrm{MEK}}D^{\alpha-1-k}_{a+;[\sigma,\eta]} g\right)(x)\Bigg|_a^b.\end{aligned} \tag{2.54}$$

A similar relationship can be obtained relating the right MEKFD and the left MCEKFD.

2.5. *Weighted/Scaled Fractional Integrals and Fractional Derivatives of a Function with Respect to Another Function*

In this section we define the left/forward and the right/backward fractional integrals and fractional derivatives of a function $f(t)$ with respect to another function $z(t)$ and weight/scale $w(t)$, and investigate some of their properties. We assume that $z(t)$ is an increasing positive monotone function on $(a,b]$ having a continuous derivative $z'(t)$ on (a,b). We further assume that function $w(t)$ is "sufficiently good."

We now define the left/forward and the right/backward weighted/scaled fractional integrals of a function with respect to another function as follows.

Left/Forward weighted/scaled fractional integral of order $\alpha > 0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$

$$\left(I^{\alpha}_{a+;[z;w]} f\right)(x) = \frac{[w(x)]^{-1}}{\Gamma(\alpha)}\int_a^x \frac{w(t)z'(t)f(t)dt}{[z(x)-z(t)]^{1-\alpha}}. \tag{2.55}$$

Right/Backward weighted fractional integral of order $\alpha > 0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$

$$\left(I^{\alpha}_{b-;[z;w]} f\right)(x) = \frac{[w(x)]}{\Gamma(\alpha)}\int_x^b \frac{[w(t)]^{-1}z'(t)f(t)dt}{[z(t)-z(x)]^{1-\alpha}}. \tag{2.56}$$

We shall denote them as the left/forward and the right/backward generalized fractional integrals (GFIs). It should be pointed out that these integrals also contain the scaling/weight function $w(t)$ and the function $z(t)$ with respect to which the function $f(t)$ is integrated. In our case, functions $w(t)$ and $z(t)$ will often remain the same, and therefore, it is not necessary to explicitly mention of the presence of these functions in the definitions of the integrals.

Like the GFIs, we define the left/forward and the right/backward weighted/scaled fractional derivatives of a function $f(t)$ with respect to another function $z(t)$ and weight/scale $w(t)$ as follows.

Left/Forward weighted fractional derivative of order $\alpha > 0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$

$$\left(D^{\alpha}_{a+;[z;w,1]}f\right)(x) = [w(x)]^{-1}\left(\frac{1}{z'(x)}D_x\right)^{m} w(x)\left(I^{m-\alpha}_{a+;[z;w]}f\right)(x). \tag{2.57}$$

Right/Backward weighted fractional derivative of order $\alpha > 0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$

$$\left(D^{\alpha}_{b-;[z;w,1]}f\right)(x) = [w(x)]\left(\frac{-1}{z'(x)}D_x\right)^{m} [w(x)]^{-1}\left(I^{m-\alpha}_{b-;[z;w]}f\right)(x), \tag{2.58}$$

where $m-1 < \alpha < m$, and $D_x = d/dx$. These derivatives contain one parameter only, namely the order of the derivative. (In reality, functions $w(t)$ and $z(t)$ would also introduce additional parameters. For the time being, we shall keep these functions the same, and therefore, calling these derivatives as one parameter derivatives is justified.) In later sections, we shall define derivatives containing many more parameters. Furthermore, note that in these derivatives, the GFI Operators (GFIOs) are applied first and some derivative operators are applied next, and accordingly, they are like Riemann-Liouville fractional derivatives. Later, we shall consider Caputo type derivatives in which the derivative operators would be applied first, and the GFIOs next. A name that describes all these would be very long. For brevity, we denote these derivatives as One-Parameter Type-1 generalized fractional derivatives (1PT1GFDs). We choose not to call these derivatives as the Riemann-Liouville fractional derivatives for the reason given below. We further define

$$\begin{aligned}\left(D_{[z,w,L]}f\right)(x) &= [w(x)]^{-1}\left[\left(\frac{1}{z'(x)}D_x\right)\left(w(x)f(x)\right)\right](x),\\ \left(D_{[z,w,R]}f\right)(x) &= [w(x)]\left[\left(\frac{-1}{z'(x)}D_x\right)\left([w(x)]^{-1}f(x)\right)\right](x).\end{aligned} \tag{2.59}$$

Here, $D_{[z,w,L]}$ and $D_{[z,w,R]}$ are like the left/forward and the right/backward operators D and $-D$, respectively, except that they also contain functions $w(t)$ and $z(t)$. These are new integer order operators. variational calculus in terms of these operators will be considered somewhere else. In this paper, we shall focus on the role of these operators in formulating generalized fractional variational calculus. Using (2.59), (2.57), and (2.58) can be written as

$$\begin{aligned}\left(D^{\alpha}_{a+;[z;w,1]}f\right)(x) &= D^{m}_{[z,w,L]}\left(I^{m-\alpha}_{a+;[z;w]}f\right)(x),\\ \left(D^{\alpha}_{b-;[z;w,1]}f\right)(x) &= D^{m}_{[z,w,R]}\left(I^{m-\alpha}_{b-;[z;w]}f\right)(x).\end{aligned} \tag{2.60}$$

Here subscript “1” is added to indicate that these are Type-1 fractional derivatives. In the same faashion, we introduce One-parameter Type-2 generalized fractional derivatives (1PT2GFDs) as

$$\left(D^{\alpha}_{a+;[z;w,2]}f\right)(x) = \left(I^{m-\alpha}_{a+;[z;w]}D^{m}_{[z,w,L]}f\right)(x), \tag{2.61}$$

$$\left(D^{\alpha}_{b-;[z;w,2]}f\right)(x) = \left(I^{m-\alpha}_{b-;[z;w]}D^{m}_{[z,w,R]}f\right)(x). \tag{2.62}$$

Note that in these fractional derivatives, the differential operators are applied first, and the GFIOs are applied next, accordingly they are Caputo type fractional derivatives.

Before we proceed further, we would like to note that the fractional integrals and derivatives of a function with respect to another function defined in [12, 19] does not consider the weight/scale function. Further, by taking $w(t) = 1$ in (2.55) to (2.58), we obtain the fractional integrals and derivatives $(I^{\alpha}_{a+;z}f)(x)$, $(I^{\alpha}_{b-;z}f)(x)$, $(D^{\alpha}_{a+;z}f)(x)$, and $(D^{\alpha}_{b-;z}f)(x)$ defined in [12, 19]. However, in our case, $w(t)$ need not be 1. Thus, the fractional integrals and derivatives defined here are more general than those given in [12, 19]. Also note that in [19], the authors define a substitution operator Q_g and its inverse Q_g^{-1} such that $(Q_g f)(x) = f[g(x)]$. One of the major advantages of these operators is that they link the fractional integrals and derivatives $(I^{\alpha}_{a+;z}f)(x)$, $(I^{\alpha}_{b-;z}f)(x)$, $(D^{\alpha}_{a+;z}f)(x)$, and $(D^{\alpha}_{b-;z}f)(x)$ to the Riemann-Liouville fractional integrals and derivatives defined by (2.1), (2.2), (2.6), and (2.7). Accordingly, the properties of $(I^{\alpha}_{a+;z}f)(x)$, $(I^{\alpha}_{b-;z}f)(x)$, $(D^{\alpha}_{a+;z}f)(x)$, and $(D^{\alpha}_{b-;z}f)(x)$ could be obtained from the operators defined in (2.1), (2.2), (2.6), and (2.7). Operators similar to Q_g and Q_g^{-1} could be defined for the present case. However, in this paper, we shall use the definitions directly to obtain the properties of the operators defined in (2.55) to (2.62).

We now list several properties of operators $I^{\alpha}_{a+;[z;w]}$, $I^{\alpha}_{b-;[z;w]}$, $D^{\alpha}_{a+;[z;w,1]}$, $D^{\alpha}_{b-;[z;w,1]}$, $D^{\alpha}_{a+;[z;w,2]}$, and $D^{\alpha}_{b-;[z;w,2]}$. Operators $I^{\alpha}_{a+;[z;w]}$ and $I^{\alpha}_{b-;[z;w]}$ satisfy the following semigroup properties:

$$I^{\alpha}_{a+;[z;w]}I^{\beta}_{a+;[z;w]} = I^{\alpha+\beta}_{a+;[z;w]} = I^{\beta}_{a+;[z;w]}I^{\alpha}_{a+;[z;w]}, \tag{2.63}$$

$$I^{\alpha}_{b-;[z;w]}I^{\beta}_{b-;[z;w]} = I^{\alpha+\beta}_{b-;[z;w]} = I^{\beta}_{b-;[z;w]}I^{\alpha}_{b-;[z;w]}. \tag{2.64}$$

Operators $D^{\alpha}_{a+;[z;w,1]}$ and $D^{\alpha}_{b-;[z;w,1]}$ are left inverse of the operators $I^{\alpha}_{a+;[z;w]}$ and $I^{\alpha}_{b-;[z;w]}$, that is, they satisfy the following relations:

$$\left(D^{\alpha}_{a+;[z;w,1]}I^{\alpha}_{a+;[z;w]}f\right)(x) = f(x), \qquad \left(D^{\alpha}_{b-;[z;w,1]}I^{\alpha}_{b-;[z;w]}f\right)(x) = f(x). \tag{2.65}$$

Operators $D^{\alpha}_{a+;[z;w,1]}$ and $D^{\alpha}_{a+;[z;w,2]}$ are related by the following formula

$$\left(D^{\alpha}_{a+;[z;w,1]}f\right)(x) = \left(D^{\alpha}_{a+;[z;w,2]}f\right)(x) + \frac{w(a)}{w(x)}\sum_{k=0}^{m-1}\frac{\left(D^{(k)}_{[z;w]}f\right)(a)}{\Gamma(k-\alpha+1)}(z(x)-z(a))^{k-\alpha}. \tag{2.66}$$

One can develop a similar relation relating $D^{\alpha}_{b-;[z;w,1]}$ to $D^{\alpha}_{b-;[z;w,2]}$. The above relation shows that like in the case of Riemann-Liouville and Caputo derivatives, if $f(x)$ and all its derivatives of order upto $m-1$ are 0 at $x=a$, then Type-1 and Type-2 derivatives $(D^{\alpha}_{a+;[z;w,1]}f)(x)$ and $({}^{C}D^{\alpha}_{a+;[z;w,2]}f)(x)$ are the same, that is

$$\left(D^{\alpha}_{a+;[z;w,1]}f\right)(x) = \left(D^{\alpha}_{a+;[z;w,2]}f\right)(x), \quad \text{if } f^{(k)}(a)=0,\ k=0,\ldots,m-1. \tag{2.67}$$

Similarly, the following relationship holds:

$$\left(D^{\alpha}_{b-;[z;w,1]}f\right)(x) = \left(D^{\alpha}_{b-;[z;w,2]}f\right)(x), \quad \text{if } f^{(k)}(b)=0,\ k=0,\ldots,m-1. \tag{2.68}$$

Operators $I^{\alpha}_{a+;[z;w]}$, $I^{\alpha}_{b-;[z;w]}$, $D^{\alpha}_{a+;[z;w,1]}$, $D^{\alpha}_{b-;[z;w,1]}$, $D^{\alpha}_{a+;[z;w,2]}$, and $D^{\alpha}_{b-;[z;w,2]}$ satisfy the following integration by parts formulas:

$$\int_a^b z'(x)f(x)\left(I^{\alpha}_{a+;[z;w]}g\right)(x)dx = \int_a^b z'(x)g(x)dx\left(I^{\alpha}_{b-;[z;w]}f\right)(x), \tag{2.69}$$

$$\int_a^b z'(x)f(x)\left(D^{\alpha}_{a+;[z;w,1]}g\right)(x)dx = \int_a^b z'(x)g(x)\left(D^{\alpha}_{b-;[z,w,1]}f\right)(x)dx, \tag{2.70}$$

$$\begin{aligned}\int_a^b z'(x)f(x)\left(D^{\alpha}_{a+;[z;w,1]}g\right)(x)dx = &\int_a^b z'(x)g(x)\left(D^{\alpha}_{b-;[z;w,2]}f\right)(x)dx \\ &+ \sum_{k=0}^{m-1}\left[\left(D^{(k)}_{[z;w,R]}f\right)(x)\left(D^{\alpha-1-k}_{a+;[z;w,1]}g\right)(x)\right]\Bigg|_a^b,\end{aligned} \tag{2.71}$$

$$\begin{aligned}\int_a^b z'(x)f(x)\left(D^{\alpha}_{b-;[z;w,1]}g\right)(x)dx = &\int_a^b z'(x)g(x)\left(D^{\alpha}_{a+;[z;w,2]}f\right)(x)dx \\ &- \sum_{k=0}^{m-1}\left[\left(D^{(k)}_{[z;w,L]}f\right)(x)\left(D^{\alpha-1-k}_{b-;[z;w,1]}g\right)(x)\right]\Bigg|_a^b.\end{aligned} \tag{2.72}$$

Equations (2.69) to (2.72) would play a key role in deriving a generalized fractional variational formulation. Equation (2.70) will implicitly account for some boundary conditions. In contrast, (2.71) and (2.72) would provide either natural or the boundary conditions. Equations (2.69) to (2.72) would play a key role in developing adjoint equations. However, this issue will be considered elsewhere.

References [12, 19] obtain some special cases of fractional integrals and derivatives of a function with respect to another function by setting $a=-\infty$ or 0 and $b=\infty$. Similarly, one can obtain special cases of (2.55) to (2.58) by setting $a=-\infty$ or 0 and $b=\infty$. In particular, by setting $(a=0, b=\infty)$ and $(a=-\infty, b=\infty)$ we obtain generalized Liouville type fractional integrals and derivatives on the half-axis $\mathbb{R}^+$ and on the whole axis $\mathbb{R}$.

The generalized fractional integrals and derivatives defined in (2.55) to (2.62) encompass many of the fractional integrals and derivatives defined earlier. For example, for $w(t) = 1$ and $z(t) = t$, the generalized fractional integrals and derivatives reduce to Riemann-Liouville and Caputo fractional integrals and derivatives, that is, we get

$$\begin{gathered}\left(I^{\alpha}_{a+;[t;\,1]}f\right)(x) = \left(I^{\alpha}_{a+}f\right)(x), \qquad \left(I^{\alpha}_{b-;[t;\,1]}f\right)(x) = \left(I^{\alpha}_{b-}f\right)(x), \\ \left(D^{\alpha}_{a+;[t;\,1,1]}f\right)(x) = \left(D^{\alpha}_{a+}f\right)(x), \qquad \left(D^{\alpha}_{b-;[t;\,1,1]}f\right)(x) = \left(D^{\alpha}_{b-}f\right)(x), \\ \left(D^{\alpha}_{a+;[t;\,1,2]}f\right)(x) = \left({}^{C}D^{\alpha}_{a+}f\right)(x), \qquad \left(D^{\alpha}_{b-;[t;\,1,2]}f\right)(x) = \left({}^{C}D^{\alpha}_{b-}f\right)(x).\end{gathered} \tag{2.73}$$

For $w(t) = 1$ and $z(t) = \log(t)$, the generalized fractional integrals and derivatives reduce to Hadamard type fractional integrals and derivatives, that is, we get

$$\begin{gathered}\left(I^{\alpha}_{a+;\left[\log(t);\,1\right]}f\right)(x) = \left({}^{H}I^{\alpha}_{a+}f\right)(x), \qquad \left(I^{\alpha}_{b-;\left[\log(t);\,1\right]}f\right)(x) = \left({}^{H}I^{\alpha}_{b-}f\right)(x), \\ \left(D^{\alpha}_{a+;\left[\log(t);\,1,1\right]}f\right)(x) = \left({}^{H}D^{\alpha}_{a+}f\right)(x), \qquad \left(D^{\alpha}_{b-;\left[\log(t);\,1,1\right]}f\right)(x) = \left({}^{H}D^{\alpha}_{b-}f\right)(x), \\ \left(D^{\alpha}_{a+;\left[\log(t);\,1,2\right]}f\right)(x) = \left({}^{\mathrm{HC}}D^{\alpha}_{a+}f\right)(x), \qquad \left(D^{\alpha}_{b-;\left[\log(t);\,1,2\right]}f\right)(x) = \left({}^{\mathrm{HC}}D^{\alpha}_{b-}f\right)(x).\end{gathered} \tag{2.74}$$

For $w(t) = t^{\sigma\eta}$ and $z(t) = t^{\sigma}$, the generalized fractional integrals and derivatives reduce to Modified Erdélyi-Kober type fractional integrals and derivatives, that is, we get

$$\begin{gathered}\left(I^{\alpha}_{a+;[t^{\sigma};t^{\sigma\eta}]}f\right)(x) = \left({}^{\mathrm{MEK}}I^{\alpha}_{a+;\left[\sigma,\eta\right]}f\right)(x), \qquad \left(I^{\alpha}_{b-;[t^{\sigma};t^{\sigma\eta}]}f\right)(x) = \left({}^{\mathrm{MEK}}I^{\alpha}_{b-;\left[\sigma,\eta\right]}f\right)(x), \\ \left(D^{\alpha}_{a+;[t^{\sigma};t^{\sigma\eta},1]}f\right)(x) = \left({}^{\mathrm{MEK}}D^{\alpha}_{a+;\left[\sigma,\eta\right]}f\right)(x), \qquad \left(D^{\alpha}_{b-;[t^{\sigma};t^{\sigma\eta},1]}f\right)(x) = \left({}^{\mathrm{MEK}}D^{\alpha}_{b-;\left[\sigma,\eta\right]}f\right)(x), \\ \left(D^{\alpha}_{a+;[t^{\sigma};t^{\sigma\eta},2]}f\right)(x) = \left({}^{\mathrm{MCEK}}D^{\alpha}_{a+;\left[\sigma,\eta\right]}f\right)(x), \qquad \left(D^{\alpha}_{b-;[t^{\sigma};t^{\sigma\eta},2]}f\right)(x) = \left({}^{\mathrm{MCEK}}D^{\alpha}_{b-;[\sigma,\eta]}f\right)(x).\end{gathered} \tag{2.75}$$

It could also be verified that for $(z(t), w(t))$ equal to $(t, 1)$, $(\log(t), 1)$ and $(t^{\sigma}, t^{\sigma\eta})$ the semi-group, the left inverse, and the integration by parts type identities for the generalized fractional integrals and derivatives, namely (2.63) to (2.72), reduce to those for Riemann-Liouville, Hadamard, and Modified Erdélyi-Kober fractional integrals and derivatives. It should be pointed out that functions $(z(t), w(t))$ are not limited to $(t, 1)$, $(\log(t), 1)$ and $(t^{\sigma}, t^{\sigma\eta})$. In fact this choice is significantly large, and for each choice, one would obtain a different set of fractional integrals and derivatives. The rational for not calling $(D^{\alpha}_{a+;[z;w,1]}f)(x)$ and $(D^{\alpha}_{b-;[z;w,1]}f)(x)$ as the Riemann-Liouville fractional derivatives should now be clear; for different $z(t)$ and $w(t)$, they lead to different fractional integrals and derivatives. In literature, these fractional integrals and derivatives have been called by different names. Thus, to avoid confusion, we prefer to call $(D^{\alpha}_{a+;[z;w,1]}f)(x)$ and $(D^{\alpha}_{b-;[z;w,1]}f)(x)$ as the 1PT1GFDs, and $(D^{\alpha}_{a+;[z;w,2]}f)(x)$ and $(D^{\alpha}_{b-;[z;w,2]}f)(x)$ as the 1PT2GFDs.

3. Fractional Variational Formulation in Terms of One-Parameter Generalized Fractional Derivatives

In this section, we present several fractional variational formulations in terms of one parameter generalized fractional derivatives. We shall consider formulations in terms of one variable and one fractional derivative term, specified and unspecified terminal conditions, one variable and multiple fractional derivative terms with different order of derivatives, multiple variables and multiple derivative terms but the same order of derivatives, geometric constraints, and parametric constrains to name a few. The approach presented here will also be applicable to multivariables and multiple fractional derivative terms with different order of derivatives, free end points, free end-point constraints, multidimensions, and many other formulations can also be considered. As a matter of fact almost all variational formulations can be recast in terms of generalized fractional derivatives, and these will be considered in the future. The derivations of almost all formulations follow the same pattern, and for this reason, a fractional variational formulation would be given in detail for a simple fractional variational problem only. For other fractional variational problems, the final Euler-Lagrange equation will be given but the details would be omitted.

3.1. A Simple Fractional Variational Formulation

In this subsection, we develop an Euler-Lagrange formulation for a simple fractional variational formulation. The functional considered in this case may contain the left and the right integrals and derivatives, and the derivatives could be of Type-1, Type-2, or both. In the simplest fractional variational problem considered here, we take only one fractional derivative term, namely the term $D^{\alpha}_{a+;[z,w,2]}y$. The approach for functional containing other fractional integrals and derivatives would be the same. Accordingly, the fractional variational problem is defined as follows: among all functions $y(t)$ which are continuously differentiable on (a,b) find the function $y^{\star}(t)$ for which the functional

$$J[y] = \int_a^b F\left(t, y, D^{\alpha}_{a+;[z;w,2]}y\right)(t)z'(t)dt \tag{3.1}$$

is an extremum. For simplicity, we assume for the time being that $0 < \alpha < 1$. In case the functional is given in terms of $D^{\alpha}_{a+;[z;w,1]}y$, one can use (2.66) to write the functional in terms of $D^{\alpha}_{a+;[z;w,2]}y$.

At this stage, two points can be made which are the same as those made in [38] for another set of fractional derivatives. We repeat them here for completeness and to indicate that those points equally apply here. First, note that fractional integration introduces some degree of continuity, and therefore the differentiability requirements of $y(t)$ could be relaxed. For certain class of problems and in several numerical schemes, the differentiability of $y(t)$ would be required on only a finite subset of $[a,b]$, and the derivative of $y(t)$ could be discontinuous at a finite set of points. Further, we have implicitly assumed that function $F(t, y, D^{\alpha}_{a+;[z;w,2]}y)(t)$ has continuous first and second partial derivatives with respect to all its arguments. In all formulation to follow, these conditions will be implicitly assumed. In many cases, mathematical operations performed will determine the class of functions being considered. Second, we have not specified the terminal conditions yet, because in fractional variational formulations, the forms of the necessary conditions are tied to the specified terminal conditions. This will be demonstrated shortly, and subsequently the boundary conditions will be specified.

To derive the necessary conditions, we define

$$y(t) = y^{\star}(t) + \epsilon\eta(t), \quad \epsilon \in \mathbb{R} \tag{3.2}$$

and substitute it in (3.1) to obtain J in terms of ϵ and $\eta(t)$. Here $y^{\star}(t)$ is the desired solution, $\eta(t)$ is an arbitrary function consistent with the boundary conditions, and ϵ is a small real number. Thus, for the specified $\eta(t)$, $J = J[\epsilon]$ would be a function of ϵ only. We differentiate $J = J[\epsilon]$ with respect to ϵ, and set ϵ and the resulting equation to 0 to obtain

$$\frac{dJ}{d\epsilon}\Big|_{\epsilon=0} = \int_a^b \left[\frac{\partial F}{\partial y}\eta(t) + \frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}\left(D^{\alpha}_{a+;[z;w,2]}\eta\right)(t)\right] z'(t)dt = 0. \tag{3.3}$$

Using integration by parts formula given by (2.72), we obtain

$$\int_a^b \left[\frac{\partial F}{\partial y} + D^{\alpha}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}\right]\eta(t)z'(t)dt - D^{\alpha-1}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}\eta\Bigg|_a^b = 0. \tag{3.4}$$

Here, $(D^{\alpha-1}_{b-;[z;w,1]}\eta) = (I^{1-\alpha}_{b-;[z;w,1]}\eta)$. Typically, in variational formulations the boundary terms suggest the geometric and the natural boundary conditions. Thus, (3.4) suggests that for this case, $y(a) = y_a$ and $y(b) = y_b$ would be the appropriate geometric boundary conditions. These boundary conditions are the same as those considered in ordinary variational calculus. However, this leads to fractional natural boundary conditions. It can be verified that if (2.71) is used for integration by parts formula, then one obtains fractional geometric boundary conditions and regular natural boundary conditions.

Let us assume that $y(a) = y_a$ and $y(b) = y_b$ are specified. In this case, $\eta(a) = \eta(b) = 0$, and using the fundamental lemma of variational calculus, (3.4) leads to the following Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} + D^{\alpha}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y} = 0. \tag{3.5}$$

It is the necessary condition for the extremum. Let us now assume that $y(b)$ is not specified. In this case, (3.4) leads to

$$D^{\alpha-1}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,1]}y}\Bigg|_{t=b} = 0. \tag{3.6}$$

A similar condition is obtained if $y(a)$ is not specified. Equation (3.6) and its variations are known as the natural boundary conditions.

As a quick modification of the above problem, assume that the functional $J[y]$ contains an additional term $\phi(y(b), b)$. Clearly, $y(b)$ would be unknown, otherwise $\phi(y(b), b)$ would be a constant which could be removed from the functional without altering the problem. Term $\phi(y(b),b)$ would lead to an additional term in (3.4) of type

$$\frac{\partial \phi}{\partial y(b)}\eta(b) \tag{3.7}$$

and the natural boundary conditions as

$$\left[D^{\alpha-1}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,1]}y}+\frac{\partial \phi}{\partial y(b)}\right]\Bigg|_{t=b}=0. \tag{3.8}$$

In the case of α greater than 1, (3.4) is replaced with

$$\int_a^b\left[\frac{\partial F}{\partial y}+D^{\alpha}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}\right]\eta(t)z'(t)dt-\sum_{k=0}^{n-1}D^{\alpha-1-k}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}D^{(k)}_{[z;w,L]}\eta\Bigg|_a^b=0, \tag{3.9}$$

where $n-1<\alpha<n$. Equation (3.9) leads to the same Euler-Lagrange equation as that given by (3.5). It further suggests that the geometric boundary conditions $D^{(k)}_{[z;w,L]}y(c)$ should be specified or the natural boundary conditions

$$D^{\alpha-1-k}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}(c)=0, \tag{3.10}$$

should be considered. Here $c=a,b$ and $k=0,\ldots,n-1$. The above discussion assumes that all $D^{(k)}_{[z;w,L]}y(c)$, $c=a,b$ and $k=0,\ldots,n-1$ are independent. In case they are not independent, then the natural boundary conditions are modified accordingly.

It should be pointed out that $D^{(k)}_{[z;w,L]}$ and $D^{(k)}_{[z;w,R]}$ are not ordinary differential operators. Therefore, derivatives $D^{(k)}_{[z;w,L]}y(c)$, $c=a,b$ are not the same as the ordinary derivatives $D^{(k)}y(c)$. To demonstrate this, take $z(t)=\log(t)$, and $w(t)=t$. In this case, we have $D^{(1)}_{[z;w,L]}y(t)=(1/t)*((tD)(t*y))=D(ty)=y+tDy$, that is, $D^{(1)}_{[z;w,L]}y(t)$ is a weighted/scaled combination of $y(t)$ and $Dy(t)$, and $D^{(1)}_{[z;w,L]}y(c)=y(c)+cDy(c)$. Therefore, when writing the geometric and/or natural boundary conditions, one must exercise caution.

In the above functional, we considered only the left fractional derivative. If the functional contains the right fractional derivative also, then the functional is written as

$$J[y]=\int_a^b F\left(t,y,D^{\alpha}_{a+;[z;w,2]}y,D^{\beta}_{b-;[z;w,2]}y\right)(t)z'(t)dt, \tag{3.11}$$

where $n-1<\alpha<n$ and $m-1<\beta<m$. For simplicity, we shall assume that $n-1<\alpha,\beta<n$. It can be shown that for this functional the above approach leads to the following Euler-Lagrange equation

$$\frac{\partial F}{\partial y}+D^{\alpha}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}+D^{\beta}_{a+;[z;w,1]}\frac{\partial F}{\partial D^{\beta}_{b-;[z;w,2]}y}=0, \tag{3.12}$$

and the following condition at the boundary points

$$\sum_{k=0}^{n-1}\left[D^{\alpha-1-k}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}D^{(k)}_{[z;w,L]}\eta-D^{\beta-1-k}_{a+;[z;w,1]}\frac{\partial F}{\partial D^{\beta}_{b-;[z;w,2]}y}D^{(k)}_{[z;w,R]}\eta\right]\Bigg|_a^b=0. \tag{3.13}$$

Note that in this case, we have left the natural boundary conditions, $D^{(k)}_{[z;w,L]}$ and $D^{(k)}_{[z;w,R]}$ together because we cannot treat both $D^{(k)}_{[z;w,L]}$ and $D^{(k)}_{[z;w,R]}$ as independent and they are not equal either. As a result, we cannot even write

$$\left[D^{\alpha-1-k}_{b-;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y}D^{(k)}_{[z;w,L]}\eta-D^{\alpha-1-k}_{a+;[z;w,1]}\frac{\partial F}{\partial D^{\alpha}_{b-;[z;w,2]}y}D^{(k)}_{[z;w,R]}\eta\right](c)=0, \tag{3.14}$$

where $c=a,b$, $k=0,\ldots,n-1$. However, for $k=0,\ldots,n-1$, $(D^{(k)}_{[z;w,L]}y)(a)$, $(D^{(k)}_{[z;w,L]}y)(b)$, $(D^{(k)}_{[z;w,R]}y)(a)$, and $(D^{(k)}_{[z;w,R]}y)(b)$ are linearly related to $(D^{(k)}y)(a)$ and $(D^{(k)}y)(b)$. These linear relations would be necessary to separate the natural boundary conditions. Terms $D^{(k)}(a)$ and $D^{(k)}(b)$ could still be taken as the geometric boundary conditions.

In the formulation above, we have considered Type-2 fractional derivatives in the functional. Using (2.66), the functional can be written in terms of Type-1 fractional derivatives. However, Type-2 fractional derivatives were considered for two reasons. First, Type-2 fractional derivatives lead to geometric boundary conditions in terms of the desired function and its ordinary derivatives at the boundary points. In contrast, Type-1 fractional derivatives lead to fractional geometric boundary conditions. Many engineers and scientists avoid fractional geometric boundary conditions with remarks that these conditions are nonphysical. For this reason, Type-2 fractional derivatives would be more appealing. Second, treatment of the geometric and the natural boundary conditions is quite involved due to the presence of $z(t)$ and $w(t)$. Fractional geometric boundary conditions and the associated variational formulations will be considered elsewhere.

3.2. Multiorder and Multiterm Fractional Variational Formulation

We now consider several variations of the above formulation. As a first variation, assume that the functional contains m left Type-2 fractional derivatives where the order of the derivatives α_k, $k = 1, \ldots, m$ satisfy the following condition, $0 < \alpha_1, \ldots, \alpha_m < 1$. For this case, the functional is given as

$$J[y] = \int_a^b F\Big(t, y, D^{\alpha_1}_{a+;[z;w,2]} y, \ldots, D^{\alpha_m}_{a+;[z;w,2]} y\Big)(t) z'(t) dt. \tag{3.15}$$

For this functional, following the above approach, we obtain the Euler-Lagrange equation as

$$\frac{\partial F}{\partial y} + \sum_{k=1}^{m} D^{\alpha_k}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha_k}_{a+;[z;w,2]} y} = 0. \tag{3.16}$$

This is a straight forward generalization of (3.6). Further, assume that both $y(a)$ and $y(b)$ are independent. In this case, at point b either $y(b)$ must be specified (geometric boundary condition), or the identity

$$\sum_{k=1}^{m} D^{\alpha_k - 1}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha_k}_{a+;[z;w,1]} y} \Bigg|_{t=b} = 0 \tag{3.17}$$

(natural boundary condition) must be satisfied. A similar condition applies at point a.

Assume now that the functional is of the following type:

$$J[y] = \int_a^b F\Big(t, y, D^{\alpha_1}_{a+;[z;w,2]} y, \ldots, D^{\alpha_n}_{a+;[z;w,2]} y, D^{\beta_1}_{a+;[z;w,2]} y, \ldots, D^{\beta_m}_{a+;[z;w,2]} y\Big)(t) z'(t) dt, \tag{3.18}$$

such that $n_j - 1 < \alpha_j < n_j$, $j = 1, \ldots, n$ and $m_j - 1 < \beta_j < m_j$, $j = 1, \ldots, m$. In this case, the Euler-Lagrange equation is given as

$$\frac{\partial F}{\partial y} + \sum_{k=1}^{n} D^{\alpha_k}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha_k}_{a+;[z;w,2]} y} + \sum_{k=1}^{m} D^{\beta_k}_{a+;[z;w,1]} \frac{\partial F}{\partial D^{\beta_k}_{b-;[z;w,2]} y} = 0, \tag{3.19}$$

and the boundary terms must satisfy the identity,

$$\begin{aligned} &\sum_{k=0}^{N-1} \sum_{j=1}^{n} \left[D^{\alpha_j - 1 - k}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha_j}_{a+;[z;w,2]} y} H(\alpha_j - k) D^{(k)}_{[z;w,L]} \eta \right] \Bigg|_a^b \\ &\quad - \sum_{k=0}^{M-1} \sum_{j=1}^{m} \left[D^{\beta_j - 1 - k}_{a+;[z;w,1]} \frac{\partial F}{\partial D^{\beta_j}_{b-;[z;w,2]} y} H(\beta_j - k) D^{(k)}_{[z;w,R]} \eta \right] \Bigg|_a^b = 0, \end{aligned} \tag{3.20}$$

where $N = \max\{n_1, \ldots, n_n\}$, $M = \max\{m_1, \ldots, m_m\}$, and $H(x)$ is the Heavyside unit step function such that for $x > 0$, $H(x) = 1$, otherwise $H(x) = 0$. Note that like in (3.14), the geometric and the natural boundary conditions are coupled, and they can be separated only by applying some transformations. Here the two cases, namely (3.15) (for $0 < \alpha_1, \dots, \alpha_m < 1$) and (3.18) (for $n_j - 1 < \alpha_j < n_j$, $j = 1, \dots, n$ and $m_j - 1 < \beta_j < m_j$, $j = 1, \dots, m$) are considered separately to emphasize the fact that only for the prior case the natural boundary conditions can be written directly, and in the later case, one must use some transformation. Equation (3.20) can also be written as

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{k=0}^{n_j-1}\left[D_{b-;[z;w,1]}^{\alpha_j-1-k}\frac{\partial F}{\partial D_{a+;[z;w,2]}^{\alpha_j}y}D_{[z;w,L]}^{(k)}\eta\right]\Bigg|_a^b\\ &\quad-\sum_{j=1}^{m}\sum_{k=0}^{m_j-1}\left[D_{a+;[z;w,1]}^{\beta_j-1-k}\frac{\partial F}{\partial D_{b-;[z;w,2]}^{\beta_j}y}D_{[z;w,R]}^{(k)}\eta\right]\Bigg|_a^b=0.\end{aligned}\tag{3.21}$$

Equation (3.20) is preferred here because in some cases, it allows writing the natural boundary conditions directly. For example, assume that (3.18) does not contain the right fractional derivative terms and $(D_{[z;w,L]}^{(k)}y)(t)$, $k = 0, \dots, N-1$ are all independent. In this case, for a specific k, if $(D_{[z;w,L]}^{(k)}y)(b)$ is not specified, then the following natural boundary conditions must be satisfied:

$$\sum_{j=1}^{n}\left[D_{b-;[z;w,1]}^{\alpha_j-1-k}\frac{\partial F}{\partial D_{a+;[z;w,2]}^{\alpha_j}y}H(\alpha_j-k)\right]\Bigg|_{t=b}=0.\tag{3.22}$$

Thus, the reason for considering (3.20) over (3.21) should be clear.

We now consider the following functional containing multiple functions,

$$J[\mathbf{y}]=\int_a^b F\left(t,\mathbf{y},D_{a+;[z;w,2]}^{\alpha}\mathbf{y},D_{b-;[z;w,2]}^{\beta}\mathbf{y}\right)(t)z'(t)dt,\tag{3.23}$$

where α, β, $0 < \alpha$, $\beta < 1$, are the order of the forward and the backward derivatives, and $\mathbf{y} = [y_1, \dots, y_n]^T$ is an n-dimensional vector function. Here, the order of the forward (backward) derivative applied to all functions $y_k(t)$, $k = 1, \dots, n$ is the same. In general, both α and β could be any positive number, and each $y_k(t)$, $k = 1, \dots, n$ could have different order of derivatives. The functional in (3.23) is considered for simplicity.

Following the above approach, it can be shown that for this functional the Euler-Lagrange equation is

$$\frac{\partial F}{\partial \mathbf{y}}+D_{b-;[z;w,1]}^{\alpha}\frac{\partial F}{\partial D_{a+;[z;w,2]}^{\alpha}\mathbf{y}}+D_{a+;[z;w,1]}^{\beta}\frac{\partial F}{\partial D_{b-;[z;w,2]}^{\beta}\mathbf{y}}=0.\tag{3.24}$$

Assume that $y_k(a)$ and $y_k(b)$, $k = 1, \ldots, n$ are all independent. In this case, for specific k and boundary, (say $t = b$), we must have $y_k(b)$ (the geometric boundary condition) specified, or

$$\left[D^{\alpha-1}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]} y} - D^{\beta-1-k}_{a+;[z;w,1]} \frac{\partial F}{\partial D^{\beta}_{b-;[z;w,2]} y} \right] \Bigg|_{t=b} = 0, \tag{3.25}$$

(the natural boundary condition). Similar conditions apply for other ks and the boundary $t = a$. If α, β are greater than 1, then the geometric and the natural boundary conditions become more complex.

3.3. Fractional Variational Formulation for Constrained Systems

We now consider fractional variational formulation for constrained systems. First, we consider an isoperimetric problem defined as follows: find the curve $y = y(t)$ for which the functional

$$J[y] = \int_a^b F\left(t, y, D^{\alpha}_{a+;[z;w,2]} y,\, D^{\beta}_{b-;[z;w,2]} y\right)(t) z'(t) dt, \tag{3.26}$$

$0 < \alpha,\ \beta < 1$ has an extremum, where the admissible curves satisfy the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, and are such that another functional

$$K[y] = \int_a^b G\left(t, y, D^{\alpha}_{a+;[z;w,2]} y,\, D^{\beta}_{b-;[z;w,2]} y\right)(t) z'(t) dt, \tag{3.27}$$

takes a fixed value A. For this case, we define another functional

$$\begin{aligned} \overline{F}\left(t, y, D^{\alpha}_{a+;[z;w,2]} y,\, D^{\beta}_{b-;[z;w,2]} y\right) &= F\left(t, y, D^{\alpha}_{a+;[z;w,2]} y,\, D^{\beta}_{b-;[z;w,2]} y\right) \\ &\quad + \lambda G\left(t, y,\, D^{\alpha}_{a+;[z;w,2]} y,\, D^{\beta}_{b-;[z;w,2]} y\right), \end{aligned} \tag{3.28}$$

where λ is a constant known as Lagrange multiplier. In terms of $\overline{F}$, the Euler-Lagrange equation for this problem is given as

$$\frac{\partial \overline{F}}{\partial y} + D^{\alpha}_{b-;[z;w,1]} \frac{\partial \overline{F}}{\partial D^{\alpha}_{a+;[z;w,2]} y} + D^{\beta}_{a+;[z;w,1]} \frac{\partial \overline{F}}{\partial D^{\beta}_{b-;[z;w,2]} y} = 0. \tag{3.29}$$

To prove this, we define

$$y(t) = y^{\star}(t) + \epsilon_1 \eta_1(t) + \epsilon_2 \eta_2(t), \tag{3.30}$$

substitute it in (3.26) and (3.27) to obtain $J = J[\epsilon_1, \epsilon_2]$ and $K = K[\epsilon_1, \epsilon_2] = A$, and follow a Lagrange multiplier-based optimization technique. Here, $y^{\star}(t)$ is the desired function,

$\eta_1(t)$ and $\eta_2(t)$ are arbitrary functions consistent with the constraints, and ϵ_1 and ϵ_2 are two real numbers. In case the geometric boundary condition is not specified at $t = b$, then the following natural boundary conditions must be satisfied:

$$\left[D^{\alpha-1}_{b-;[z;w,1]} \frac{\partial \overline{F}}{\partial D^{\alpha}_{a+;[z;w,2]} y} - D^{\beta-1}_{a+;[z;w,1]} \frac{\partial \overline{F}}{\partial D^{\beta}_{b-;[z;w,2]} y} \right] \Bigg|_{t=b} = 0. \tag{3.31}$$

Note that here F has been replaced with $\overline{F}$. Other conditions such as arbitrary order of derivatives, and unspecified and mixed boundary conditions are handled as discussed above.

Next, we consider a system subjected to holonomic constraints. For simplicity, we consider the functional defined by (3.23), subjected to an m-dimensional constraint defined as

$$\Phi(\mathbf{y}, t) = \left[\phi_1(y_1, \ldots, y_n, t), \ldots, \phi_m(y_1, \ldots, y_n, t) \right]^T = [0, \ldots, 0]^T, \tag{3.32}$$

where $m < n$. For this case, the Euler-Lagrange equation is given as

$$\frac{\partial F}{\partial \mathbf{y}} + D^{\alpha}_{b-;[z;w,1]} \frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]} \mathbf{y}} + D^{\beta}_{a+;[z;w,1]} \frac{\partial F}{\partial D^{\beta}_{b-;[z;w,2]} \mathbf{y}} + \lambda^T \left(\frac{\partial \Phi}{\partial \mathbf{y}} \right) = 0, \tag{3.33}$$

where λ is an m-dimensional vector of Lagrange multipliers. A simple approach to obtain (3.33) is to augment Φ to F using Lagrange multiplier, and use the technique for multiple functions. Note that in this case, only $n - m$ functions $y_k(t)$ are independent, and the rest are determined from (3.32). Accordingly, only $n - m$ geometric or natural boundary conditions need to be specified at each boundary. Of course, we require each constraint to be independent (i.e., we require the rank of the Jacobian $\partial \Phi / \partial \mathbf{y}$ to be full).

The constraints can also be given in the form of fractional differential equations which govern the dynamics of the system. Such problems arise in optimal controls. For simplicity, we consider one state variable $x(t)$ and one control variable $u(t)$, and define the functional and the dynamic equations as

$$J[u] = \phi(x(b), b) + \int_a^b F(x, u, t) z'(t) dt, \tag{3.34}$$

$$\left(D^{\alpha}_{a+;[z;w,2]} x \right)(t) = G(x, u, t), \tag{3.35}$$

and the inital condition as

$$x(a) = x_a. \tag{3.36}$$

Once again, we assume that $0 < \alpha < 1$. In optimal control literature, the functional given by (3.34) is typically known as performance index. Equation (3.34) is called the Bolza's form. If the boundary terms are not present in (3.34), the form is known as the Lagrange form. On the other hand, if the integral term is not present, then the form is known as the Mayer form.

Several forms of this problem are possible. The functional may have differential terms, the dynamic constraint may be a nonlinear function of the fractional derivative terms, the boundary conditions may be specified implicitly, and we may have inequality constraints. The form is considered here for simplicity.

An approach to the above problem is to redefine the functional as

$$J[y] = \overline{\phi}(y(b), b) + \int_a^b \overline{F}\left(y, D^{\alpha}_{a+;[z;w,2]}y, t\right)dt, \tag{3.37}$$

where $\overline{F} = F+\lambda(G-D^{\alpha}_{a+;[z;w,2]}x)$, $y = [x, u]$, and $\overline{\phi}(y(b), b) = \phi(x(b), b)$. Here, λ is the Lagrange multiplier. The boundary condition is given as $y_1(a) = x_a$. The functionals in (3.23) and (3.37) are in the same form. Thus, the approach discussed above could be used to obtain the necessary fractional differential equations and the natural boundary conditions. In particular, for this case, the necessary differential equations are given as

$$\begin{gathered} D^{\alpha}_{b-;[z;w,1]}\lambda = \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}, \\ \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} = 0, \end{gathered} \tag{3.38}$$

and the natural boundary condition is given as

$$I^{1-\alpha}_{b-;[z;w,1]}\lambda(b) + \frac{\partial \phi}{\partial x(b)} = 0. \tag{3.39}$$

Many prefer to write these equations in terms of a Hamiltonian. Thus, if we define a Hamiltonian H as

$$H = F + \lambda G, \tag{3.40}$$

then the necessary fractional differential equations are given as

$$\begin{gathered} D^{\alpha}_{b-;[z;w,1]}\lambda = \frac{\partial H}{\partial x}, \\ \frac{\partial H}{\partial u} = 0, \\ D^{\alpha}_{a+;[z;w,1]}x = \frac{\partial H}{\partial \lambda}. \end{gathered} \tag{3.41}$$

3.4. Fractional Hamiltonian Principle

The fractional variational principle discussed above allows us to develop fractional Hamilton principles. However, depending on the form and the boundary conditions considered, the resulting fractional differential equations would be different, and accordingly the Hamilton equation would also be different. Here we discuss one such principle. Thus, one form of the fractional Hamilton principle can be stated as follows: the fractional Hamilton's principle states that the path traced by a system of particles which are described by n generalized coordinates $\mathbf{y}(t) = [y_1(t), \ldots, y_n(t)]$ between two states $\mathbf{y}(a) = \mathbf{y}_a$ and $\mathbf{y}(b) = \mathbf{y}_b$ at two times a and b is a stationary point of the action functional

$$I[\mathbf{y}] = \int_a^b L\Big(t, \mathbf{y}, D^{\alpha}_{a+;[z;w,2]}\mathbf{y}, D^{\beta}_{b-;[z;w,2]}\mathbf{y}\Big)(t) z'(t) dt, \tag{3.42}$$

where $L(t, \mathbf{y}, D^{\alpha}_{a+;[z;w,2]}\mathbf{y}, D^{\beta}_{b-;[z;w,2]}\mathbf{y})$ is the Lagrangian function. If this is true, then following the discussion above, the dynamics of the system is described by the fractional Euler-Lagrange equation

$$\frac{\partial L}{\partial \mathbf{y}} + D^{\alpha}_{b-;[z;w,1]} \frac{\partial L}{\partial D^{\alpha}_{a+;[z;w,2]}\mathbf{y}} + D^{\beta}_{a+;[z;w,1]} \frac{\partial L}{\partial D^{\beta}_{b-;[z;w,2]}\mathbf{y}} = 0. \tag{3.43}$$

As a matter of fact it has been implicitly assumed here that the dynamics of the system is governed by (3.43). In this case, we can define a fractional Hamiltonian function as

$$H = -L + \mathbf{p}_{\alpha}^{T} D^{\alpha}_{a+;[z;w,2]}\mathbf{y} + \mathbf{p}_{\beta}^{T} D^{\alpha}_{b-;[z;w,2]}\mathbf{y}, \tag{3.44}$$

where $\mathbf{p}_{\alpha}$ and $\mathbf{p}_{\beta}$ are vector of generalized momenta defined as

$$\mathbf{p}_{\alpha} = \frac{\partial L}{\partial D^{\alpha}_{a+;[z;w,2]}\mathbf{y}}, \qquad \mathbf{p}_{\beta} = \frac{\partial L}{\partial D^{\beta}_{b-;[z;w,2]}\mathbf{y}}. \tag{3.45}$$

These equations lead to the following generalized fractional canonical system of Euler equations

$$\begin{gathered} D^{\alpha}_{a+;[z;w,2]}\mathbf{y} = \frac{\partial H}{\partial \mathbf{p}_{\alpha}}, \qquad D^{\beta}_{b-;[z;w,2]}\mathbf{y} = \frac{\partial H}{\partial \mathbf{p}_{\beta}}, \\ \frac{\partial H}{\partial \mathbf{y}} = D^{\alpha}_{b-;[z;w,1]}\mathbf{p}_{\alpha} + D^{\beta}_{a+;[z;w,1]}\mathbf{p}_{\beta}, \\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \end{gathered} \tag{3.46}$$

Equation (3.46) is very similar to those developed in Section 4.3 of [38], and as stated in [38], (3.46) could also be used to develop fractional mechanics, both classical and quantum. However, the advantage of formulation presented here is that $z(t)$ and $w(t)$ allow one to develop a wide spectrum of formulations to fit the needs of the problems under consideration.

4. Illustrative Example

To demonstrate an application of the formulations presented above, consider the following problem: *find a function $y(t)$ in the domain $[0,1]$ that minimizes the functional*

$$J[y] = \int_0^1 \left[\frac{1}{2}\left(D^{\alpha}_{0+;[z;w,2]}y\right)^2 - c(t)y\right] z'(t)dt, \tag{4.1}$$

where $0 < \alpha < 1$, and $y(0)$ and $y(1)$ may or may not be specified.

For this case, function $F(t, y, D^{\alpha}_{0+;[z;w,2]}y)$ is given as

$$F\left(t, y, D^{\alpha}_{0+;[z;w,2]}y\right) = \left[\frac{1}{2}\left(D^{\alpha}_{0+;[z;w,2]}y\right)^2 - c(t)y\right], \tag{4.2}$$

and using (3.5), we obtain the Euler-Lagrange equation for the problem as

$$D^{\alpha}_{1-;[z;w,1]}\left(D^{\alpha}_{0+;[z;w,2]}y\right) - c(t) = 0. \tag{4.3}$$

Further, (3.4) suggests the following geometric and natural boundary conditions at $t = 1$: $y(1)$ must be specified (geometric boundary condition) or the following condition must be satisfied:

$$I^{1-\alpha}_{1-;[z;w]}\left(D^{\alpha}_{0+;[z;w,2]}y\right)(1) = 0 \tag{4.4}$$

(natural boundary condition). A similar condition is predicted at $t = 0$.

We now consider 4 different cases of this example.

Case 1. As a first case, take $w(t) = 1$ and $z(t) = t$. In this case, the generalized fractional derivative operators $D^{\alpha}_{0+;[1;\,1,2]}$ and $D^{\alpha}_{1-;[1;\,1,1]}$ reduce to the Caputo fractional derivative operator ${}^{C}D^{\alpha}_{0+}$ and the Riemann-Liouville fractional derivative operator D^{α}_{1-}, $I^{1-\alpha}_{1-;[1;\,1]}$ reduces to $I^{1-\alpha}_{1-}$, and we get the Euler-Lagrange equation as

$$D^{\alpha}_{1-;[z;w,1]}\left(D^{\alpha}_{0+;[z;w,2]}y\right) - c(t) = 0. \tag{4.5}$$

Further, at $t = 1$, $y(1)$ must be specified, or one must have

$$I^{1-\alpha}_{1-}\left({}^{C}D^{\alpha}_{0+}y\right)(1) = 0. \tag{4.6}$$

A similar condition is given at $t = 0$.

Case 2. As a second case, take $w(t) = 1$ and $z = \log(t)$. In this case, the generalized fractional derivative and integral operators $D^{\alpha}_{0+;[log(t);\,1,2]}$, $D^{\alpha}_{1-;[log(t);\,1,1]}$ and $I^{1-\alpha}_{1-;[log(t);\,1]}$ reduce to the left Hadamard-Caputo and the right Hadamard fractional derivative operators ${}^{HC}D^{\alpha}_{0+}$ and

${}^{H}D^{\alpha}_{1-}$ and the right Hadamard fractional integral operator ${}^{H}I^{1-\alpha}_{1-}$, respectively, and we get the Euler-Lagrange equation as

$$ {}^{H}D^{\alpha}_{1-}\left({}^{\mathrm{HC}}D^{\alpha}_{0+}y\right) - c(t) = 0. \tag{4.7}$$

Further, at $t = 1$, $y(1)$ must be specified, or one must have

$$ {}^{H}I^{1-\alpha}_{1-}\left({}^{\mathrm{HC}}D^{\alpha}_{0+}y\right)(1) = 0. \tag{4.8}$$

A similar condition is given at $t = 0$.

Case 3. Next, consider $z(t) = t^{\sigma}$ and $w(t) = t^{\sigma\eta}$. In this case, the generalized fractional derivative and integral operators $D^{\alpha}_{0+;[t^{\sigma};t^{\sigma\eta},2]}$, $D^{\alpha}_{1-;[t^{\sigma};t^{\sigma\eta},1]}$, and $I^{1-\alpha}_{1-;[t^{\sigma};t^{\sigma\eta}]}$ reduce to the modified left Caputo-Erdélyi-Kober and the modified right Erdélyi-Kober type fractional derivative operators ${}^{\mathrm{MCEK}}D^{\alpha}_{0+;[\sigma,\eta]}$ and ${}^{\mathrm{MEK}}D^{\alpha}_{1-;[\sigma,\eta]}$ and the modified right Erdélyi-Kober type fractional integral operator ${}^{\mathrm{MEK}}I^{1-\alpha}_{1-;[\sigma,\eta]}$, respectively, and we get the Euler-Lagrange equation as

$$ {}^{\mathrm{MEK}}D^{\alpha}_{1-;[\sigma,\eta]}\left({}^{\mathrm{MCEK}}D^{\alpha}_{0+;[\sigma,\eta]}y\right) - c(t) = 0. \tag{4.9}$$

Further, at $t = 1$, $y(1)$ must be specified, or one must have

$$ {}^{\mathrm{MEK}}I^{1-\alpha}_{1-;[\sigma,\eta]}\left({}^{\mathrm{MCEK}}D^{\alpha}_{0+;[\sigma,\eta]}y\right)(1) = 0. \tag{4.10}$$

A similar condition is given at $t = 0$.

Case 4. We now consider $z(t) = t$, $w(t) = (t+1)$, $\alpha = 1$, and $y(0) = y_0$, but $y(1)$ is not specified. Earlier we specified that $0 < \alpha < 1$. Condition $\alpha = 1$ leads to an integer order system. Strictly speaking, replacing $\alpha = 1$ in a formulation for $0 < \alpha < 1$ is not straight forward, but it works. Alternatively, for this case, one can derive the Euler-Lagrange equation and the terminal conditions using variational calculus for integer order system. A more general case of $\alpha = 1$ but arbitrary $z(t)$ and $w(t)$ would be considered in the future. The above condition is considered here to emphasize a point related to terminal condition. In this case, we have $D^{1}_{0+;[1;t,2]} = D_{[1;t,L]}$, $D^{1}_{1-;[1;t+1,1]} = D_{[1;t+1,R]}$ and $I^{0}_{1-;[1;t+1]} = 1$. The Euler-Lagrange equation is

$$ D_{[1;t+1,R]}\left(D_{[1;t+1,L]}y\right) - c(t) = 0. \tag{4.11}$$

and we get the natural boundary condition at $t = 1$ as

$$ \left(D_{[1;t+1,L]}y\right)(1) = 0. \tag{4.12}$$

We further have $D_{[1;t+1,L]}y = (1/(t+1))D((t+1)y) = y'(t) + ((y(t))/(t+1))$. Thus, the natural boundary condition at $t = 1$ is given as

$$y'(1) + \frac{y(1)}{2} = 0. \tag{4.13}$$

Note that in this case, the boundary condition at $t = 1$ contains both $y(1)$ and $y'(1)$ even when terms y and $D_{[1;t+1,L]}y$ do not appear in coupled form in the functional. This is typical of the generalized fractional derivatives introduced here.

At this stage we would like to emphasize the following two points. First, we have considered only a few type of $z(t)$ and $w(t)$ functions. However, these functions can be selected from a larger set of functions. Second, generally, finding a closed form solution of an Euler-Lagrange equation resulting from a fractional variational formulation is difficult. Finding a closed form solution to a fractional variational problem formulated in terms of generalized fractional derivatives would be even more difficult, and in most cases, it will depend on the functions $z(t)$ and $w(t)$.

5. Additional Remarks

The fractional variational formulations developed here can be extended in many directions. In this regard, we emphasize that the additional remarks made in [38] are equally applicable here also. Many of the extensions of the formulations presented here will be considered in the papers to follow. We list some of them here.

First, we have considered one dimensional domain only. The formulation above could be easily extended to multi dimensional domains. For example, by replacing t with x_μ, $\mu = 1, 2, 3$ and 4, we will develop formulations for field problems which will allow us to develop fractional classical and quantum field theory.

Second, the formulations developed here could be extended to symmetric and antisymmetric fractional derivatives. Depending on whether Type-1 or Type-2 fractional derivatives are selected, two types of symmetric and two types of antisymmetric fractional derivatives can be defined. We call them Type-3 and Type-4 fractional derivatives, and they are defined as follows.

Type-3 symmetric fractional derivatives

$${}^{S}D^{\alpha}_{;[z;w,3]}y = \frac{1}{2}\left(D^{\alpha}_{a+;[z;w,1]} + (-1)^{n} D^{\alpha}_{b-;[z;w,1]}\right)y, \tag{5.1}$$

Type-3 antisymmetric fractional derivatives

$${}^{A}D^{\alpha}_{;[z;w,3]}y = \frac{1}{2}\left(D^{\alpha}_{a+;[z;w,1]} - (-1)^{n} D^{\alpha}_{b-;[z;w,1]}\right)y, \tag{5.2}$$

Type-4 symmetric fractional derivatives

$${}^{S}D^{\alpha}_{;[z;w,4]}y = \frac{1}{2}\left(D^{\alpha}_{a+;[z;w,2]} + (-1)^{n} D^{\alpha}_{b-;[z;w,2]}\right)y, \tag{5.3}$$

Type-4 antisymmetric fractional derivatives

$$ {}^{A}D^{\alpha}_{;[z;w,4]}y = \frac{1}{2}\Big(D^{\alpha}_{a+;[z;w,2]} - (-1)^{n} D^{\alpha}_{b-;[z;w,2]}\Big)y, \tag{5.4}$$

where $n-1 < \alpha < n$. The symmetric fractional derivatives ${}^{S}D^{\alpha}_{;[z;w,3]}y$ and ${}^{S}D^{\alpha}_{;[z;w,4]}y$ are similar to the Riesz and Riesz-Caputo fractional derivatives defined in [38]. To demonstrate how a problem can be defined and formulated in terms of these fractional derivatives, let us define a functional as

$$ J[y] = \int_a^b F\Big(t, y, {}^{S}D^{\alpha}_{;[z;w,4]}y\Big)(t) z'(t)dt. \tag{5.5}$$

Since ${}^{S}D^{\alpha}_{;[z;w,4]}y$ is a linear combination of $D^{\alpha}_{a+;[z;w,2]}y$ and $D^{\alpha}_{b-;[z;w,2]}y$, and therefore it can be given by (3.11), and accordingly the corresponding Euler-Lagrange equation is given by (3.12). However, in this case, we have

$$ \frac{\partial F}{\partial D^{\alpha}_{a+;[z;w,2]}y} = \frac{1}{2}\frac{\partial F}{\partial {}^{S}D^{\alpha}_{;[z;w,4]}y}, \qquad \frac{\partial F}{\partial D^{\alpha}_{b-;[z;w,2]}y} = \frac{1}{2}(-1)^{n}\frac{\partial F}{\partial {}^{S}D^{\alpha}_{;[z;w,4]}y}. \tag{5.6}$$

Using (3.12), (5.1), and (5.4), we obtain the Euler-Lagrange equation for this case as

$$ \frac{\partial F}{\partial y} + (-1)^{n} {}^{S}D^{\alpha}_{;[z;w,3]}\frac{\partial F}{\partial {}^{S}D^{\alpha}_{;[z;w,4]}y} = 0. \tag{5.7}$$

The natural boundary conditions can be obtained by some similar manipulations.

The above approach could also be applied to functionals defined in terms of sequential generalized fractional derivatives. To demonstrate this, consider the following functional:

$$ J[y] = \int_a^b F\Big(t, y, D^{\alpha}_{a+;[z;w,2]}y, D^{\alpha}_{a+;[z;w,2]}D^{\alpha}_{a+;[z;w,2]}y\Big)(t)z'(t)dt, \tag{5.8}$$

where $D^{\alpha}_{a+;[z;w,2]}D^{\alpha}_{a+;[z;w,2]}y$ is a generalized sequential fractional derivative. To find the extremum of this functional, define, $y_1(t) = y(t)$, and $y_2(t) = (D^{\alpha}_{a+;[z;w,2]}y)(t)$. Substituting these definitions in (5.8), we obtain the functional as

$$ J[y] = \int_a^b F\Big(t, y_1, D^{\alpha}_{a+;[z;w,2]}y_1, D^{\alpha}_{a+;[z;w,2]}y_2\Big)(t)z'(t)dt. \tag{5.9}$$

This equation is in the same form as that given by (3.23). Accordingly, (3.24) and (3.25) provide the necessary Euler-Lagrange equation and the natural boundary conditions for this problem.

Third, we define the Hilfer type two parameter generalized fractional derivatives as follows.

Left/Forward Hilfer type generalized two parameter fractional derivative

$$\left({}^{H}D^{\alpha,\beta}_{a+;[z;w]}y\right)(t) = \left(I^{(1-\beta)(n-\alpha)}_{a+;[z;w]}D^{n}_{[z;w,L]}I^{\beta(n-\alpha)}_{a+;[z;w]}\right)y(t). \tag{5.10}$$

Right/Backward Hilfer type generalized two parameter fractional derivative

$$\left({}^{H}D^{\alpha,\beta}_{b-;[z;w]}y\right)(t) = \left(I^{(1-\beta)(n-\alpha)}_{b-;[z;w]}D^{n}_{[z;w,R]}I^{\beta(n-\alpha)}_{b-;[z;w]}\right)y(t). \tag{5.11}$$

Note that for $\beta = 0$ and 1, (5.10) and (5.11) lead to, respectively, Type-1 and Type-2 generalized fractional derivative. Therefore, (5.10) and (5.11) can be thought of as an interpolation (in an extended sense—due to lack of a proper terminology) between Type-1 and Type-2 fractional derivatives. By setting $z(t) = w(t) = 1$, it can be shown that (5.10) and (5.11) reduce to the left/forward and right/backward two parameter fractional derivatives defined in [38], in which it was demonstrated that by taking different values for the two parameters α and β, we obtain Riemann-Liouville and Caputo fractional derivatives as special cases of the two-parameter fractional derivatives. Since $z(t)$ and $w(t)$ can be selected from a large set of functions, the generalized two-parameter fractional derivatives defined by (5.10) and (5.11) provide a larger set of two-parameter fractional derivatives.

One of the most important keys to developing a fractional variational formulation or for that matter any variational formulation is to develop an integration by parts formula for the associated differential operators. Once that is done, many of the standard techniques from variational calculus can be used to develop the desired formulations. The fractional integration by parts formula for the generalized two parameter fractional derivatives can be developed using (2.69) to (2.72), (5.10), and (5.11). Note that the operators $({}^{H}D^{\alpha,\beta}_{a+;[z;w,1]})[({}^{H}D^{\alpha,\beta}_{b-;[z;w,1]})]$ could be thought of as a combination of three sequential operators $I^{(1-\beta)(n-\alpha)}_{a+;[z;w]}$, $D^{n}_{[z;w,L]}$, and $I^{\beta(n-\alpha)}_{a+;[z;w]}$ $[I^{(1-\beta)(n-\alpha)}_{b-;[z;w]}$, $D^{n}_{[z;w,R]}$, and $I^{\beta(n-\alpha)}_{b-;[z;w]}]$. These operators could be combined differntly, and (2.69) to (2.72) could be applied to the resulting forms to obtain different integration by parts formulas. Accordingly, one would also obtain a few different sets of geometric and natural boundary conditions. This issue will be further considered in the future. However, we present one such integration by parts formula here. For simplicity, we take $0 < \alpha < 1$. It can be shown that the Hilfer type generalized two-parameter fractional derivatives $({}^{H}D^{\alpha,\beta}_{a+;[z;w,1]}y)$ and $({}^{H}D^{\alpha,\beta}_{b-;[z;w,1]}y)$ satisfy the following integration by parts formula:

$$\begin{aligned}\int_a^b z'(t)f(t)\left({}^{H}D^{\alpha,\beta}_{a+;[z;w]}g\right)(t)dt &= \int_a^b z'(t)g(t)\left({}^{H}D^{\alpha,(1-\beta)}_{b-;[z;w]}f\right)(t)dt \\ &\quad + \left[\left(I^{(1-\beta)(1-\alpha)}_{b-;[z;w]}f\right)(t)\left(I^{\beta(1-\alpha)}_{a+;[z;w]}g\right)(t)\right]\Big|_a^b.\end{aligned} \tag{5.12}$$

Integration by parts formulas for arbitrary $\alpha > 0$ and their other forms can be derived in a similar fashion. Extension of this to a sequential generalized two-parameter fractional derivatives is straight forward.

Fourth, we now define a generalized three-parameter fractional derivative as follows

$$\left(D_{[z;w]}^{\alpha,\beta,\gamma}y\right)(t) = \gamma\left({}^{H}D_{a+;[z;w]}^{\alpha,\beta}y\right)(t) + (1-\gamma)(-1)^{n}\left({}^{H}D_{b-;[z;w]}^{\alpha,\beta}y\right)(t). \tag{5.13}$$

This derivative is a weighted average of the left and the right Hilfer type generalized two-parameter fractional derivative. By setting $\gamma = 1/2$, we obtain a generalized symmetric two-parameter fractional derivative. An integration by parts formula for this derivative can be developed by realizing that the derivative operator $D_{b-;[z;w]}^{\alpha,\beta,\gamma}$ is a linear combination of the operators ${}^{H}D_{a+;[z;w]}^{\alpha,\beta}$, and ${}^{H}D_{b-;[z;w]}^{\alpha,\beta}$ and following the technique used to derive (5.7). Since many forms are possible, integration by parts formula for Hilfer type generalized two-parameter fractional derivatives, we will also have several forms for integration by parts formula for the generalized three-parameter fractional derivative. In particular, (5.12) and (5.13) lead to

$$\begin{aligned}\int_{a}^{b} z'(t)f(t)\left(D_{[z;w]}^{\alpha,\beta,\gamma}g\right)(t)dt = \int_{a}^{b} &z'(t)g(t)\left(D_{[z;w]}^{\alpha,(1-\beta),(1-\gamma)}f\right)(t)dt\\ &+\left[\gamma\left(I_{b-;[z;w]}^{(1-\beta)(1-\alpha)}f\right)(t)\left(I_{a+;[z;w]}^{\beta(1-\alpha)}g\right)(t)\right.\\ &\left.\quad-(1-\gamma)\left(I_{b-;[z;w]}^{\beta(1-\alpha)}g\right)(t)\left(I_{a+;[z;w]}^{(1-\beta)(1-\alpha)}f\right)(t)\right]\Big|_{a}^{b}.\end{aligned} \tag{5.14}$$

Equations (5.12) and (5.14) and their equivalent forms can be used to develop the entire generalized fractional variational calculus in terms of generalized two- and three-parameter fractional derivatives.

In [38], it was discussed that one can develop fractional derivatives containing many more parameters. For example, one can define a further generalization of the Hilfer type fractional derivatives as follows.

Left/Forward Hilfer type generalized multiparameter fractional derivative

$$\left({}^{H}D_{a+;[z;w]}^{\alpha,\beta_0\cdots\beta_n}y\right)(t) = \left(I_{a+;[z;w]}^{\beta_0(n-\alpha)}D_{[z;w,L]}I_{a+;[z;w]}^{\beta_1(n-\alpha)}D_{[z;w,L]}\cdots I_{a+;[z;w]}^{\beta_n(n-\alpha)}y\right)(t). \tag{5.15}$$

Right/Backward Hilfer type generalized multiparameter fractional derivative

$$\left({}^{H}D_{b-;[z;w]}^{\alpha,\beta_0\cdots\beta_n}y\right)(t) = \left(I_{b-;[z;w]}^{\beta_0(n-\alpha)}D_{[z;w,R]}I_{b-;[z;w]}^{\beta_1(n-\alpha)}D_{[z;w,R]}\cdots I_{b-;[z;w]}^{\beta_n(n-\alpha)}y\right)(t), \tag{5.16}$$

where one may require that

$$\beta_0 + \beta_1 + \cdots + \beta_n = 1. \tag{5.17}$$

Similarly, one can derive multiparameter fractional derivatives of type given by (5.13) as follows:

$$\left(D_{[z;w]}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2}y\right)(t) = \left(\gamma_1{}^{H}D_{a+;[z;w]}^{\alpha_1,\beta_1}y + \gamma_2{}^{H}D_{b-;[z;w]}^{\alpha_2,\beta_2}y\right)(t). \tag{5.18}$$

Here, parameters α_1, α_2, β_1, β_2, γ_1, and γ_2 could all be arbitrary. Further, the generalized fractional derivative defined in (5.18) could have more than two Hilfer type fractional derivative terms. Integration by parts formulas and the associated fractional variational formulations for these multi-parameter fractional derivatives can be obtained using the procedure discussed above. It should be emphasized here that the domains of these generalized fractional derivatives are much larger than those derivatives which do not consider $z(t)$ and $w(t)$.

Fifth, the fractional integrals and the derivatives are defined here using kernels of type $K(t,\tau) = (t-\tau)^{\alpha-1}$. However, this need not be the case. In [40], several formulations are presented that use kernels other than $K(t,\tau) = (t-\tau)^{\alpha-1}$. Such kernels can also be considered here, and using these kernels one can develop a more general variational calculus. To demonstrate this, let us define two operators K_L^α and K_R^α as

$$K_L^\alpha f(t) = [w(t)]^{-1}\int_a^t w(\tau)z'(\tau)k_\alpha[\phi(t),\phi(\tau)]f(\tau)d\tau,$$
$$K_L^\alpha f(t) = [w(t)]\int_a^t [w(\tau)]^{-1}z'(\tau)k_\alpha[\phi(\tau),\phi(t)]f(\tau)d\tau, \tag{5.19}$$

where $k_\alpha[\phi(t),\phi(\tau)]$ is a kernel which may depend on α. Operators K_L^α and K_R^α are like generalized fractional integral operators. Strictly speaking, the operators resulting from arbitrary kernels do not qualify to be fractional integral operators. For example, for an arbitrary kernel, the semi-group property may not be satisfied. However, in special cases, $K_{P_1}^\alpha$ and $K_{P_2}^\alpha$ do lead to fractional integral operators. As a matter of fact, for $k_\alpha[z(t),z(\tau)] = [z(t)-z(\tau)]^{\alpha-1}/\Gamma(\alpha)$, K_L^α and K_R^α are generalized fractional integral operators.

It can also be demonstrated that operators K_L^α and K_R^α satisfy the following integration by parts formula:

$$\int_a^b z'(t)g(t)K_L^\alpha f(t)dt = \int_a^b z'(t)f(t)K_R^\alpha g(t)dt. \tag{5.20}$$

For simplicity, let us consider $\alpha \in (0,1)$. We can further define differential operators A_L^α, A_R^α, B_L^α, and B_R^α as follows:

$$A_L^\alpha f(t) = D_{[z,w,L]}K_L^{1-\alpha}f(t),$$
$$A_R^\alpha f(t) = D_{[z,w,R]}K_R^{1-\alpha}f(t), \tag{5.21}$$

$$B_L^\alpha f(t) = K_L^{1-\alpha}D_{[z,w,L]}f(t),$$
$$B_R^\alpha f(t) = K_R^{1-\alpha}D_{[z,w,R]}f(t). \tag{5.22}$$

These operators are like generalized differential operators, but they are not differential operators, largely because K_L^α and K_R^α may not satisfy the semigroup property. However, when $k_\alpha[\phi(t),\phi(\tau)] = [z(t)-z(\tau)]^{\alpha-1}/\Gamma(\alpha)$, operators A_L^α, A_R^α, B_L^α, and B_R^α indeed represent generalized fractional differential operators.

It can be demonstrated that operators A_L^α, A_R^α, B_L^α, and B_R^α satisfy the following integration by parts formula:

$$\int_a^b z'(t)g(t)A_L^\alpha f(t)dt = \int_a^b z'(t)f(t)B_R^\alpha g(t)dt + \text{boundary terms},$$
$$\int_a^b z'(t)g(t)A_R^\alpha f(t)dt = \int_a^b z'(t)f(t)B_L^\alpha g(t)dt + \text{boundary terms}. \tag{5.23}$$

Here, we have considered $\alpha \in (0,1)$. However, one can define the above operators and derive their properties for $\alpha > 0$. Once integration by parts formula is developed, many aspects of fractional variational calculus, fractional mechanics (including fractional Lagrangian, Hamiltonian, action principle, and adjoint operator theory), fractional optimal control can be developed in terms of operators K_L^α, K_R^α, A_L^α, A_R^α, B_L^α, and B_P^α. Several such formulations are presented in [40]. However, the operators defined here contain two additional functions $z(t)$ and $w(t)$, and therefore, they are more general than those defined in [40].

Sixth, several different kernel functions have been considered to develop generalized fractional operators and generalized fractional calculus (see [12, 13, 39, 41]). All these operators can be recast and the fractional calculus can be further generalized using the weight/scaling function proposed here. For example, most of the integral equations given in [41] can be recast in terms of generalized operators. To demonstrate this, we consider two examples both of which are given in [41].

As a first example, consider the integral equation

$$\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x-t}} y(t)dt = f(x), \tag{5.24}$$

the solution of which is given as

$$y(x) = \frac{1}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{e^{-\lambda t}}{\sqrt{x-t}} f(t)dt. \tag{5.25}$$

Equation (5.24) can be written in terms of a generalized fractional integral operator as

$$\Gamma\left(\frac{1}{2}\right)\left(I^{1/2}_{a;[t,e^{-\lambda t}]} y\right)(x) = f(x). \tag{5.26}$$

By applying the operator $D^{1/2}_{a;[t,e^{-\lambda t},1]}$ from left on both sides of (5.26) and noticing that this operator is the left inverse of the operator $I^{1/2}_{a;[t,e^{-\lambda t}]}$, we obtain

$$y(x) = \frac{1}{\Gamma(1/2)}\left(D^{1/2}_{a;[t,e^{-\lambda t},1]} f\right)(x). \tag{5.27}$$

It can be verified that (5.25) and (5.27) are indeed the same. Note that for $\lambda = 0$, (5.24) and (5.26) represent two forms of Abel's equation and (5.25) and (5.27) represent two forms of the solution.

As a second example, consider the integral equation

$$\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{\ln(x/t)}} y(t)dt = f(x), \tag{5.28}$$

the solution of which is given as

$$y(x) = \frac{1}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{e^{-\lambda t}}{t\sqrt{\ln(x/t)}} f(t)dt. \tag{5.29}$$

Equation (5.28) can be written in terms of a generalized fractional integral operator as

$$\Gamma(1/2)\left(I^{1/2}_{a;[\ln(t),e^{-\lambda t}]} y\right)(x) = f(x). \tag{5.30}$$

By applying the operator $D^{1/2}_{a;[\ln(t),e^{-\lambda t},1]}$ from left on both sides of (5.29), dividing the result by $x\Gamma(1/2)$, and noticing that this operator is the left inverse of the operator $I^{1/2}_{a;[\ln(t),e^{-\lambda t}]}$, we obtain

$$y(x) = \frac{1}{x\Gamma(1/2)}\left(D^{1/2}_{a;[\ln(t),e^{-\lambda t},1]} f\right)(x). \tag{5.31}$$

It can be verified that (5.29) and (5.31) are indeed the same.

The above observation will have several consequences: (a) it will allow us to write many integral equations in terms of generalized fractional integral and differential operators, and use the properties of these operators to find the solution of the integral equations using the properties of the generalized fractional operators in elegant way. (b) It will initiate a new class of generalized differential equations, and blur the distinction between differential and integral equations. (c) It will allow us to write many of the equations, physical, and social laws, and so forth, in the field of science, engineering, economics, and bioengineering in terms generalized fractional operators, and thus broaden the area where fractional calculus could be applied. (d) It will also impact the history of fractional calculus. In the history of fractional calculus, Abel is attributed for solving a practical fractional calculus problem. Recently, many integral equations have been recast in terms of fractional integrals and derivative operators. The new operators and their properties proposed above will allow one to write equations and physical and social laws in terms of generalized fractional operators. Thus, while the researchers were proposing these equations and physical and social laws, they were indeed, indirectly, proposing applications of generalized fractional calculus; and when they solved the associated integral equations, they were indeed developing analytical tools for fractional calculus.

Seventh, we have largely focused here on the discrete systems. However, it could be easily extended to field/distributed-order-dynamic-systems (see [21]). In addition, we have largely dealt with finite domains. However, operators such as Weyl fractional derivatives could be generalized in the same way.

Finally, note that we developed a generalized variational calculus in [38] which opened several new areas for further investigations. In particular, we listed the following areas: analytical and numerical solutions of the resulting equations, applications to fractional optimal control, time delay systems and applications in physics (i.e., Lagrangian and Hamiltonian formulations, Fractional Schrödinger equation and fractional quantum mechanics, relativistic fractional quantum mechanics, quantization of fractional systems containing fractional derivatives of multiple order, fractional statistical mechanics, fractional Maxwell's and wave equations, and fractional Lie algebra). As a matter of fact, fractional variational calculus has already been applied in most of these fields. Since most of the fractional derivatives used in these fields are special cases of the generalized derivatives, it is clear that the generalized fractional derivatives proposed here would also apply to many of the fields listed above. These research areas will be pursued in the future.

6. Conclusions

In this paper, we first introduced some one-parameter FDs, and listed some of their properties useful in developing FVC. We then introduced new one-parameter GFDs, developed their properties, and used them to develop several parts of FVC. These parts include fractional variational formulations for functionals containing one function and multi functions, specified and unspecified terminal conditions, multiorder of FDs, holonomic, parametric, and dynamic constraints. These parts also include formulations for fractional Lagrangian and Hamiltonians and fractional optimal controls. Subsequently, we introduced two- and three-parameters GFDs and developed some Euler-Lagrange type necessary conditions, and pointed out how other multiparameter fractional derivatives could be developed. We also discussed many areas where the formulations developed here could be applied.

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