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Research Article

An Oseen Two-Level Stabilized Mixed Finite-Element Method for the 2D/3D Stationary Navier-Stokes Equations

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We investigate an Oseen two-level stabilized finite-element method based on the local pressure projection for the 2D/3D steady Navier-Stokes equations by the lowest order conforming finite-element pairs (i.e., $Q_1 - P_0$ and $P_1 - P_0$). Firstly, in contrast to other stabilized methods, they are parameter free, no calculation of higher-order derivatives and edge-based data structures, implemented at the element level with minimal cost. In addition, the Oseen two-level stabilized method involves solving one small nonlinear Navier-Stokes problem on the coarse mesh with mesh size H, a large general Stokes equation on the fine mesh with mesh size $h = O(H)^2$. The Oseen two-level stabilized finite-element method provides an approximate solution (u^h, p^h) with the convergence rate of the same order as the usual stabilized finite-element solutions, which involves solving a large Navier-Stokes problem on a fine mesh with mesh size h. Therefore, the method presented in this paper can save a large amount of computational time. Finally, numerical tests confirm the theoretical results. Conclusion can be drawn that the Oseen two-level stabilized finite-element method is simple and efficient for solving the 2D/3D steady Navier-Stokes equations.

1. Introduction

There are numerous works devoted to the development of efficient stable mixed finite-element methods for solving the Navier-Stokes equations. It is a well-known fact that the lowest-order conforming elements spaces $P_1 - P_0$ (linear velocity, constant pressure) and $Q_1 - P_0$ (bilinear velocity, constant pressure), which are the most attractive choice from an implementation point of view, fail to satisfy the inf-sup condition [1]. In order to make full use of these finite-element spaces, some kinds of stabilized finite-element methods for the two- or three-dimensional Stokes flows and two dimensional Navier-Stokes flows appeared in [2–13]. However, these stabilized methods depend on the stabilization parameters, the derivatives of the pressure, the edge-based data structures, and nested mesh. Examples include the local and global pressure jump formulations for the bilinear-constant pair where the constraint is relaxed by using the jumps of the pressure across element interfaces. Moreover, there is no satisfactory way to choose the optimal parameters for any given mesh.

The idea of the stabilized finite-element method based on the local pressure projection is derived from [14] for the Stokes equations. This method differs from existing stabilization techniques and avoids approximation of derivatives, specification of mesh-dependent parameters, interface boundary data structures, and evaluated locally at the element level. This stabilized technique has been extended to solve the two-dimensional Navier-Stokes equation by Wang et al. [15].

In the last two decades, two-level strategy has been studied for efficiently solving the nonlinear partial differential and sometimes also some linear problems. The basic idea of two-level method is to compute an initial approximation on a very coarse mesh (involving the solution of a very small number of nonlinear equations). Moreover, the fine structures are captured by solving one linear system. If we choose the appropriate proportion between the coarse and fine scale, then the two-level stabilized methods have the same convergence rate as the usual stabilized finite-element methods. Some ideas of two-level method can be found in the works of Xu [16, 17], Layton and Tobiska [18], Layton [19], Layton and Lenferink [20], He et al. [21–24], and Li [25].

Recently, He and Li [22] and Li [25] have combined simplified and Newton's two-level iterative techniques with different stabilized finite-element methods for solving the 2D Navier-Stokes problems. The method we study in this paper is to combine the new stabilized finite-element method in [14] with the two-level method based on the Oseen iterative technique by the lowest-order conforming finite-element pairs $Q_1 - P_0$ and $P_1 - P_0$ for solving the 2D/3D stationary Navier-Stokes problems. We present theoretical analysis under the assumption of the uniqueness condition. The results of Theorem 4.1 in Section 4 show that if we choose the coarse mesh scale H and the fine mesh scale h satisfying $h = O(H^2)$, the method we study is of the convergence rate of the same order as the usual stabilized finite-element method. However, our method is more simple and efficient. The numerical experiments further confirm the theoretical results.

This paper is organized as follows. In Section 2, an abstract functional setting for the steady Navier-Stokes equations is given, together with some basic notations and assumptions. Section 3 is to recall the stabilized finite-element method based on the local pressure projection [14] for the steady Stokes equations. we borrow the well-posedness and the optimal error estimate of the stabilized finite-element method for the steady Navier-Stokes equations in [15]. In Section 4, the uniform stability and convergence of the Oseen two-level stabilized finite method are proved. In Section 5, a series of numerical experiments is given to illustrate the theoretical results.

2. Functional Setting of the Navier-Stokes Equations

Let Ω be a bounded domain in R^d $(d = 2, 3)$, assumed to have a Lipschitz continuous boundary Γ and to satisfy a further condition stated in (H_0) below. The steady incompressible Navier-Stokes equations are considered as follows:

$$-\nu\Delta u + (u\cdot\nabla)u + \frac{1}{2}(\operatorname{div} u)u + \nabla p = f \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad \int_\Omega p dx = 0. \tag{2.2}$$

Here $u : \Omega \to R^d$ and $p : \Omega \to R$ are the velocity and pressure, $\nu > 0$ is the viscosity, and f represents the body forces.

For the mathematical setting of problem (2.1)-(2.2), we introduce the following Sobolev spaces:

$$X = H_0^1(\Omega)^d, \qquad Y = L^2(\Omega)^d, \qquad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q(x)dx = 0\right\},$$
$$D(A) = H^2(\Omega)^d \cap X. \tag{2.3}$$

We make a regularity assumption on the Stokes problem as follows.

Assumption 2.1 (H_0). For a given $g \in Y$ and the Stokes problem

$$-\Delta v + \nabla q = g, \qquad \operatorname{div} v = 0 \text{ in } \Omega, \quad v|_\Gamma = 0, \tag{2.4}$$

we assume that (v, q) satisfy the following regularity result:

$$\|v\|_2 + \|q\|_1 \le \kappa\|g\|_0, \tag{2.5}$$

where $\|\cdot\|_i$ is the norm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^d$, $i = 0, 1, 2$, as appropriate, and κ is a positive constant depending only on Ω, which may stand for different value at its different occurrences. Subsequently, the positive constants κ and c (with or without a subscript) will depend only on the data (ν, Ω, f). Because the norm and seminorm are equivalent on $H_0^1(\Omega)^d$, we use the same notation $\|u\|_1$ for them.

Now, the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$ are defined, respectively, by

$$a(u, v) = \nu(\nabla u, \nabla v), \quad \forall u, v \in X, \qquad d(v, q) = (q, \operatorname{div} v), \quad \forall (v, q) \in (X, M). \tag{2.6}$$

Also, a generalized bilinear form $\mathcal{B}((\cdot,\cdot);(\cdot,\cdot))$ on $(X, M) \times (X, M)$ is defined by

$$B((u, p);(v, q)) = a(u, v) - d(v, p) + d(u, q). \tag{2.7}$$

Then there hold the following estimates for the bilinear term $B((\cdot,\cdot);(\cdot,\cdot))$ ([2–4]):

$$\begin{aligned} \left|\mathcal{B}((u,p);(u,p))\right| &= \nu\|u\|_1^2, \\ \left|\mathcal{B}((u,p);(v,q))\right| &\le \alpha(\|u\|_1+\|p\|_0)(\|v\|_1+\|q\|_0), \\ \sup_{(v,q)\in(X,M)}\frac{\left|\mathcal{B}(u,p);(v,q)\right|}{\|v\|_1+\|q\|_0} &\ge \beta(\|u\|_1+\|p\|_0), \end{aligned} \tag{2.8}$$

for all $(u,p),(v,q)\in(X,M)$ and constants $\alpha,\beta>0$. Moreover, we define the trilinear form

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v,w)+\frac{1}{2}((\operatorname{div}\ u)v,w) \\ &= \frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v), \quad \forall u,v,w\in X. \end{aligned} \tag{2.9}$$

By the above notations and Hölder's inequality, there hold the following estimates:

$$b(u,v,w)=-b(u,w,v), \quad \forall u\in X,\ v,w\in X, \tag{2.10}$$

$$|b(u,v,w)|\le\frac{1}{2}c_0\|u\|_0^{1/2}\|u\|_1^{1/2}\left(\|v\|_1\|w\|_0^{1/2}\|w\|_1^{1/2}+\|v\|_0^{1/2}\|v\|_1^{1/2}\|w\|_1\right), \quad \forall u,v,w\in X, \tag{2.11}$$

$$|b(u,v,w)|+|b(v,u,w)|+|b(w,u,v)|\le c_1\|u\|_1\|v\|_2\|w\|_0, \quad \forall u\in X,\ v\in D(A),\ w\in Y. \tag{2.12}$$

Also, the Poincare inequality holds:

$$\|v\|_0\le\gamma_0\|v\|_1, \tag{2.13}$$

where c_0, c_1, and γ_0 are positive constants depending only on Ω.

For a given $f\in Y$, the variational formulation of problem (2.1)-(2.2) reads as follows: find $(u,p)\in(X,M)$ such that

$$\mathcal{B}((u,p);(v,q))+b(u,u,v)=(f,v), \quad \forall(v,q)\in(X,M). \tag{2.14}$$

The following existence and uniqueness result is classical [26, 27].

Theorem 2.2. *Assume that $\nu>0$ and $f\in Y$ satisfy the following uniqueness condition:*

$$1-\frac{c_0\gamma_0^2}{\nu^2}\|f\|_{-1}>0. \tag{2.15}$$

Then the variational problem (2.14) *admits a unique solution* $(u, p) \in (D(A) \cap X, H^1(\Omega) \cap M)$ *such that*

$$\|u\|_1 \leq \frac{\gamma_0}{\nu} \|f\|_{-1}, \qquad \|u\|_2 + \|p\|_1 \leq \kappa \|f\|_0. \tag{2.16}$$

3. The Stabilized Finite-Element Method

In this section, we focus on the stabilized method proposed by [14] for the Stokes equations. Let h be a real positive parameter tending to zero. Finite-element subspace (X_h, M_h) of (X, M) is characterized by $\tau_h = \tau_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles, quadrilaterals, tetrahedrons, or hexahedrons K, assumed to be regular in the usual sense. The set of all interelement boundaries will be denoted by Γ_h, and the norm will be endowed as follows:

$$\|u\|_{\Gamma_h} = \left(\sum_{\gamma_f \in \Gamma_h} \int_{\gamma_f} u^2 dS \right)^{1/2}. \tag{3.1}$$

In this paper, the finite element of velocity is defined by setting

$$R_1(K) = \begin{cases} P_1(K), & \text{if } K \text{ is triangular or tetrahedron}, \\ Q_1(K), & \text{if } K \text{ is the quadrilateral or hexahedron}. \end{cases} \tag{3.2}$$

Then, the finite-element pairs are coupled as follows:

$$\begin{gathered} X_h = \{ v \in X ; v_i|_K \in R_1(K),\ i, \ldots, d \}, \\ M_h = \left\{ q \in M : q\big|_K \in P_0(K),\ \forall K \in \tau_h \right\}. \end{gathered} \tag{3.3}$$

Also, well-known approximation results are presented as follows [26, 27].

(H1) Approximation properties

$$\begin{gathered} \|v - I_h v\|_0 + h^{1/2} \|v - I_h v\|_{\Gamma_h} + h \|v - I_h v\|_1 \leq \kappa h^2 \|v\|_2, \quad \forall v \in D(A), \\ \|q - J_h q\|_0 \leq \kappa h \|q\|_1, \quad \forall q \in H^1(\Omega) \cap M. \end{gathered} \tag{3.4}$$

(H2) Inverse inequality

$$\begin{gathered} \|v_h\|_1 \leq \kappa h^{-1} \|v_h\|_0, \quad \forall v_h \in X_h, \\ \| [q_h] \|_{\Gamma_h} \leq \kappa h^{-1/2} \|q_h\|_0, \quad \forall q_h \in M_h. \end{gathered} \tag{3.5}$$

Here, $I_h : D(A) \rightarrow X_h$ is the interpolation operator, and $J_h : H^1(\Omega) \cap M \rightarrow M_h$ is the L^2-orthogonal projection.

Note that neither of these methods are stable in the standard Babuška-Brezzi sense; $P_1 - P_0$ triangle or tetrahedron "locks" on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral or hexahedron is the most infamous example of unstable mixed method [28]. Obviously, the following Lemma 3.1 manifest the insufficiency of the Babuška-Brezzi condition.

Lemma 3.1 (see [14]). *There exist positive constants κ_1 and κ_2 whose values are independent of h and such that*

$$\sup_{v_h \in X_h} \frac{\int_\Omega p_h \nabla \cdot v_h d\Omega}{\|v_h\|_1} \geq \kappa_1 \|p_h\|_0 - \kappa_2 h^{1/2} \|[p_h]\|_{\Gamma_h}, \quad \forall p_h \in M_h. \tag{3.6}$$

In order to counterbalance the terms $h^{1/2}\|[p_h]\|_{\Gamma_h}$ in (3.6) and compensate the inf-sup deficiency of the lowest-order conforming finite-element pairs, the bilinear form $G(\cdot,\cdot)$ can be defined:

$$G(p,q) = (p - \Pi p, q - \Pi q), \tag{3.7}$$

where the local pressure projection $\Pi : L^2(\Omega) \rightarrow R_1$.

Lemma 3.2 (see [14]). *There exist positive constants C_1 and C_2 whose values are independent of h and such that*

$$C_2 h^{1/2} \|[p_h]\|_{\Gamma_h} \leq \|p_h - \Pi p_h\|_0, \quad \forall p_h \in M_h. \tag{3.8}$$

Furthermore, it holds that

$$\sup_{v_h \in X_h} \frac{\int_\Omega p_h \nabla \cdot v_h d\Omega}{\|v_h\|_1} \geq C_1 \|p_h\|_0 - C_2 \|p_h - \Pi p_h\|_0, \quad \forall p_h \in M_h. \tag{3.9}$$

With the above notations and the choice of the velocity-pressure finite-element spaces $(X_h, M_h) \subset (X, M)$, we define the stabilized bilinear form as follows:

$$\mathcal{B}_h((u_h,p_h);(v,q)) = a(u_h,v) - d(v,p_h) + d(u_h,q) + G(p_h,q), \quad \forall (v,q) \in (X_h, M_h). \tag{3.10}$$

Then the stabilized discrete variational formulation of the Navier-Stokes problem (2.14) reads as follows: find $(u_h,p_h) \in (X_h, M_h)$ such that, for all $(v,q) \in (X_h, M_h)$,

$$\mathcal{B}_h((u_h,p_h);(v,q)) + b(u_h,u_h,v) = (f,v). \tag{3.11}$$

The stability of the stabilized finite-element method based on the local pressure projection for the Stokes equation is presented as follows.

Theorem 3.3 (see [2, 14]). *Let* (X_h, M_h) *be the lowest-order conforming finite-element spaces. Assume that*

$$\|\Pi p\|_0 \le \kappa \|p\|_0, \quad \forall p \in L^2(\Omega), \qquad \|p - \Pi p\|_0 \le \kappa h \|p\|_1, \quad \forall p \in H^1(\Omega), \tag{3.12}$$

hold. Then there exists a positive constant β, *independent of* h *and satisfying*

$$|\mathcal{B}_h((u,p);(v,q))| \le \alpha(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \quad \forall (u,p),(v,q) \in (X,M), \tag{3.13}$$

$$\beta(\|u_h\|_1 + \|p_h\|_0) \le \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{B}_h((u_h,p_h);(v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0}, \quad \forall (u_h,p_h) \in (X_h,M_h). \tag{3.14}$$

Applying Brouwer's fixed-point theorem, the following theorem can be directly given.

Theorem 3.4 (see [2, 15]). *Under the assumptions of Theorems 2.2 and 3.3, problem* (3.11) *admits a unique solution* $(u_h, p_h) \in (X_h, M_h)$ *satisfying*

$$\|u_h\|_1 \le \frac{\gamma_0}{\nu}\|f\|_0, \qquad \|p_h\|_0 \le \beta^{-1}\left(c_0\nu^{-2}\gamma_0^3\|f\|_0^2 + \gamma\|f\|_0\right). \tag{3.15}$$

For simpleness, the Stokes projection $(R_h, Q_h) : (X,M) \to (X_h, M_h)$ is defined by

$$\mathcal{B}_h((R_h(v,q), Q_h(v,q));(v_h,q_h)) = \mathcal{B}((\mathrm{v},q);(v_h,q_h)), \quad \forall (v_h,q_h) \in (X_h,M_h), \tag{3.16}$$

for all $(v,q) \in (X,M)$, and satisfies [2, 15]

$$\begin{aligned} &\|v - R_h(v,q)\|_0 + h(\|v - R_h(v,q)\|_1 + \|q - Q_h(v,q)\|_0) \\ &\quad \le \kappa h^2(\|v\|_2 + \|q\|_1), \quad \forall (v,q) \in \left(D(A), H^1(\Omega)\cap M\right). \end{aligned} \tag{3.17}$$

Thus, we can easily obtain the following optimal error estimate for the steady Navier-Stokes equations.

Theorem 3.5 (see [15]). *Under the assumptions of Theorems 2.2 and 3.3, one has*

$$\|u - u_h\|_0 + h(\|u - u_h\|_1 + \|p - p_h\|_0) \le \kappa h^2. \tag{3.18}$$

4. Oseen Two-Level Stabilized Finite-Element Method

In this section, we will present Oseen two-level stabilized finite-element method and derive optimal bound of the errors. Let H and $h \ll H$ be two real positive parameters tending to 0. Also, a coarse mesh partition $\tau_H(\Omega)$ of Ω is made as in Section 3, and a fine mesh partition $\tau_h(\Omega)$ is generated by a mesh refinement process to $\tau_H(\Omega)$. The conforming finite-element space pais (X_h, M_h) and $(X_H, M_H) \subset (X_h, M_h)$ based on the partition $\tau_h(\Omega)$ and $\tau_H(\Omega)$, respectively, are constructed in Section 3. The Oseen two-level stabilized finite approximations is applied by using the lowest finite-element pairs $P_1 - P_0$ and $Q_1 - P_0$ as follows.

Step I. Solve the nonlinear Navier-Stokes problem on a coarse mesh: find $(u_H, p_H) \in (X_H, M_H)$ such that

$$\mathcal{B}_h((u_H, p_H); (v_H, q_H)) + b(u_H, u_H, v_H) = (f, v_H), \quad \forall (v_H, q_H) \in (X_H, M_H). \tag{4.1}$$

Step II. Solve the linear Stokes problem on a fine mesh: find $(u_h, p_h) \in (X_h, M_h)$ such that

$$\mathcal{B}_h\Big(\Big(u^h, p^h\Big); (v_h, q_h)\Big) + b\Big(u_H, u^h, v_h\Big) = (f, v_h), \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{4.2}$$

Obviously, the Oseen two-level stabilized finite-element method admits a unique solution by applying the Lax-Milgram theorem. For convenience, we set $e_h = u_h - u^h$, $\eta_h = p_h - p^h$, then a sequence of error estimates for the Oseen two-Level errors $u - u^h$ and $p - p^h$ are derived as follows.

Theorem 4.1. *Under the assumptions of Theorems 2.2 and 3.3, the Oseen two-level stabilized finite-element scheme* (4.1)-(4.2) *admits a unique solution* $(u^h, p^h) \in (X_h, M_h)$ *such that*

$$\left\|u - u^h\right\|_1 + \left\|p - p^h\right\|_0 \leq \kappa\Big(h + H^2\Big). \tag{4.3}$$

Proof. Subtracting (3.11) from (4.2) yields that

$$a(e_h, v_h) - d(v_h, \eta_h) + d(e_h, q_h) + G(\eta_h, q_h) + b(u_h - u_H, u_h, v_h) + b(u_H, e_h, v_h) = 0. \tag{4.4}$$

Using (2.10) and taking $(v_h, q_h) = (e_h, \eta_h)$, then

$$\nu\|e_h\|_1^2 + G(\eta_h, \eta_h) = b(u_H - u_h, u_h, e_h). \tag{4.5}$$

For the trilinear terms, by (2.11), (2.12), (2.16), and (3.18), we estimate the following

$$\begin{aligned} |b(u_H - u_h, u_h, e_h)| &= |b(u_H - u_h, u_h - u, e_h) + b(u_H - u_h, u, e_h)| \\ &\leq \kappa(\|u_H - u_h\|_1\|u - u_h\|_1 + \|u\|_2\|u_h - u_H\|_0)\|e_h\|_1 \\ &\leq \kappa H^2\|e_h\|_1^2 \leq \frac{\nu}{2}\|e_h\|_1^2 + \kappa H^4. \end{aligned} \tag{4.6}$$

Thanks to (3.14), (4.6), and the relation $h < H$, it follows that

$$\begin{aligned} \|e\|_1 + \|\eta\|_0 &\leq \beta^{-1} \sup_{(v_h, q_h) \in (X_h, M_h)} \frac{|\mathcal{B}_h((e, \eta); (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \\ &\leq cH^2. \end{aligned} \tag{4.7}$$

Then, we deduce from a triangle inequality, (3.18), and (4.7) that

$$\begin{aligned} \left\|u-u^h\right\|_1+\left\|p-p^h\right\|_0 &\leq \|e_h\|_1+\|u-u_h\|_1+\|\eta_h\|_0+\|p-p_h\|_0 \\ &\leq \kappa\left(h+H^2\right). \end{aligned} \tag{4.8}$$

□

5. Numerical Examples

In this section, we concentrate on the performance of the one-level finite-element method and Oseen two-level finite element method described in this paper.

In order to implement these methods presented in this paper, a suitable choice of local pressure projection Π is important. The simplest way to accomplish this is to use standard finite-element projection or interpolation operators.

From a practical viewpoint, the main factors in the choice of Π are simplicity and locality, that is, computation of its action must be done at the element level using only standard nodal data structures. We recall the technique presented in Pavel Bochev [14], which is different from references [29, 30]. For a given node N_i in τ_h, let $\widehat{\Omega}_i$ denote its dual volume. Given a function $q\in L^2(\Omega)$, let q_i be the constant function on $\widehat{\Omega}_i$ that minimizes the functional

$$J_i(q)=\frac{1}{2}\int_{\widehat{\Omega}_i}(q_i-q)^2 d\Omega. \tag{5.1}$$

Then set

$$\Pi q=\sum_{i=1}^{N_{\text{nodes}}} q_i N_i(x)\in R_1, \tag{5.2}$$

where $N_i(x)$ denotes the nodal basis of R_1 and N_{nodes} is the number of nodes in τ_h. The action of the operator defined in (5.2) can be computed locally at the element level and has the same properties as the usual Clement interpolation, that is, (3.12) is satisfied. For $q\in q_h\in R_0$, the functional in (5.1) further simplifies to

$$J_i(q_h)=\sum_{K\cap\widehat{\Omega}_i\neq 0} V_i(K)(q_i-q_{kh})^2, \tag{5.3}$$

where q_{kh} is the constant value of q_h on K and $V_i(K)$ is the volume fraction of the element K that belongs to the dual cell $\widehat{\Omega}_i$ associated with the node N_i. Minimization of J_i then yields the formula

$$q_i=\frac{\sum_{K\in\widehat{\Omega}_i} V_i(K)q_{kh}}{\sum_{K\in\widehat{\Omega}_i} V_i(K)}; \tag{5.4}$$

that is, the nodal values of Πq_h are area weighted averages of the surrounding constant pressure values of q_h.

Table 1: Comparison of the one-level stabilized method with the Oseen two-level stabilized method.

1/H	1/h	$\|u-u_h\|_{H_1}/\|u\|_{H_1}$	rate	$\|p-p_h\|_{L^2}/\|p\|_{L^2}$	rate	Time (s)
3	9	0.573428	/	0.140335	/	1.00
	9	0.573514	/	0.140348	/	0.45
4	16	0.250137	1.4419	0.0709911	/	6.00
	16	0.250247	1.4414	0.071002	1.1843	1.88
5	25	0.134685	1.3872	0.0430539	1.1181	50.00
	25	0.13479	1.3864	0.0430626	1.1205	24.85
6	36	0.0832799	1.3184	0.029072	1.0769	421.20
	36	0.0833677	1.3176	0.0290787	1.0768	214.00
7	49	0.056536	1.2563	0.0210298	1.0504	2526.50
	49	0.0566021	1.2560	0.0210348	1.0504	1255.30

For the purpose of numerical comparisons, we consider the spatial domain in R^2 as $[0,1]\times[0,1]$ and viscosity $\nu = 1.0$. The velocity and pressure are designed on the same uniform triangulation of Ω. The exact solution is given by

$$\begin{aligned} u &= (u_1(x,y), u_2(x,y)), \qquad p(x,y) = 10(x-0.5), \\ u_1(x,y) &= 10x^2(1-x)^2 y(1-y)(1-2y), \\ u_2(x,y), &= -10x(1-x)(1-2x)y^2(1-y)^2, \end{aligned} \tag{5.5}$$

and f is determined by (2.1).

For simplicity, the unit square is divided into N^2 small squares, where N dedicates the number of the partition in each direction. When K is the square, $V_i(K) = 1/N^2$. Then the formula (5.4) can be given as follows:

$$q_i = \frac{\sum_{K\in\widehat{\Omega}_i} q_{kh}}{n_k}, \tag{5.6}$$

where n_k is the number of elements in $\widehat{\Omega}_i$.

To establish a reference point for the evaluation of the possible impact from the Oseen two-level stabilized finite-element method, we compute the one-level stabilized finite-element method for the steady Navier-Stokes equations on the fine mesh. The Oseen two-level stabilized finite-element solutions (u^h, p^h) are obtained by solving the solution (u_H, p_H) of the stationary Navier-Stokes equations on a coarse mesh and then solving a linear system on a fine mesh with mesh size h.

Table 1 shows the relative energy error estimates of the numerical velocity and pressure for these methods. In particular, the results of the one-level stabilized finite-element method are presented on the first line; those of the Oseen two-level stabilized finite-element method with relationship $h = O(H^2)$ on second line. From Table 1, we find that these two methods have a convergence rate of the same order, as shown in Theorem 4.1. However, the Oseen method is probably two times faster than the one-level stabilized method.

In conclusion, the Oseen two-level stabilized finite-element method is an efficient and potential method for solving the 2D/3D steady Navier-Stokes equation. It is also suitable to solve some practical engineering problems arising in the fluid dynamics. Furthermore, the methods can help to solve the two-dimensional and three-dimensional nonstationary incompressible viscous flows which will be discussed in our further work.

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