Research Article
A Generalization of Ćirić Quasicontractions

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We proved a fixed point theorem for a class of maps that satisfy Ćirić's contractive condition dependent on another function. We presented an example to show that our result is a real generalization.

1. Introduction and Preliminaries

The fixed point theorems have extensive applications in many disciplines such as in mathematics, economics, and computer science (see, e.g., [1–5]). The Banach contraction mapping principle [6] is the main motivation of this theory. A self-mapping T on a metric space X is called a contraction if, for each $x, y \in X$, there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$. Banach [6] proved that every contraction on a complete metric space has a unique fixed point.

Many authors have studied on various generalizations of Banach contraction mapping principle. For important and famous examples of such generalizations, we may cite Kannan [7], Reich [8–10], Chatterjea [11], Hardy and Rogers [12], and Ćirić [13, 14]. In this manuscript, we focused on Ćirić type contractions and generalized some of his results [13, 14].

In [14], Ćirić introduced a class of self-maps on a metric space (X, d) which satisfy the following condition:

$$d(Sx, Sy) \leq q \max\{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\}, \tag{1.1}$$

for every $x, y \in X$ and $0 \leq q < 1$. The maps satisfying Condition (1.1) are said to be *quasicontractions*.

Let T be a self-mapping on a metric space (X, d). For $Y \subset X$ and for each $x \in X$, we set (cf. [14, 15])

(1) $\delta(Y) = \sup\{d(x, y) : x, y \in Y\}$,

(2) $O(x, n) = \{x, Tx, T^2x, \ldots, T^nx\}$ for $n \in \mathbb{N}$,

(3) $O(x, \infty) = \{x, Tx, T^2x, \ldots, \}$.

A metric space (X, d) is called T-orbitally complete if and only if every Cauchy sequence of $O(x, \infty)$ (for some $x \in X$) converges to a point in X. It is clear that if X is complete, then X is S-orbitally complete for every $T : X \to X$. We now state the theorem of Ćirić [14].

Theorem 1.1. *Let T be a quasicontraction on T-orbitally complete metric space X. Then*

(1) *T has a unique fixed point u in X,*

(2) *$\lim_{n \to \infty} T^nx = u$, for $x \in X$.*

Other generalizations of Banach's contraction principle and analogous results for metric spaces are abundantly present in the literature (see [7–19] and the references given therein). Rhoades [19] considered many types of contractive definitions and analyzed the relationships among them. Recently, some authors have provided a result on the existence of fixed points for a new class of contractive mappings.

Definition 1.2 (see, e.g., [20, 21]). Let (X, d) be a metric and $T, S : X \to X$ be two functions. A mapping S is called to be a T *contraction* if there exists $q \in (0, 1)$ such that

$$d(TSx, TSy) \leq qd(Tx, Ty), \quad \forall x, y \in X. \tag{1.2}$$

In the definition above, if we choose $Tx = x$ for all $x \in X$, then the T *contraction* becomes a contraction. In [20, 22], some fixed point theorems are proved for T contractions in metric spaces and in 2-metric spaces. In this paper, by the using the same techniques, we shall extend the Ćirić's theorem (Theorem 1.1). For this purpose, we recall the following elementary definition.

Definition 1.3 (see, e.g., [21, 22]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called *sequentially convergent* if the statement $\{Ty_n\}$ is convergent and implies that $\{y_n\}$ is a convergent sequence for every sequence $\{y_n\}$.

2. Main Results

The aim of this section is to prove the following result.

Theorem 2.1. *Let (X, d) be a metric space. Let $T : X \to X$ be a one-to-one, continuous, and sequentially convergent mapping and S be a self-mapping of X. Suppose that there exists $q \in [0, 1)$,*

$$d(TSx, TSy) \leq q \max\{d(Tx, Ty), d(Tx, TSx), d(Ty, TSy), d(Tx, TSy), d(Ty, TSx)\}, \tag{2.1}$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X. Then

(i) $d(TS^i, TS^j) \leq q\delta[O(Tx, n)]$, *for all* $i, j \in \{1, 2, \ldots, n\}$, *for all* $x \in X$ *and* $n \in \mathbb{N}$,

(ii) $\delta[O(Tx, \infty)] \leq 1/(1-q)d(Tx, TSx)$, *for all* $x \in X$,

(iii) *S has a unique fixed point* $b \in X$,

(iv) $\lim TS^n x = Tb$.

Proof. We will mainly follow the arguments in the proof of the Ćirić's theorem. Let T, S be defined as in Theorem 2.1. We start with the proof of (i). Let $x \in X$ be arbitrary element. Let $n \geq 1$ be integer and $i, j \in \{1, 2, \ldots, n\}$. Then

$$TS^{i-1}x, TS^i, TS^{j-1}x, TS^j x \in O(Tx, n). \tag{2.2}$$

Note that we use S^0 as the identity self-mapping. Due to (2.1), we have

$$
\begin{aligned}
d\left(TS^i x, TS^j x\right) &= d\left(TS\left(S^{i-1}x\right), TS\left(S^{j-1}x\right)\right) \\
&\leq q \max\left\{d\left(TS^{i-1}x, TS^{j-1}x\right); d\left(TS^{i-1}x, TS^i x\right),\right. \\
&\qquad \left. d\left(TS^{j-1}x, TS^j x\right), d\left(TS^{i-1}x, TS^j x\right), d\left(TS^i x, TS^{j-1}x\right)\right\} \\
&\leq q\delta[O(Tx, n)].
\end{aligned}
\tag{2.3}
$$

Thus, (i) is proved. Let us show (ii). Observe that

$$\delta[O(Tx, 1)] \leq \delta[O(Tx, 2)] \leq \cdots \leq \delta[O(Tx, n)] \leq \delta[O(Tx, n+1)] \leq \cdots, \tag{2.4}$$

which implies that

$$\delta[O(Tx, \infty)] = \sup\{\delta[O(Tx, n)] : n \in \mathbb{N}\}. \tag{2.5}$$

So we only need to show that $\delta[O(Tx, n)] \leq 1/(1-q)d(Tx, TSx)$ for all $n \in \mathbb{N}$. Indeed, since $O(Tx, n) = \{Tx, TSx, \ldots, TS^n x\}$ is a finite set for each n and

$$\left(TS^i, TS^j\right) \leq q\delta[O(Tx, n)], \quad \forall i, j \in \{1, 2, \ldots, n\} \tag{2.6}$$

with $0 \leq q < 1$, we infer that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$d\left(Tx, TS^k x\right) = \delta[O(Tx, n)]. \tag{2.7}$$

Applying the triangle inequality and by (i), we get that

$$
\begin{aligned}
d\left(Tx, TS^k x\right) &\leq d(Tx, TSx) + d\left(TSx, TS^k x\right) \\
&\leq d(Tx, TSx) + q\delta[O(Tx, n)] \\
&= d(Tx, TSx) + qd\left(Tx, TS^k x\right).
\end{aligned}
\tag{2.8}
$$

Hence, $\delta[O(Tx, n)] = d(Tx, TS^k x) \leq 1/(1-q)d(Tx, TSx)$. Thus, (ii) is proved. $\qquad\square$

To prove (iii), choose $x_0 \in X$ and define the iterative sequence $\{x_n\}$ and $\{y_n\}$ as follows:

$$
x_{n+1} = Sx_n = S^{n+1}x_0, \qquad y_n = Tx_n = TS^n x_0, \quad n = 0, 1, 2, \ldots.
\tag{2.9}
$$

We assert that $\{y_n\}$ is a Cauchy sequence. Without loss of generality, we may assume that $n < m$, where $n, m \in \mathbb{N}$. By (i), we obtain

$$
\begin{aligned}
d(y_n, y_m) &= d(TS^n x_0, TS^m x_0) \\
&= d\left(TSS^{n-1}x_0, TS^{m-n+1}S^{n-1}x_0\right) \\
&\leq q\delta\left[O\left(TS^{n-1}x_0, m-n+1\right)\right].
\end{aligned}
\tag{2.10}
$$

According to the assumption $n < m$, there exists $l \in \{1, m-n+1\}$ such that

$$
\delta\left[O\left(TS^{n-1}x_0, m-n+1\right)\right] = d\left(TS^{n-1}x_0, TS^l S^{n-1}x_0\right).
\tag{2.11}
$$

Again, by the mentioned assumption, we have

$$
\begin{aligned}
d\left(TS^{n-1}x_0, TS^l S^{n-1}x_0\right) &= d\left(TSS^{n-2}x_0, TS^{l+1}S^{n-2}x_0\right) \\
&\leq q\delta\left[O\left(TS^{n-2}x_0, l+1\right)\right].
\end{aligned}
\tag{2.12}
$$

So we can get the following system of inequalities:

$$
d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq q\delta\left[O\left(TS^{n-1}x_0, m-n+1\right)\right] \leq q^2\delta\left[O\left(TS^{n-2}x_0, m-n+2\right)\right].
\tag{2.13}
$$

By the same argument, we obtain

$$
d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq \cdots \leq q^n \delta[O(TSx_0, m)].
\tag{2.14}
$$

Since $\delta[O(TSx_0, m)] \leq \delta[O(TSx_0, \infty)] \leq 1/(1-q)d(Tx_0, TSx_0)$, we have

$$d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq \frac{q^n}{1-q} d(Tx_0, TSx_0). \tag{2.15}$$

This implies that $\{y_n\} = \{TS^n x_0\}$ is a Cauchy sequence. Since every Cauchy sequence of the form $\{TS^n x_0\}$ converges in X, we deduce that

$$\lim TS^n x_0 = a \in X. \tag{2.16}$$

Since T is sequentially convergent, it follows that $\lim S^n x_0 = b \in X$. By the hypothesis that T is continuous, we get

$$\lim TS^n x_0 = Tb. \tag{2.17}$$

Therefore, $Tb = a$. We shall show that $Sb = b$. To achieve this, we consider the following inequalities:

$$\begin{aligned}
d(TSb, Tb) &\leq d\left(Tb, TS^{n+1} x_0\right) + d\left(TS^{n+1} x_0, TSb\right) \\
&\leq d\left(Tb, TS^{n+1} x_0\right) + q \max\Big\{ d(Tb, TS^n x_0), d(Tb, TSb), \\
&\qquad\qquad d\left(TS^n x_0, TS^{n+1} x_0\right), d\left(Tb, TS^{n+1} x_0\right), d(TS^n x_0, TSb) \Big\} \\
&\leq d\left(Tb, TS^{n+1} x_0\right) + q \max\Big\{ d(Tb, TS^n x_0), d(Tb, TSb), d\left(TS^n x_0, TS^{n+1} x_0\right), \\
&\qquad\qquad d\left(Tb, TS^{n+1} x_0\right), d(TS^n x_0, Tb) + d(Tb, TSb) \Big\} \\
&\leq d\left(Tb, TS^{n+1} x_0\right) + q\Big[d\left(Tb, TS^{n+1} x_0\right) + d\left(TS^n x_0, TS^{n+1} x_0\right) \\
&\qquad\qquad + d(TS^n x_0, Tb) + d(Tb, TSb) \Big].
\end{aligned} \tag{2.18}$$

Hence,

$$d(Tb, TSb) \leq \frac{1}{1-q}\Big[(1+q)d\left(Tb, TS^{n+1} x_0\right) + qd(Tb, TS^n x_0) + qd\left(TS^n x_0, TS^{n+1} x_0\right) \Big]. \tag{2.19}$$

Letting $n \to \infty$ and combining with the fact that $\lim TS^n x_0 = Tb$, we can conclude that

$$d(TSb, Tb) = 0. \tag{2.20}$$

Therefore, $TSb = Tb$. Since T is one to one, we obtain that $Sb = b$. Now we will show that b is the unique fixed point of S. Suppose that b and b' are fixed points of S. Then $Sb = b$, $Sb' = b'$, and applying (2.1),

$$
\begin{aligned}
d(Tb, Tb') &= d(TSb, TSb') \\
&\leq q \max\{d(Tb, TSb), d(Tb', TSb'), d(Tb, TSb'), d(Tb', TSb), d(Tb, Tb')\} \qquad (2.21) \\
&= q\,d(Tb, Tb').
\end{aligned}
$$

Since $0 \leq q < 1$, we infer that $d(Tb, Tb') = 0$. Hence, $Tb = Tb'$. Since T is one-to-one, we may conclude that $b = b'$. This proves (iii) of **Theorem 2.1**. Proof of (iv) is directly obtained from (2.17). The theorem is proved.

Remark 2.2. **Theorem 1.1** is obtained from **Theorem 2.1** if we choose $Tx = x$ for every $x \in X$.

The following example shows that **Theorem 2.1** is indeed an extension of **Theorem 1.1**.

Example 2.3. Let $X = [0, +\infty)$ with metric induced by $d(x, y) = |x - y|$. Then X is a complete metric space. Consider the function $Sx = x^2/(x + 1)$ for all $x \in X$. It is easy to compute that

$$
\begin{aligned}
d(Sx, S2x) &= \frac{4x^2}{2x + 1} - \frac{x^2}{x + 1} = \frac{x^2(2x + 3)}{(2x + 1)(x + 1)}, \\
d(x, Sx) &= x - \frac{x^2}{x + 1} = \frac{x}{x + 1}, \\
d(2x, S2x) &= 2x - \frac{4x^2}{2x + 1} = \frac{2x}{2x + 1}, \\
d(x, S2x) &= \left| \frac{4x^2}{2x + 1} - x \right| = \left| \frac{2x^2 - x}{2x + 1} \right|, \\
d(2x, Sx) &= 2x - \frac{x^2}{x + 1} = \frac{x^2 + 2x}{x + 1}, \\
d(x, 2x) &= x.
\end{aligned}
\qquad (2.22)
$$

Therefore, if x is large enough, we have

$$
\max\{d(x, 2x), d(x, Sx), d(2x, S2x), d(x, S2x), d(2x, Sx)\} = \frac{x^2 + 2x}{x + 1}. \qquad (2.23)
$$

It follows that the quasicontractive condition

$$
d(Sx, S2x) \leq q \max\{d(x, 2x), d(x, Sx), d(2x, S2x), d(x, S2x), d(2x, Sx)\} \qquad (2.24)
$$

is equivalent to

$$
\frac{x(2x + 3)}{(2x + 1)(x + 2)} \leq q. \qquad (2.25)
$$

Letting $x \to \infty$, we get that $q \geq 1$ which contradicts the fact that $q \in (0,1)$. Hence S is not a quasicontraction. Therefore, we can not apply Theorem 1.1 for S. However, it is easy to see that 0 is the unique fixedpoint of S.

Now we shall show that S satisfies Theorem 2.1 with $Tx = e^x - 1$ and $q = 1/2$. Indeed, T is one-to-one, continuous and sequentially convergent on $[0, +\infty)$. We have

$$d(TSx, TSy) = \left| e^{x^2/(x+1)} - e^{y^2/(y+1)} \right|,$$

$$d(Tx, Ty) = |e^x - e^y|. \tag{2.26}$$

Now, we will show that

$$d(TSx, TSy) \leq \frac{1}{2} d(Tx, Ty). \tag{2.27}$$

If $x = y$, then the inequality above holds. So we can assume that $x > y$. Then (2.27) is equivalent to

$$\left| e^{x^2/(x+1)} - e^{y^2/(y+1)} \right| \leq \frac{1}{2} |e^x - e^y|, \quad \forall x > y. \tag{2.28}$$

It follows from the fact that ϕ and ψ are increasing, (2.28) reduces to

$$e^{x^2/(x+1)} - \frac{e^x}{2} \leq e^{y^2/(y+1)} - \frac{e^y}{2}, \quad \forall x > y. \tag{2.29}$$

Consider the function $\varphi(t) = e^{t^2/(t+1)} - e^t/2$, $(t \geq 0)$. It is easy to check that φ is decreasing on $[0, +\infty)$. So we can deduce that

$$d(TSx, TSy) \leq \frac{1}{2} d(Tx, Ty)$$

$$\leq \frac{1}{2} \max\{d(Tx, Ty); d(Tx, TSx); \tag{2.30}$$

$$d(Ty, TSy), d(Tx, TSy), d(Ty, TSx)\}$$

for every $x, y \in X$. Therefore, we can apply Theorem 2.1 for S.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let (X, d) be a metric space. Let $T : X \to X$ be a one-to-one, continuous, and sequentially convergent mapping, and let S be a self-mapping of X. Suppose that there exists $q \in [0, 1)$,*

$$d(TSx, TSy) \leq q \max\left\{ d(Tx, Ty), d(Tx, TSx), d(Ty, TSy), \frac{d(Tx, TSy) + d(Ty, TSx)}{2} \right\} \tag{2.31}$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X. Then,

   (a) *S has a unique fixed point $b \in X$,*

   (b) $\lim TS^n x = Tb$ *for $x \in X$.*

This corollary is also a consequence of the main result of [21], which is established by a contractive condition of integral type.

We also get the following corollary.

Corollary 2.5. *Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping and S be a self-mapping of X. Suppose that there exists $q \in [0, 1)$,*

$$d\left(TS^k x, TS^k y\right) \le q \max\Big\{d(Tx, Ty), d\left(Tx, TS^k x\right),$$
$$d\left(Ty, TS^k y\right), d\left(Tx, TS^k y\right), d\left(Ty, TS^k x\right)\Big\} \tag{2.32}$$

for every $x, y \in X$, for some positive integer k and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X. Then, S has a unique fixed point $b \in X$.

Proof. Applying Theorem 2.1 for the map S^k, we can deduce that S^k has a unique fixed point $b \in X$. This yields that $S^k b = b$ and $S^k(Sb) = S^{k+1}b = S(S^k b) = Sb$. It follows that Sb is a fixed point of S^k. By the uniqueness of fixed point of S^k, we infer that $Sb = b$. Therefore, b is a fixed point of S. Now, if a is a fixed point of S, then $S^k a = S^{k-1} a = \cdots = Sa = a$. Hence, a is a fixed point of S^k, so that $a = b$. This proves the uniqueness of fixed point of S. $\square$

The following corollary is an extension of the main result of [12].

Corollary 2.6. *Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping, and let S be a self-mapping of X. Suppose that there exists $a_i \ge 0$, $i = 1, 2, \ldots, 5$ satisfying that $\sum_{i=1}^{n} a_i < 1$ such that*

$$d(TSx, TSy) \le a_1 d(Tx, Ty) + a_2 d(Tx, TSx) + a_3 d(Ty, TSy) + a_4 d(Tx, TSy) + a_5 d(Ty, TSx) \tag{2.33}$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X. Then

   (a) *S has a unique fixed point $b \in X$.*

   (b) $\lim TS^n x = Tb$.

Proof. It is easy to see that the condition (2.1) is a consequence of the condition (2.33). This is enough to prove the corollary. $\square$

Remark 2.7. In the same way in Corollary 2.6, we can get the extensions of Kannan's contractions (see [7]), and Reich's contractions (see [8–10]).

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