Research Article

On Weakly (C, ψ, ϕ)-Contractive Mappings in Ordered Partial Metric Spaces

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We introduce the notion of weakly (C, ψ, ϕ)-contractive mappings in ordered partial metric spaces and prove some common fixed point theorems for such contractive mappings in the context of partially ordered partial metric spaces under certain conditions. We give some common fixed point results of integral type as an application of our main theorem. Also, we give an example and an application of integral equation to support the useability of our results.

1. Introduction and Preliminaries

In 1994, Matthews [1] introduced the notion of a partial metric space as a generalization of the usual metric space. In partial metric space self distance, that is. $d(x, x)$ is not necessarily equal a zero. In this interesting paper, Matthews [1] prove the Banach contraction mapping principle in the frame of partial metric spaces. After this initial paper, many authors have studied various type contractions and related fixed point results in partial metric spaces (see, [2–32]).

Definition 1.1 (see [1]). A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A pair (X, p) is called a partial metric space where X is a nonempty set and p is a partial metric on X.

Each partial metric p on X generates a T_0 topology τ_p on X. The set $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$ forms the base of τ_p.

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a metric on X.

Definition 1.2 (see [1]). Let (X, p) be a partial metric space. Then one has the following.

(1) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

(2) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n, m \to \infty} p(x_n, x_m)$.

(3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$.

The following lemma is crucial in proving our main results.

Lemma 1.3 (see [1]). *Let (X, p) be a partial metric space.*

(1) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*

(2) *A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n \to \infty} d_p(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n, m \to \infty} p(x_n, x_m). \tag{1.2}$$

In 1972, Chatterjea [33] introduced the concept of C-contraction as follows.

Definition 1.4 (see [33]). Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is called a C-contraction if there exists $k \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq k(d(x, Ty) + d(Tx, y)) \tag{1.3}$$

holds for all $x, y \in X$.

In this interesting paper, Chatterjea [33] proved the following theorem.

Theorem 1.5 (see [33]). *Every C-contraction in a complete metric space has a unique fixed point.*

Choudhury [34] introduced the concept of weakly C-contractive mapping as a generalization of C-contractive mapping and prove that every weakly C-contractive mapping in a complete metric space has a unique fixed point.

Definition 1.6 (see [34]). Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. Then T is called a weakly C-contractive if there exists a continuous function $\phi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ such that for all $x, y \in X$, we have

$$d(Tx, Ty) \leq \frac{1}{2} (d(x, Ty) + d(Tx, y)) - \phi(d(x, Ty), d(Tx, y)). \tag{1.4}$$

Harjani et al. [35] announced some fixed point results for weakly C-contractive mappings in a complete metric space endowed with a partial order. Meanwhile, Shatanawi [36] proved some fixed point and coupled fixed point theorems for a nonlinear weakly C-contraction type mapping in metric and ordered metric spaces.

In this paper, we introduce the concept of weakly (C, ψ, ϕ)-contractive mappings in ordered partial metric spaces, and we prove some existence theorems of common fixed point for such mapping in the context of complete partial metric spaces under certain conditions.

2. The Main Result

We start this section with the following definitions.

Definition 2.1. Suppose that (X, p) is a partial metric space. A mapping $T : X \rightarrow X$ is said to be continuous at $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x, \delta)) \subseteq B_p(Tx, \epsilon)$. We say that T is continuous if T is continuous at all $x \in X$.

It is easy to see that if (X, p) is a partial metric space, $T : X \rightarrow X$ is continuous, (x_n) is a sequence in X, $x \in X$ and

$$\lim_{n \to +\infty} p(x_n, x) = p(x, x), \quad \text{then} \lim_{n \to +\infty} p(Tx_n, Tx) = p(Tx, Tx). \tag{2.1}$$

Altun and Simsek [37] introduce the notion of weakly increasing of two mappings $T, S : X \rightarrow X$ in the following way.

Definition 2.2 (see [37]). Let $(X, \preceq)$ be a partially ordered set. Two mappings $T, S : X \rightarrow X$ are said to be weakly increasing if $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in X$.

For more details on weakly increasing mappings, we refer the reader to [24, 38–41]. Let ϕ denote all functions $\phi : [0, \infty) \times [0, +\infty) \rightarrow [0, \infty)$ such that

ϕ is continuous,

$\phi(t, s) = 0$ if and only if $t = s = 0$.

Similarly, we denote by Ψ all functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that

ψ is continuous and nondecreasing,

$\psi(t) = 0$ if and only if $t = 0$.

Inspired the definitions above, we introduce the following definition.

Definition 2.3. Let $(X, \preceq, p)$ be an partially ordered metric space. Then the mappings $T, S :$ $X \to X$ are said to be weakly (C, ψ, ϕ)-contractive mappings if T and S are weakly increasing with respect to $\preceq$ and for any comparable x and y, we have

$$\psi(p(Tx, Sy)) \leq \psi\left(\frac{1}{2}(p(Tx, y) + p(x, Sy))\right) - \phi(p(Tx, y), p(x, Sy)), \qquad (2.2)$$

where $\phi \in \Phi$ and $\psi \in \Psi$.

Now, we introduce and prove our first results.

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \to X$ are weakly (C, ψ, ϕ)-contractive mappings. If T and S are continuous, then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.*

Proof. Given $x_0 \in X$. Set $Tx_0 = x_1$ and $Sx_1 = x_2$. Continuing this process, we construct sequences (x_n) in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Using the fact that that S and T are weakly increasing with respect to $\preceq$, we obtain that

$$x_1 = Tx_0 \preceq S(Tx_0) = Sx_1 = x_2 \preceq \cdot \preceq x_{2n+1} \preceq x_{2n+2}. \qquad (2.3)$$

Now, we will prove that

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = 0. \qquad (2.4)$$

Since x_{2n+1} and x_{2n+2} are comparable, by (2.2), we have

$$
\begin{aligned}
\psi(p(x_{2n+1}, x_{2n+2})) &= \psi(p(Tx_{2n}, Sx_{2n+1})) \\
&\leq \psi\left(\frac{1}{2}(p(Tx_{2n}, x_{2n+1}) + p(x_{2n}, Sx_{2n+1}))\right) \\
&\quad - \phi(p(Tx_{2n}, x_{2n+1}), p(x_{2n}, Sx_{2n+1})) \\
&= \psi\left(\frac{1}{2}(p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2}))\right) \\
&\quad - \phi(p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})).
\end{aligned}
\qquad (2.5)
$$

By (p_4), we have

$$p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}). \qquad (2.6)$$

Thus (2.5) becomes

$$\psi\left(p(x_{2n+1}, x_{2n+2})\right) \leq \psi\left(\frac{1}{2}\left(p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\right)\right)$$
$$- \phi\left(p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})\right) \qquad (2.7)$$
$$\leq \psi\left(\frac{1}{2}\left(p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\right)\right).$$

Using the fact that ψ is nondecreasing, we get that

$$p(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}\left(p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\right). \qquad (2.8)$$

Hence, we have

$$p(x_{2n+1}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}). \qquad (2.9)$$

Similarly, we may show that

$$p(x_{2n+1}, x_{2n}) \leq p(x_{2n}, x_{2n-1}). \qquad (2.10)$$

From (2.9) and (2.10), we have

$$p(x_{n+1}, x_n) \leq p(x_n, x_{n-1}) \qquad \forall \, n \in \mathbf{N}. \qquad (2.11)$$

By (2.11), we get that $\{p(x_{n+1}, x_n) : n \in \mathbf{N}\}$ is a non increasing sequence. Hence there is $r \geq 0$ such that

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = r. \qquad (2.12)$$

Letting $n \to +\infty$ in (2.7), we get

$$\psi(r) \leq \psi(r) - \lim_{n \to +\infty} \inf \phi\left(p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})\right) \leq \psi(r). \qquad (2.13)$$

Thus

$$\psi(r) - \lim_{n \to +\infty} \inf \phi\left(p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})\right) = \psi(r), \qquad (2.14)$$

and hence

$$\lim_{n \to +\infty} \inf \phi\left(p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})\right) = 0. \qquad (2.15)$$

Using the continuity of ϕ, we conclude that

$$
\begin{aligned}
\liminf_{n \to +\infty} p(x_{2n+1}, x_{2n+1}) &= 0, \\
\liminf_{n \to +\infty} p(x_{2n}, x_{2n+2}) &= 0.
\end{aligned}
\tag{2.16}
$$

Again, on taking limit sup in (2.5), we have $\psi(r) = 0$ and hence $r = 0$. From the definition of d_p, we have

$$
\lim_{n \to +\infty} d_p(x_n, x_{n+1}) = 0.
\tag{2.17}
$$

Next, we will show that (x_n) is a Cauchy sequence in the metric space (X, d_p). It is sufficient to show that (x_{2n}) is a Cauchy sequence in (X, d_p). Suppose to the contrary, that is, (x_{2n}) is not a Cauchy sequence in (X, d_p). Then there exists $\epsilon > 0$ for which we can find two subsequences of positive integers $(x_{2m(i)})$ and $(x_{2n(i)})$ such that $n(i)$ is the smallest index for which

$$
n(i) > m(i) > i, \qquad d_p(x_{2m(i)}, x_{2n(i)}) \geq \epsilon.
\tag{2.18}
$$

This means that

$$
d_p(x_{2m(i)}, x_{2n(i)-2}) < \epsilon.
\tag{2.19}
$$

From (2.18), (2.19), and the triangular inequality, we get that

$$
\begin{aligned}
\epsilon &\leq d_p(x_{2m(i)}, x_{2n(i)}) \\
&\leq d_p(x_{2m(i)}, x_{2n(i)-2}) + d_p(x_{2n(i)-2}, x_{2n(i)-1}) \\
&\quad + d_p(x_{2n(i)-1}, x_{2n(i)}) \\
&< \epsilon + d_p(x_{2n(i)-2}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2n(i)}).
\end{aligned}
\tag{2.20}
$$

Letting $i \to +\infty$ in above inequalities and using (2.17), we have

$$
\lim_{i \to +\infty} d_p(x_{2m(i)}, x_{2n(i)}) = \epsilon.
\tag{2.21}
$$

Again, from (2.18) and the triangular inequality, we get that

$$
\begin{aligned}
\epsilon &\leq d_p(x_{2m(i)}, x_{2n(i)}) \\
&\leq d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2m(i)}) \\
&\leq d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2m(i)+1}) + d_p(x_{2m(i)+1}, x_{2m(i)})
\end{aligned}
$$

$$\leq d_p\left(x_{2n(i)}, x_{2n(i)-1}\right) + d_p\left(x_{2n(i)-1}, x_{2m(i)}\right) + 2d_p\left(x_{2m(i)+1}, x_{2m(i)}\right)$$

$$\leq 2d_p\left(x_{2n(i)}, x_{2n(i)-1}\right) + d_p\left(x_{2n(i)}, x_{2m(i)}\right) + 2d_p\left(x_{2m(i)+1}, x_{2m(i)}\right). \tag{2.22}$$

Letting $i \to +\infty$ in above inequalities and using (2.4) and (2.21), we get that

$$\begin{aligned}
\lim_{i \to +\infty} d_p\left(x_{2m(i)}, x_{2n(i)}\right) &= \lim_{i \to +\infty} d_p\left(x_{2m(i)+1}, x_{2n(i)-1}\right) \\
&= \lim_{i \to +\infty} d_p\left(x_{2m(i)+1}, x_{2n(i)}\right) \\
&= \lim_{i \to +\infty} d_p\left(x_{2m(i)}, x_{2n(i)-1}\right) \\
&= \epsilon.
\end{aligned} \tag{2.23}$$

By the fact that

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{2.24}$$

for all $x, y \in X$, and the expression above, we conclude that

$$\begin{aligned}
\lim_{i \to +\infty} p\left(x_{2m(i)}, x_{2n(i)}\right) &= \lim_{i \to +\infty} p\left(x_{2m(i)+1}, x_{2n(i)-1}\right) \\
&= \lim_{i \to +\infty} p\left(x_{2m(i)+1}, x_{2n(i)}\right) \\
&= \lim_{i \to +\infty} p\left(x_{2m(i)}, x_{2n(i)-1}\right) \\
&= \lim_{i \to +\infty} d_p\left(x_{2m(i)}, x_{2n(i)}\right) \\
&= \frac{\epsilon}{2}.
\end{aligned} \tag{2.25}$$

By (2.2), we have

$$\begin{aligned}
\psi\left(p\left(x_{2m(i)+1}, x_{2n(i)}\right)\right) &= \psi\left(p\left(Tx_{2m(i)}, Sx_{2n(i)-1}\right)\right) \\
&\leq \psi\left(\frac{1}{2}\left(p\left(Tx_{2m(i)}, x_{2n(i)-1}\right) + p\left(x_{2m(i)}, Sx_{2n(i)-1}\right)\right)\right) \\
&\quad - \phi\left(p\left(Tx_{2m(i)}, x_{2n(i)-1}\right), p\left(x_{2m(i)}, Sx_{2n(i)-1}\right)\right) \\
&\leq \psi\left(\frac{1}{2}\left(p\left(x_{2m(i)+1}, x_{2n(i)-1}\right) + p\left(x_{2m(i)}, x_{2n(i)}\right)\right)\right) \\
&\quad - \phi\left(p\left(x_{2m(i)+1}, x_{2n(i)-1}\right), p\left(x_{2m(i)}, x_{2n(i)}\right)\right).
\end{aligned} \tag{2.26}$$

Letting $i \to +\infty$ and using the continuity of ϕ and ψ, we get that

$$\psi\left(\frac{\epsilon}{2}\right) \leq \psi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right). \tag{2.27}$$

Therefore, we get that $\phi(\epsilon/2, \epsilon/2) = 0$. Hence $\epsilon = 0$ which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in (X, d_p). From Lemma 1.3, the sequence (x_n) converges in the metric space (X, d_p), say $\lim_{n \to \infty} d_p(x_n, u) = 0$. Again from Lemma 1.3, we have

$$p(u, u) = \lim_{n \to \infty} p(x_n, u) = \lim_{n,m \to \infty} p(x_n, x_m). \tag{2.28}$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p), we have

$$\lim_{n,m \to \infty} d_p(x_n, x_m) = 0. \tag{2.29}$$

From the definition of d_p, we have

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m). \tag{2.30}$$

Letting $n, m \to +\infty$ in the above equality and using (2.4) and (2.29), we get

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{2.31}$$

Thus by (2.28), we have

$$\lim_{n \to +\infty} p(x_n, u) = p(u, u) = 0. \tag{2.32}$$

Now,

$$p(x_n, Tu) \leq p(x_n, u) + p(u, Tu) - p(u, u)$$
$$\leq p(x_n, u) + p(u, x_n) + p(x_n, Tu) - p(x_n, x_n) - p(u, u). \tag{2.33}$$

Letting $n \to +\infty$ in above inequalities and using (2.4) and (2.29), we get that

$$\lim_{n \to +\infty} p(x_n, Tu) = p(u, Tu). \tag{2.34}$$

Similarly, we may show that

$$\lim_{n \to +\infty} p(x_n, Su) = p(u, Tu). \tag{2.35}$$

Since T is continuous and

$$\lim_{n \to +\infty} p(x_{2n}, u) = p(u, u) = 0, \qquad (2.36)$$

then by (2.34), we have

$$p(u, Tu) = \lim_{n \to +\infty} p(x_{2n+1}, Tu) = \lim_{n \to +\infty} p(Tx_{2n}, Tu) = p(Tu, Tu). \qquad (2.37)$$

Similarly, we may show that $p(u, Su) = p(Su, Su)$. By (p_3) and (p_4) we derive that

$$
\begin{aligned}
p(u, Tu) &= p(Tu, Tu) \\
&\le P(Tu, Su) + P(Su, Tu) - P(Su, Su) \\
&= 2p(Tu, Su) - P(u, Su).
\end{aligned}
\qquad (2.38)
$$

The above inequality yields that

$$\frac{1}{2}(p(u, Tu) + p(u, Su)) \le p(Tu, Su). \qquad (2.39)$$

Since u and u are comparable and ψ is nondecreasing, then by (2.2), we have

$$
\begin{aligned}
\psi\left(\frac{1}{2}(p(u, Tu) + p(u, Su))\right) &\le \psi(p(Tu, Su)) \\
&\le \psi\left(\frac{1}{2}(P(u, Tu) + p(u, Su))\right) - \phi(p(u, Tu), p(u, Su)).
\end{aligned}
\qquad (2.40)
$$

Thus, we have $\phi(p(u, Tu), p(u, Su)) = 0$ and hence $p(u, Tu) = p(u, Su) = 0$. By using (p_1) and (p_2) of Definition 1.1, we find that $u = Su = Tu$. That is, u is a common fixed point of T and S. $\qquad \square$

The continuity of T and S in Theorem 2.4 can be dropped. For this instance, suppose that X satisfies the following property:

(P): if (x_n) is a nondecreasing sequence in X such that $\lim_{n \to +\infty} p(x_n, u) = p(u, u)$, then $x_n \preceq$ for all $n \in \mathbb{N}$.

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \to X$ be weakly (C, ψ, ϕ)-contractive mappings. If X satisfies property (P), then T and S have a common fixed point, that is, there exists $u \in X$ such that $u = Tu = Su$.*

Proof. Following the proof of Theorem 2.4 step by step to construct a nondecreasing sequence (x_n) in X such that $\lim_{n \to +\infty} p(x_n, u) = p(u, u) = 0$,

$$
\begin{aligned}
\lim_{n \to +\infty} p(x_n, Tu) &= p(u, Tu), \\
\lim_{n \to +\infty} p(x_n, Su) &= p(u, Tu).
\end{aligned} \tag{2.41}
$$

By property, we have $(P)x_n \preceq x$ for all $n \in \mathbb{N}$. By (2.2), we have

$$
\begin{aligned}
\psi(p(x_{2n+1}, Su)) &= \psi(p(Tx_{2n}, Su)) \\
&\leq \psi\left(\frac{1}{2}(p(x_{2n}, Su) + p(Tx_{2n}, u))\right) - \phi(p(x_{2n}, Su), p(Tx_{2n}, u)) \\
&\leq \psi\left(\frac{1}{2}(p(x_{2n}, Su) + p(Tx_{2n}, u))\right) \\
&= \psi\left(\frac{1}{2}(p(x_{2n}, Su) + p(x_{2n+1}, u))\right).
\end{aligned} \tag{2.42}
$$

Letting $n \to +\infty$ in above inequalities, and using (2.41) we get $\psi(p(u, Su)) \leq \psi((1/2)p(u, Su))$. Since ψ is nondecreasing we get that $p(u, Su) \leq (1/2)p(u, Su)$. Hence $p(u, Su) = 0$. By (p_1) and (p_2), we conclude that $u = Su$. By similar arguments, we can show that $u = Tu$. Thus u is a common fixed point of T and S. $\qquad\square$

By taking $\psi = i$ (the identity function on $[0, +\infty)$) and defining $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ via $\phi(s, t) = (1/2 - q/2)(s + t)$ where $q \in [0, 1)$ in Theorems 2.4 and 2.5, we have the following results.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \to X$ be weakly increasing mappings with respect to $\preceq$ such that for any comparable x and y, one has*

$$
p(Tx, Sy) \leq \frac{q}{2}(p(Tx, y) + p(x, Ty)). \tag{2.43}
$$

If T and S are continuous and $q < 1$, then T and S have a common fixed point, that is, there exists $u \in X$ such that $u = Tu = Su$.

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \to X$ be weakly increasing mappings with respect to $\preceq$ such that for any comparable x and y, one has*

$$
p(Tx, Sy) \leq \frac{q}{2}(p(Tx, y) + p(x, Ty)). \tag{2.44}
$$

If X satisfies property (P) and $q < 1$, then T and S have a common fixed point, that is, there exists $u \in X$ such that $u = Tu = Su$.

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T : X \rightarrow X$ are mapping such that for any comparable x and y in X, one has*

$$\psi(p(Tx, Ty)) \leq \psi\left(\frac{1}{2}(p(Tx, y) + p(x, Ty))\right) - \phi(p(Tx, y), p(x, Ty)), \tag{2.45}$$

where $\phi : [0, \infty) \times [0, +\infty) \rightarrow [0, \infty)$ is a continuous function such that $\phi(t, s) = 0$ if and only if $t = s = 0$ and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Also, suppose that $Tx \preceq T(Tx)$ for all $x \in X$. If T is continuous, then T has a fixed point, that is, there exists $u \in X$ such that $u = Tu$.

Proof. It follows from **Theorem 2.4** by taking $S = T$ and noting that S and T are weakly (C, ψ, ϕ)-contractive mappings. $\qquad\square$

Corollary 2.9. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T : X \rightarrow X$ be mapping such that for any comparable x and y in X, one has*

$$\psi(p(Tx, Ty)) \leq \psi\left(\frac{1}{2}(p(Tx, y) + p(x, Ty))\right) - \phi(p(Tx, y), p(x, Ty)), \tag{2.46}$$

where $\phi : [0, \infty) \times [0, +\infty) \rightarrow [0, \infty)$ is a continuous function such that $\phi(t, s) = 0$ if and only if $t = s = 0$ and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Also, suppose that $Tx \preceq T(Tx)$ for all $x \in X$. If X satisfies property (P), then T has a fixed point, that is, there exists $u \in X$ such that $u = Tu$.

Proof. It follows from **Theorem 2.5** by taking $S = T$ and noting that S and T are weakly (C, ψ, ϕ)-contractive mappings. $\qquad\square$

We present the following common fixed points of integral type as an application of our results.

Denote by Ω the set of functions $\mu : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following hypotheses:

(1) μ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,

(2) for every $\epsilon > 0$, we have $\int_0^\epsilon \mu(s)ds > 0$.

It is easy to see that the mapping

$$\psi(t) = \int_0^t \mu(s)ds \in \Phi, \qquad \phi(t_1, t_2) = \int_0^{(t_1+t_2)/2} \mu(s)ds \in \Psi. \tag{2.47}$$

We have the following result.

Corollary 2.10. *Let* $(X, \preceq, p)$ *be an partially ordered metric space. Suppose that* $T, S : X \to X$ *are weakly increasing mappings with respect to* $\preceq$ *and for any comparable* x *and* y, *one has*

$$\int_0^{p(Tx,Sy)} \mu(s)ds \leq \int_0^{1/2(d(x,Sy)+d(Tx,y))} \mu(s)ds - \int_0^{(1/4)((d(x,Sy)+d(y,fx))/4)} \mu(s)ds. \qquad (2.48)$$

If T *and* S *are continuous, then* T *and* S *have a common fixed point.*

Proof. It follows from **Theorem 2.4** by defining $\psi : [0, +\infty) \to [0, +\infty)$ via $\psi(t) = \int_0^t \mu(s)ds$ and $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ via

$$\phi(t_1, t_2) = \int_0^{(t_1+t_2)/2} \mu(s)ds. \qquad (2.49)$$
$\square$

Corollary 2.11. *Let* $(X, \preceq, p)$ *be an partially ordered metric space. Suppose that* $T, S : X \to X$ *are weakly increasing mappings with respect to* $\preceq$ *and for any comparable* x *and* y, *one has*

$$\int_0^{p(Tx,Sy)} \mu(s)ds \leq \int_0^{(1/2)(d(x,Sy)+d(Tx,y))} \mu(s)ds - \int_0^{(1/4)((d(x,Sy)+d(y,fx))/4)} \mu(s)ds. \qquad (2.50)$$

If X *satisfies property* (P), *then* T *and* S *have a common fixed point.*

Proof. It follows from **Theorem 2.5** by defining $\psi : [0, +\infty) \to [0, +\infty)$ via $\psi(t) = \int_0^t \mu(s)ds$ and $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ via

$$\phi(t_1, t_2) = \int_0^{(t_1+t_2)/2} \mu(s)ds. \qquad (2.51)$$
$\square$

Example 2.12. Let $X = [0, 1]$. Define the partial metric space on X by $p(x, y) = \max\{x, y\}$ and the relation $\preceq$ on X by $y \preceq x$ if and only if $x \leq y$. Also, define the mappings $T, S : X \to X$ by $Tx = 1/4\ x^2$, $Sx = 1/5\ x^2$ and the functions $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t^3$ and $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \phi(s, t) = (7/64)(s + t)^3$. Then one has the following.

(1) $(X, p, \preceq)$ is a complete ordered partial metric space.

(2) T and S are continuous.

(3) S and T are weakly increasing.

(4) For any two comparable elements x and y in X, we have

$$\psi(p(Tx, Sy)) \leq \psi\left(\frac{1}{2}(p(y, Tx) + p(x, Sy))\right) - \phi(p(y, Tx), p(x, Sy)). \qquad (2.52)$$

Proof. The proof of (1) and (2) is clear. To prove (3), given $x \in X$. Since

$$S(Tx) = S\left(\frac{1}{4}x^2\right) = \frac{1}{80}x^4 \leq \frac{1}{4}x^2 = Tx, \qquad (2.53)$$

we have $Tx \preceq S(Tx)$. Similarly, we can show that $Sx \preceq T(Sx)$. Thus T and S are weakly increasing mappings. To prove (4), given two comparable elements x and y in X. Without loss of generality, we assume that $x \preceq y$, that is, $y \leq x$. So,

$$
\begin{aligned}
\psi(p(Tx, Sy)) &= \psi\left(p\left(\frac{1}{4}x^2, \frac{1}{5}y^2\right)\right) \\
&= \psi\left(\max\left\{\frac{1}{4}x^2, \frac{1}{5}y^2\right\}\right) \\
&\leq \psi\left(\max\left\{\frac{1}{4}x^2, \frac{1}{4}y^2\right\}\right) \\
&= \psi\left(\frac{1}{4}x^2\right) = \frac{1}{64}x^6 \\
&\leq \frac{1}{64}x^3 \\
&\leq \frac{1}{64}(p(y, Tx) + x)^3 \\
&= \frac{1}{8}(p(y, Tx) + x)^3 - \frac{7}{64}(p(y, Tx) + x)^3 \\
&= \psi\left(\frac{1}{2}(p(y, Tx) + x)\right) - \phi(p(y, Tx), x) \\
&= \psi\left(\frac{1}{2}(p(Tx, y) + p(x, Sy))\right) - \phi(p(Tx, y), p(x, Sy)). \qquad \square
\end{aligned}
$$

(2.54)

From (3) and (4), we conclude that T and S are weakly (C, ψ, ϕ)-contractive mappings. Note that Example 2.12 satisfies all the hypotheses of Theorem 2.4. Thus T and S have a common fixed point. Here 0 is a common fixed point of T and S.

3. Application

In this section, we apply our results to prove an existence solution of the following integral equation:

$$
x(t) = \int_0^1 G(t, s) f(s, x(s)) ds, \quad \forall t \in [0, 1]. \tag{3.1}
$$

Let $X = C([0, 1])$ be the space of all continuous functions defined on $I = [0, 1]$. Define a partial metric space:

$$
p : C(I) \times C(I) \longrightarrow [0, +\infty), \tag{3.2}
$$

by

$$
p(x, y) = \|x - y\| := \sup\{|x(t) - y(t)| : t \in [0, 1]\}. \tag{3.3}
$$

Also, define a relation $\preceq$ on X by

$$x \preceq y \quad \text{if, and only if } x(t) \leq y(t) \quad \forall t \in I. \tag{3.4}$$

Then $(X, p, \preceq)$ is an ordered complete partial metric space.

Now, we will give an existence theorem for the solution of the integral equation (3.1).

Theorem 3.1. *Suppose the following hypotheses hold.*

(i) $f(s, x(s)) \leq f(s, \int_0^1 G(s,t) f(t, x(t)) dt)$ *for all $s \in I$.*

(ii) *There exists a continuous function $H : [0,1] \rightarrow [0, +\infty)$ such that*

$$\left| f(t, a) - f(t, b) \right| \leq H(t) |a - b|. \tag{3.5}$$

(iii) *There exists $k < 1$ such that*

$$\sup_{t \in [0,1]} \int_0^1 H(t) dt \leq \left(\frac{1}{3} \right) k. \tag{3.6}$$

(iv) $G(s, t) \leq 1$ *for all $s, t \in I$.*

Then the integral equation (3.1) has a solution $x^* \in C^2(I)$.

Proof. Define the operators:

$$T, S : C(I) \longrightarrow C(I), \tag{3.7}$$

by

$$Tx(t) = Sx(t) = \int_0^1 G(t, s) f(s, x(s)) ds \quad \forall t \in I. \tag{3.8}$$

Given $x \in C(I)$. Then from (i), we have

$$
\begin{aligned}
Tx(t) &= \int_0^1 G(t, s) f(s, x(s)) ds \\
&\leq \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) f(\tau, x(\tau)) d\tau \right) ds \\
&= T(Tx(t)).
\end{aligned}
\tag{3.9}
$$

Thus T and S are weakly increasing mappings with respect to $\preceq$. Again, for $x, y \in C(I)$ with $x \preceq y$, we have

$$
\begin{aligned}
p(Tx, Ty) &= \|Tx - Ty\| \\
&= \sup_{t \in I} |Tx(t) - Ty(t)| \\
&= \sup_{t \in I} \left| \int_0^1 G(t, s)(f(s, x(s)) - f(s, y(s))) ds \right| \\
&\leq \sup_{t \in I} \int_0^1 G(t, s) |f(s, x(s)) - f(s, y(s))| ds \\
&\leq \sup_{t \in I} \int_0^1 G(t, s) H(s) |x(s) - y(s)| ds \\
&\leq \sup_{t \in I} \int_0^1 G(t, s) H(s) \|x - y\| ds.
\end{aligned}
\tag{3.10}
$$

By (iv), we have

$$
p(Tx, Ty) \leq \sup_{t \in I} \int_0^1 H(s) \|x - y\| ds.
\tag{3.11}
$$

By using (iii), we get

$$
\begin{aligned}
p(Tx, Ty) &\leq \frac{1}{3} k \|x - y\| \\
&\leq \frac{1}{3} k (\|x - Ty\| + \|Ty - Tx\| + \|Tx - y\|) \\
&= \frac{1}{3} k (p(x, Ty) + p(Tx, Ty) + p(y, Tx)).
\end{aligned}
\tag{3.12}
$$

Hence

$$
p(Tx, Ty) \leq \frac{k}{3 - k} (p(x, Ty) + p(y, Tx)).
\tag{3.13}
$$

Take $q = 2k/(3 - k)$. Since $k < 1$, we have $q < 1$. Also, we have

$$
p(Tx, Ty) \leq \frac{q}{2} (p(x, Ty) + p(Tx, y)).
\tag{3.14}
$$

Moreover if (f_n) is a nondecreasing sequence in $C(I)$ such that $f_n \to f$ as $n \to +\infty$, then $f_n \preceq f$ for all $n \in \mathbb{N}$ (see [25]). Thus, all the required hypotheses of Corollary 2.7 are satisfied. Therefore, T has a fixed point and hence the integral equation (3.1) has a solution. $\qquad \square$

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