

Research Article

Modified Noor's Extragradient Method for Solving Generalized Variational Inequalities in Banach Spaces

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Motivated and inspired by Korpelevich's and Noor's extragradient methods, we suggest an extragradient method by using the sunny nonexpansive retraction which has strong convergence for solving the generalized variational inequalities in Banach spaces.

1. Introduction

In the present paper, we focus on the following generalized variational inequality:

$$\text{Find } x^* \in C \text{ such that } \langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \quad (1.1)$$

where C is a nonempty closed convex subset of a real Banach space E , $A : C \rightarrow E$ is a nonlinear mapping, and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \right\}, \quad \forall x \in E. \quad (1.2)$$

We use $S(C, A)$ to denote the solution set of (1.1). It is clear that (1.1) is reduced to the following variational inequality in Hilbert spaces:

$$\text{Find } x^* \in C \text{ such that } \langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \quad (1.3)$$

which was introduced and studied by Stampacchia [1]. Variational inequalities are being used as mathematical programming tools and models to study a wide class of unrelated problems arising in mathematical, physical, regional, engineering, and nonlinear optimization sciences. See, for instance, [2–23]. In order to solve (1.3), especially, Korpelevich [24] introduced the following well-known extragradient method:

$$\begin{aligned} y_n &= P_C(I - \lambda A)x_n, \\ x_{n+1} &= P_C(x_n - \lambda A y_n), \quad n \geq 0, \end{aligned} \quad (1.4)$$

where P_C is the metric projection from R^n onto its subset C , $\lambda \in (0, 1/k)$ and $A : C \rightarrow R^n$ is a monotone operator. He showed that the sequence $\{x_n\}$ converges to some solution of the above variational inequality (1.3). Noor [10] further suggested and analyzed the following new iterative methods:

$$\begin{aligned} x_{n+1} &= P_C(y_n - \lambda A y_n), \\ y_n &= P_C[x_n - \lambda A x_n], \quad n \geq 0, \end{aligned} \quad (1.5)$$

which is known as the modified Noor's extragradient method. We would like to point out that this algorithm (1.5) is quite different from the method of Koperlevich. However, these two algorithms fail, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

The generalized variational inequality (1.1) was introduced by Aoyama et al. [25] which is connected with the fixed point problem for nonlinear mapping. For solving the aforementioned generalized variational inequality (1.1), Aoyama et al. [25] introduced an iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C[x_n - \lambda_n A x_n], \quad n \geq 0, \quad (1.6)$$

where Q_C is a sunny nonexpansive retraction from E onto C , and $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are two real number sequences. Aoyama et al. [25] obtained on the aforementioned method (1.6) for solving variational inequality (1.1). Motivated by (1.6), Yao and Maruster [26] presented a modification of (1.5):

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) Q_C[(1 - \alpha_n)(x_n - \lambda A x_n)], \quad n \geq 0. \quad (1.7)$$

Yao and Maruster [26] proved that (1.7) converges strongly to the solution of the generalized variational inequality (1.1). Yao et al. [27] further considered the following extended extragradient method for solving (1.1):

$$\begin{aligned} y_n &= Q_C[x_n - \lambda_n A x_n], \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n Q_C[y_n - \lambda A y_n], \quad n \geq 0. \end{aligned} \quad (1.8)$$

In this paper, motivated and inspired by Korpelevich's and Noor's extragradient methods, (1.7) and (1.8), we suggest a modified Noor's extragradient method via the sunny nonexpansive retraction for solving the variational inequalities (1.1) in Banach spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space E . Recall that a mapping A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \quad (2.1)$$

for all $x, y \in C$. A mapping A of C into E is said to be α -*strongly accretive* if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2, \quad (2.2)$$

for all $x, y \in C$. A mapping A of C into E is said to be α -*inverse-strongly accretive* if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad (2.3)$$

for all $x, y \in C$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (2.4)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.5)$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.5) is attained uniformly for $x, y \in U$. The norm of E is said to be *Fréchet differentiable* if, for each $x \in U$, the limit (2.5) is attained uniformly for $y \in U$. And we define a function $\rho : [0, \infty) \rightarrow [0, \infty)$ called the *modulus of smoothness* of E as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \quad (2.6)$$

It is known that E is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} (\rho(\tau)/\tau) = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be q -*uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

We need the following lemmas for proof of our main results.

Lemma 2.1 (see [28]). *Let q be a given real number with $1 < q \leq 2$ and let E be a q -uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + 2\|Ky\|^q \quad (2.7)$$

for all $x, y \in E$, where K is the q -uniformly smoothness constant of E and J_q is the generalized duality mapping from E into 2^{E^*} defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in E. \quad (2.8)$$

Let D be a subset of C and let Q be a mapping of C into D . Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx, \quad (2.9)$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q . A subset D of C is called a *sunny nonexpansive* retract of C if there exists a sunny nonexpansive retraction from C onto D . One knows the following lemma concerning sunny nonexpansive retraction.

Lemma 2.2 (see [29]). *Let C be a closed convex subset of a smooth Banach space E , let D be a nonempty subset of C , and let Q be a retraction from C onto D . Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu) \rangle \leq 0 \quad (2.10)$$

for all $u \in C$ and $y \in D$.

Remark 2.3. (1) It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C .

(2) Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C .

The following lemma characterized the set of solution of (1.1) by using sunny nonexpansive retractions.

Lemma 2.4 (see [25]). *Let C be a nonempty closed convex subset of a smooth Banach space X . Let Q_C be a sunny nonexpansive retraction from X onto C and let A be an accretive operator of C into X . Then for all $\lambda > 0$,*

$$S(C, A) = F(Q_C(I - \lambda A)), \quad (2.11)$$

where $S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*) \rangle \geq 0, \forall x \in C\}$.

Lemma 2.5 (see [30]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly, then x is a fixed point of T .*

Lemma 2.6 (see [31]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0, \quad (2.12)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

- (a) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
- (b) $\limsup_{n \rightarrow \infty} (\delta_n / \gamma_n) \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

In this section, we will state and prove our main result.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E . Let Q_C be a sunny nonexpansive retraction from E onto C and let $A : C \rightarrow E$ be an α -strongly accretive and L -Lipschitz continuous mapping with $S(C, A) \neq \emptyset$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} y_n &= Q_C[x_n - \lambda Ax_n], \\ x_{n+1} &= Q_C[(1 - \alpha_n)(y_n - \lambda Ay_n)], \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ and λ is a constant in $[a, b]$ for some a, b with $0 < a < b < \alpha / K^2 L^2$. Assume that the following conditions hold:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (b) $\lim_{n \rightarrow \infty} (\alpha_{n+1} / \alpha_n) = 1$.

Then $\{x_n\}$ defined by (3.1) converges strongly to $Q'(0)$, where Q' is a sunny nonexpansive retraction of E onto $S(C, A)$.

Proof. First, we note that A must be α / L^2 -inverse-strongly accretive mapping. Take $p \in S(C, A)$. By using Lemmas 2.1 and 2.4, we easily obtain the following facts.

- (1) $p = Q_C[p - \lambda Ap]$ for all $\lambda > 0$; in particular,

$$p = Q_C[p - \lambda(1 - \alpha_n)Ap] = Q_C[\alpha_n p + (1 - \alpha_n)(p - \lambda Ap)], \quad n \geq 0. \tag{3.2}$$

- (2) If $\lambda \in (0, \alpha / K^2 L^2]$, then $I - \lambda A$ is nonexpansive and for all $x, y \in C$

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda \left(K^2 \lambda - \frac{\alpha}{L^2} \right) \|Ax - Ay\|^2. \tag{3.3}$$

Indeed, from Lemma 2.1, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda \langle Ax - Ay, j(x - y) \rangle + 2K^2 \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda \frac{\alpha}{L^2} \|Ax - Ay\|^2 + 2K^2 \lambda^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda \left(K^2 \lambda - \frac{\alpha}{L^2} \right) \|Ax - Ay\|^2. \end{aligned} \tag{3.4}$$

From (3.1), we have

$$\begin{aligned}\|y_n - p\| &= \|Q_C(x_n - \lambda Ax_n) - Q_C(p - \lambda Ap)\| \\ &\leq \|(x_n - \lambda Ax_n) - (p - \lambda Ap)\| \\ &\leq \|x_n - p\|.\end{aligned}\tag{3.5}$$

By (3.1) and (3.5), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|Q_C[(1 - \alpha_n)(y_n - \lambda Ay_n)] - Q_C[\alpha_n p + (1 - \alpha_n)(p - \lambda Ap)]\| \\ &\leq \|[(1 - \alpha_n)(y_n - \lambda Ay_n)] - [\alpha_n p + (1 - \alpha_n)(p - \lambda Ap)]\| \\ &\leq \alpha_n \|p\| + (1 - \alpha_n) \|(y_n - \lambda Ay_n) - (p - \lambda Ap)\| \\ &\leq \alpha_n \|p\| + (1 - \alpha_n) \|y_n - p\| \\ &\leq \alpha_n \|p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \max\{\|p\|, \|x_0 - p\|\}.\end{aligned}\tag{3.6}$$

Therefore, $\{x_n\}$ is bounded. We observe that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|Q_C[x_n - \lambda Ax_n] - Q_C[x_{n-1} - \lambda Ax_{n-1}]\| \\ &\leq \|(x_n - \lambda Ax_n) - (x_{n-1} - \lambda Ax_{n-1})\| \\ &\leq \|x_n - x_{n-1}\|,\end{aligned}\tag{3.7}$$

and hence

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|Q_C[(1 - \alpha_n)(y_n - \lambda Ay_n)] - Q_C[(1 - \alpha_{n-1})(y_{n-1} - \lambda Ay_{n-1})]\| \\ &\leq \|[(1 - \alpha_n)(y_n - \lambda Ay_n)] - [(1 - \alpha_{n-1})(y_{n-1} - \lambda Ay_{n-1})]\| \\ &= \|(1 - \alpha_n)[(y_n - \lambda Ay_n) - (y_{n-1} - \lambda Ay_{n-1})] + (\alpha_{n-1} - \alpha_n)(y_{n-1} - \lambda Ay_{n-1})\| \\ &\leq (1 - \alpha_n) \|(y_n - \lambda Ay_n) - (y_{n-1} - \lambda Ay_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|y_{n-1} - \lambda Ay_{n-1}\| \\ &\leq (1 - \alpha_n) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|y_{n-1} - \lambda Ay_{n-1}\| \\ &\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|y_{n-1} - \lambda Ay_{n-1}\|.\end{aligned}\tag{3.8}$$

By Lemma 2.6, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.\tag{3.9}$$

From (3.1), we also have

$$\begin{aligned} \|y_n - p\|^2 &= \|Q_C[x_n - \lambda Ax_n] - Q_C[p - \lambda Ap]\|^2 \\ &\leq \|(x_n - \lambda Ax_n) - (p - \lambda Ap)\|^2 \\ &\leq \|x_n - p\|^2 + 2\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ax_n - Ap\|^2. \end{aligned} \tag{3.10}$$

By (3.1) and (3.10), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|\alpha_n(-p) + (1 - \alpha_n)[(y_n - \lambda Ay_n) - (p - \lambda Ap)]\|^2 \\ &\leq \alpha_n\|p\|^2 + (1 - \alpha_n)\|(y_n - \lambda Ay_n) - (p - \lambda Ap)\|^2 \\ &\leq \alpha_n\|p\|^2 + (1 - \alpha_n)\left[\|y_n - p\|^2 + 2\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2\right] \\ &\leq \alpha_n\|p\|^2 + (1 - \alpha_n)\left[\|x_n - p\|^2 + 2\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ax_n - Ap\|^2\right] \\ &\quad + 2(1 - \alpha_n)\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2 \\ &\leq \alpha_n\|p\|^2 + \|x_n - p\|^2 + 2(1 - \alpha_n)\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ax_n - Ap\|^2 \\ &\quad + 2(1 - \alpha_n)\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2. \end{aligned} \tag{3.11}$$

Therefore, we have

$$\begin{aligned} 0 &\leq -2(1 - \alpha_n)\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ax_n - Ap\|^2 - 2(1 - \alpha_n)\lambda\left(K^2\lambda - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2 \\ &\leq \alpha_n\|p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &= \alpha_n\|p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|) \\ &\leq \alpha_n\|p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|. \end{aligned} \tag{3.12}$$

Since $\liminf_{n \rightarrow \infty} 2(1 - \alpha_n)\lambda(K^2\lambda - \alpha/L^2) > 0$, $\alpha_n \rightarrow 0$ and $\|x_n - x_{n+1}\| \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \|Ax_n - Ap\| = \lim_{n \rightarrow \infty} \|Ay_n - Ap\| = 0. \tag{3.13}$$

It follows that

$$\lim_{n \rightarrow \infty} \|Ay_n - Ax_n\| = 0. \tag{3.14}$$

Since A is α -strongly accretive, we deduce

$$\|Ay_n - Ax_n\| \geq \alpha\|y_n - x_n\|, \tag{3.15}$$

which implies that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0, \quad (3.16)$$

that is,

$$\lim_{n \rightarrow \infty} \|Q_C(x_n - \lambda Ax_n) - x_n\| = 0. \quad (3.17)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle \geq 0. \quad (3.18)$$

To show (3.18), since $\{x_n\}$ is bounded, we can choose that a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to z such that

$$\limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle = \limsup_{i \rightarrow \infty} \langle Q'(0), j(x_{n_i} - Q'(0)) \rangle. \quad (3.19)$$

We first prove $z \in S(C, A)$. It follows that

$$\lim_{i \rightarrow \infty} \|Q_C(I - \lambda A)x_{n_i} - x_{n_i}\| = 0. \quad (3.20)$$

By Lemma 2.5 and (3.20), we have $z \in F(Q_C(I - \lambda A))$; it follows from Lemma 2.4 that $z \in S(C, A)$.

Now, from (3.19) and Lemma 2.2, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle &= \limsup_{i \rightarrow \infty} \langle Q'(0), j(x_{n_i} - Q'(0)) \rangle \\ &= \langle Q'(0), j(z - Q'(0)) \rangle \\ &\geq 0. \end{aligned} \quad (3.21)$$

Since $x_{n+1} = Q_C[(1 - \alpha_n)(y_n - \lambda Ay_n)]$ and $x^* = Q_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda Ax^*)]$ for all $n \geq 0$, we can deduce from Lemma 2.2 that

$$\begin{aligned} &\langle Q_C[(1 - \alpha_n)(y_n - \lambda_n Ay_n)] - [(1 - \alpha_n)(y_n - \lambda_n Ay_n)], j(x_{n+1} - x^*) \rangle \leq 0, \\ &\langle [\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n Ax^*)] - Q_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n Ax^*)], j(x_{n+1} - x^*) \rangle \leq 0. \end{aligned} \quad (3.22)$$

Therefore, we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &= \langle Q_C[(1 - \alpha_n)(y_n - \lambda_n A y_n)] - Q_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n A x^*)], j(x_{n+1} - x^*) \rangle \\
 &= \langle Q_C[(1 - \alpha_n)(y_n - \lambda_n A y_n)] - [(1 - \alpha_n)(y_n - \lambda_n A y_n)], j(x_{n+1} - x^*) \rangle \\
 &\quad + \langle [(1 - \alpha_n)(y_n - \lambda_n A y_n)] - [\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n A x^*)], j(x_{n+1} - x^*) \rangle \\
 &\quad + \langle [\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n A x^*)] - Q_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n A x^*)], j(x_{n+1} - x^*) \rangle \\
 &\leq \langle (1 - \alpha_n)(y_n - \lambda_n A y_n) - (1 - \alpha_n)(x^* - \lambda_n A x^*) - \alpha_n x^*, j(x_{n+1} - x^*) \rangle \\
 &\leq (1 - \alpha_n) \|y_n - x^*\| \|x_{n+1} - x^*\| - \alpha_n \langle x^*, j(x_{n+1} - x^*) \rangle \\
 &\leq (1 - \alpha_n) \|x_n - x^*\| \|x_{n+1} - x^*\| - \alpha_n \langle x^*, j(x_{n+1} - x^*) \rangle \\
 &\leq \frac{1 - \alpha_n}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) - \alpha_n \langle x^*, j(x_{n+1} - x^*) \rangle,
 \end{aligned} \tag{3.23}$$

which implies that

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle -z, j(x_{n+1} - z) \rangle. \tag{3.24}$$

Finally, by Lemma 2.6 and (3.24), we conclude that x_n converges strongly to $Q'(0)$. This completes the proof. \square

4. Conclusion

Variational inequality theory provides a simple, natural, and unified framework for a general treatment of unrelated problems. These activities have motivated to generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques. A well-known method to solve the VI is the following gradient method:

$$x_{n+1} = P_C(x_n - \alpha_n A(x_n)), \quad n \geq 0, \tag{4.1}$$

This method requires some monotonicity properties of A . However, we remark that there is no chance of relaxing the assumption on A to plain monotonicity. To overcome this weakness of the method, Korpelevich proposed a so-called Korpelevich’s method which has been extensively extended and studied. Noor [10] especially, suggested another method referred as Noor’s method which is different from Korpelevich’s method. However, these two algorithms fail, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces. In the present paper, we suggested a modified Noor’s method which has strong convergence in Banach spaces. We hope that the ideas and technique of this paper may stimulate further research in this field.

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