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Research Article

Strong Global Attractors for 3D Wave Equations with Weakly Damping

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We consider the existence of the global attractor $\mathscr{A}_1$ for the 3D weakly damped wave equation. We prove that $\mathscr{A}_1$ is compact in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and attracts all bounded subsets of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ with respect to the norm of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Furthermore, this attractor coincides with the global attractor in the weak energy space $H_0^1(\Omega) \times L^2(\Omega)$.

1. Introduction

Let $\Omega \subset R^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following weakly damped wave equation:

$$u_{tt} + \alpha u_t - \Delta u + \varphi(u) = f \quad \text{in } \Omega \times R^+ \tag{1.1}$$

with the boundary condition

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

and initial conditions:

$$u(\cdot, 0) = u_0, \qquad u_t(\cdot, 0) = u_1, \quad \text{in } \Omega, \tag{1.3}$$

where $\alpha > 0$, φ is the nonlinear term, and f is a given external forcing term.

Nonlinear wave equation of the type (1.1) arises as an evolutionary mathematical model in many branched of physics, for example, (i) modeling a continuous Josephson junction with $\varphi(u) = \beta \sin u$; (ii) modeling a relativistic quantum mechanics with $\varphi(u) = |u|^{\gamma} u$. A relevant problem is to investigate the asymptotic dynamical behavior of these mathematical models. The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem is to analyse the existence of its global attractor.

The existence of global attractors for the classical wave equations in $H_0^1(\Omega)\times L^2(\Omega)$ and the regularities of the global attractors has been studied extensively in many monographs and lectures, for example, see [1–7] and references therein.

However, to our knowledge, the research about the stronger attraction of global attractors for the damped wave equations with respect to the norm of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ is fewer, only has been found in [8–10]. In the above three papers, the global attractors in strong topological space $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ were established, the attraction with respect to the norm of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ was proved by the asymptotic compactness of the operator semigroup.

Recently, we consider (1.1) in n dimensional space where the nonlinear term φ without polynomial growth is in [11].

In this paper, our aim is to prove the existence of a global attractor for (1.1) in strong topological space $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ where the nonlinear term φ with some polynomial growth. For simplicity, we consider the space dimension is 3, as we know, when the space dimension is lagerer than 3, the case is similar as in 3D, when the space dimension is 1 or 2, the case is more easier. The attraction with respect to the norm of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ will be proved by a method different from [8–10]. Furthermore, this attractor coincides with the global attractor in the weak energy space $H_0^1(\Omega) \times L^2(\Omega)$.

The basic assumptions about the external forcing term f and the nonlinear term φ are as follows. Let $f \in L^2(\Omega)$ be independent of time, and let the nonlinear term $\varphi \in C^1(R, R)$ satisfy the following assumptions:

$$\liminf_{|s|\to\infty} \frac{\Phi(s)}{s^2} \geq 0, \quad \text{here } \Phi(s) = \int_0^s \varphi(\tau)\,d\tau; \tag{1.4}$$

$$\limsup_{|s|\to\infty} \frac{|\varphi'(s)|}{s^2} = 0; \tag{1.5}$$

moreover, there exists a constant $C_0 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{s\varphi(s) - C_0\Phi(s)}{s^2} \geq 0. \tag{1.6}$$

Throughout this paper, we use the following notations. Let Ω be a bounded subset of R^n with sufficiently smooth boundary, $A = -\Delta$. $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ with the corresponding norms $\|u\| = (\int_\Omega |\nabla u(x)|^2 dx)^{1/2}$, $|u| = (\int_\Omega |u(x)|^2 dx)^{1/2}$ and

$|\Delta u| = (\int_\Omega |\Delta u(x)|^2 dx)^{1/2}$, respectively. The norms in $L^p(\Omega), 1 \le p < \infty$ are denoted by $|u|_p = (\int_\Omega |u|^p dx)^{1/p}$, the scalar products of V, H are denoted by

$$((u,v)) = \int_\Omega \nabla u(x) \nabla v(x) dx, \qquad (u,v) = \int_\Omega u(x)v(x)dx, \tag{1.7}$$

respectively. We have $D(A) \subset V \subset H = H^* \subset V^*, H^*$ and V^* are the dual spaces of H and V, respectively, and each space is dense in the following one and the injections are continuous. Then, we introduce the product Hilbert spaces $\mathscr{H}_0 = V \times H = H_0^1(\Omega) \times L^2(\Omega)$, $\mathscr{H}_1 = D(A) \times V = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, endowed with the standard product norms:

$$\|\{u_1,u_2\}\|^2_{\mathscr{H}_0} = \|u_1\|_V^2 + \|u_2\|_H^2, \qquad \|\{u_1,u_2\}\|^2_{\mathscr{H}_1} = \|u_1\|_{D(A)}^2 + \|u_2\|_V^2. \tag{1.8}$$

Denote by C any positive constant which may be different from line to line and even in the same line, we also denote the different positive constants by $C_i, i \in \mathbb{N}$, for special differentiation.

The rest of the paper is organized as follows. In the next section, for the convenience of the reader, we recall some basic concepts about the global attractors and recapitulate some abstract results. In Section 3, we present our main results.

2. Preliminaries

In this section, we first recall some basic concepts and theorems, which are important for getting our main results. We refer to [2, 5, 6, 12, 13] and the references therein for more details. Then, we outline some known results about (1.1)–(1.3).

Definition 2.1. The mappings $S(t)$, where $S : X \times [0,+\infty) \to X$, is said to be a C^0 semigroup on X, if $\{S(t)\}_{t\ge0}$ satisfies

(1) $S(0)u = u$ for all $u \in X$;

(2) $S(t_1)(S(t_2)u) = S(t_1 + S_2)(u)$ for all $u \in X$ and $t_1, t_2 \in R^+$;

(3) the mapping $S : X \times (0,\infty) \to X$ is continuous.

Definition 2.2. Let $\{S(t)\}_{t\ge0}$ be a semigroup on a metric space (E,d). A subset $\mathscr{A}$ of E is called a global attractor for the semigroup, if $\mathscr{A}$ is compact and enjoys the following properties:

(1) $\mathscr{A}$ is invariant, that is, $S(t)\mathscr{A} = \mathscr{A}$, for all $t \ge 0$;

(2) $\mathscr{A}$ attracts all bounded sets of E. That is, for any bounded subset B of E,

$$d(S(t)B, \mathscr{A}) \longrightarrow 0, \quad \text{as } t \longrightarrow 0, \tag{2.1}$$

where $d(B,A)$ is the semidistance of two sets B and A:

$$d(B,A) = \sup_{x\in B} \inf_{y \in A} d(x,y). \tag{2.2}$$

Definition 2.3. A C^0 semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X is said to satisfy the condition (C) if for any $\varepsilon > 0$ and for any bounded set B of X, there exist $t(B) > 0$ and a finite dimensional subspace X_1 of X such that $\{\|PS(t)x\|_X, x \in B, t \geq t(B)\}$ is bounded and

$$\|(I-P)S(t)x\|_X < \varepsilon, \quad t \geq t(B),\ x \in B, \tag{2.3}$$

where $P : X \to X_1$ is a bounded projector.

Definition 2.4. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a metric space (E, d). A set $B_0 \subset E$ is called an absorbing set for the semigroup $\{S(t)\}_{t\geq 0}$, if and only if for every bounded set $B \subset E$, there exists a $T_0 = T_0(B) > 0$ such that $S(t)B \subset B_0$ for all $t \geq T_0$.

Theorem 2.5. *Let X be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in X. Then, there is a global attractor for $\{S(t)\}_{t\geq 0}$ in X if the following conditions hold true:*

(1) $\{S(t)\}_{t\geq 0}$ *satisfies the condition* (C)*, and*

(2) *there is a bounded absorbing set $B \subset X$.*

In [12], the authors have discussed the relations between Condition (C) and ω-limit compact and proved that, in uniformly convex Banach space, Condition (C) is equivalent to ω-limit compact, if the semigroup has a bounded absorbing set.

Next, we recall the result about the global attractor in $\mathscr{H}_0$ whose proofs are omitted here, the reader is referred to [6] and the reference therein.

Theorem 2.6. *Under the conditions* (1.4), (1.5), (1.6)*, the solution semigroup $\{S(t)\}_{t\geq 0}$ of the problem* (1.1)–(1.3) *has a global attractor $\mathscr{A}_0$ in $\mathscr{H}_0$. $\mathscr{A}_0$ is included and bounded in $\mathscr{H}_1$.*

3. Main Results

According to the standard Fatou-Galerkin method, it is easy to obtain the existence and uniqueness of solutions and the continuous dependence to the initial value of (1.1)–(1.3). We address the reader to [6] and the reference therein. Here, we only state the result as follows.

Lemma 3.1. *Let conditions* (1.4), (1.5), (1.6) *hold, then for any $T > 0$ and $(u_0, u_1) \in \mathscr{H}_0$, there exists a unique solution of* (1.1)–(1.3) *such that*

$$\{u, u_t\} \in C([0,T]; \mathscr{H}_0). \tag{3.1}$$

If, furthermore,

$$(u_0, u_1) \in \mathscr{H}_1, \tag{3.2}$$

then u satisfies

$$\{u, u_t\} \in C([0,T]; \mathscr{H}_1). \tag{3.3}$$

We define the mappings:

$$S(t) : \{u_0, u_1\} \longrightarrow \{u(t), u_t(t)\} \quad \forall t \in R. \tag{3.4}$$

By Lemma 3.1, it is easy to see that $\{S(t)\}_{t\geq 0}$ is C_0 semigroup in the energy phase spaces $\mathscr{H}_0$ and $\mathscr{H}_1$.

In order to verify the existence of the bounded absorbing set in $\mathscr{H}_1$, we need the result about the existence of the bounded absorbing set in $\mathscr{H}_0$. First, we establish the bounded absorbing set in $\mathscr{H}_0$. Its proof is essentially established in [6] and the reference therein, and we only need to make a few minor changes for our problem. Here, we only give the following lemma.

Lemma 3.2. *Under the conditions* (1.4), (1.5), (1.6), $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set* $\mathcal{B}_0 \triangleq B_{\mathscr{H}_0}(0, \rho_0)$ *in* $\mathscr{H}_0$*, that is, for any* $\varepsilon > 0$ *and any bounded subset* $B_0 \subset \mathscr{H}_0$*, there is a positive constant* $t_0 = t(B_0, \rho_0)$ *such that*

$$S(t)B \subset \mathcal{B}_0 \quad \text{for any } t \geq t_0,\ u_0,\ u_1 \in B_0. \tag{3.5}$$

Next, let us establish the existence of the bounded absorbing set in $\mathscr{H}_1$.

Lemma 3.3. *Under the conditions* (1.4), (1.5), (1.6), $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set* $\mathcal{B} \triangleq B_{\mathscr{H}_1}(0, \rho_1)$ *in* $\mathscr{H}_1$*, that is, for any* $\varepsilon > 0$ *and any bounded subset* $B \subset \mathscr{H}_1$*, there is a positive constant* $T = T(B, \rho_1)$ *such that*

$$S(t)B \subset \mathcal{B} \quad \text{for any } t \geq T,\ u_0, u_1 \in B. \tag{3.6}$$

Proof. Take the scalar product in H of (1.1) with $Av = Au_t + \sigma Au$, we have

$$\frac{1}{2}\frac{d}{dt}\left(|Au|^2 + \|v\|^2\right) + \sigma|Au|^2 + (\alpha - \sigma)\|v\|^2 - \sigma(\alpha - \sigma)(Au, v) + \left(\varphi(u), Av\right) = \left(f, Av\right). \tag{3.7}$$

For $0 < \sigma \leq \sigma_0$, $\sigma_0 = \{\alpha/4, \lambda_1/2\alpha\}$, by Hölder inequality, Poincaré inequality, and Cauchy inequality we have

$$\begin{aligned}
&\sigma|Au|^2 + (\alpha - \sigma)\|v\|^2 - \sigma(\alpha - \sigma)(Au, v) \\
&\quad\geq \sigma|Au|^2 + (\alpha - \sigma)\|v\|^2 - \frac{\sigma(\alpha - \sigma)}{\sqrt{\lambda_1}}|Au|\|v\| \\
&\quad\geq \sigma|Au|^2 + \frac{3}{4}\alpha\|v\|^2 - \frac{\sigma\alpha}{\sqrt{\lambda_1}}|Au|\|v\| \\
&\quad\geq \sigma|Au|^2 + \frac{3}{4}\alpha\|v\|^2 - \left(\frac{\sigma}{2}|Au|^2 + \frac{\sigma\alpha^2}{2\lambda_1}\|v\|^2\right) \\
&\quad\geq \frac{\sigma}{2}|Au|^2 + \frac{\alpha}{2}\|v\|^2.
\end{aligned} \tag{3.8}$$

It follows from (1.5) that, for any $\varepsilon > 0$, there exists a constant $C_1 > 0$, such that

$$\left|\varphi'(s)\right| \le \varepsilon|s|^2 + C_1, \quad \forall s \in \mathbb{R}. \tag{3.9}$$

Hence,

$$\begin{aligned}
&\left|((\varphi(u), v))\right| \\
&\quad = \left|\int_\Omega \varphi'(u) \cdot \nabla u \cdot \nabla v\right| dx \\
&\quad \le \int_\Omega \left|\varphi'(u)\right| \cdot |\nabla u| \cdot |\nabla v| dx \\
&\quad \le \varepsilon \int_\Omega |u|^2 \cdot |\nabla u| \cdot |\nabla v| dx + C_1 \int_\Omega |\nabla u| \cdot |\nabla v| dx \\
&\quad \le \varepsilon \left(\int_\Omega |u|^4 \cdot |\nabla u|^2 dx\right)^{1/2} \cdot \|v\| + C_1\|u\| \cdot \|v\| \\
&\quad \le \varepsilon \left(\int_\Omega |u|^6 dx\right)^{1/3} \cdot \left(\int_\Omega |\nabla u|^6 dx\right)^{1/6} \cdot \|v\| + C_1\|u\| \cdot \|v\| \\
&\quad \le \varepsilon\|u\|^2 \cdot C_2^2 \cdot |Au| \cdot \|v\| + \frac{\alpha}{16}\|v\|^2 + \frac{4C_1^2}{\alpha}\|u\|^2,
\end{aligned} \tag{3.10}$$

where C_2 is the positive constant satisfying

$$\begin{aligned}
C_2\|u\|^2 &\ge \left(\int_\Omega |u|^6 dx\right)^{1/3}, \\
C_2|Au| &\ge \left(\int_\Omega |\nabla u|^6 dx\right)^{1/6},
\end{aligned} \tag{3.11}$$

$$(f, Av) = \frac{d}{dt}(f, Au) + \sigma(f, Au) \le \frac{d}{dt}(f, Au) + \frac{\sigma}{8}|Au|^2 + 2\sigma\left|f\right|^2. \tag{3.12}$$

If (u_0, v_0) belongs to a bounded set B of $\mathscr{H}_1$, then B is also bounded in $\mathscr{H}_0$, and for $t \ge t_0$, by Lemma 3.2, we have

$$\|u\|^2 + |u_t|^2 \le \rho_0^2; \tag{3.13}$$

t_0, ρ_0 are given in Lemma 3.2 Choose

$$0 < \varepsilon^2 \le \frac{\alpha\sigma}{16\rho_0^4 C_2^4}, \tag{3.14}$$

it follows from (3.10) that

$$\begin{aligned}\left|((\varphi(u), v))\right| &\leq \frac{\alpha}{8}\|v\|^2 + \frac{4\varepsilon^2}{\alpha}\|u\|^4 C_2^4 |Au|^2 + \frac{4C_1^2}{\alpha} \\ &\leq \frac{\alpha}{8}\|v\|^2 + \frac{\sigma}{4}|Au|^2 + \frac{4C_1^2}{\alpha}\rho_0^2, \quad t \geq t_0.\end{aligned} \tag{3.15}$$

Combining with (3.8), (3.12), and (3.15), by the Hölder inequality and the Young inequality, we deduce from that (3.7):

$$\frac{d}{dt}\left(|Au|^2 + \|v\|^2 - 2(f, Au)\right) + \frac{\sigma}{4}|Au|^2 + \frac{\alpha}{4}|v|^2 \leq 4\sigma\left|f\right|^2 + \frac{8C_1^2}{\alpha}\rho_0^2. \tag{3.16}$$

Let $y = |Au - f|^2 + \|v\|^2$, from the above inequality, we can obtain

$$\begin{aligned}\frac{dy}{dt} + \frac{\sigma}{8}y &\leq \frac{dy}{dt} + \frac{\sigma}{8}\left(|Au|^2 + \|v\|^2 + \left|f\right|^2\right) + \frac{\sigma}{4}|Au|\left|f\right| \\ &= \frac{dy}{dt} + \frac{\sigma}{4}\left(|Au|^2 + \|v\|^2\right) - \frac{\sigma}{8}\left(|Au|^2 + \|v\|^2\right) + \frac{\sigma}{4}|Au|\left|f\right| + \frac{\sigma}{8}\left|f\right|^2 \\ &\leq 4\sigma\left|f\right|^2 + \frac{8C_1^2}{\alpha}\rho_0^2 - \frac{\sigma}{8}\left(|Au|^2 + \|v\|^2\right) + \frac{\sigma}{8}\left(|Au|^2 + \left|f\right|^2\right) + \frac{\sigma}{8}\left|f\right|^2 \\ &\leq 4\sigma\left|f\right|^2 + \frac{8C_1^2}{\alpha}\rho_0^2 + \frac{\sigma}{4}\left|f\right|^2 - \frac{\sigma}{8}\|v\|^2 \\ &\leq \frac{\sigma}{4}\left|f\right|^2 + 4\sigma\left|f\right|^2 + \frac{8C_1^2}{\alpha}\rho_0^2 \\ &\leq \frac{9}{2}\sigma\left|f\right|^2 + \frac{8C_1^2}{\alpha}\rho_0^2.\end{aligned} \tag{3.17}$$

Let $C_3 = (9/2)\sigma|f|^2 + (8C_1^2/\alpha)\rho_0^2$. By the Gronwall lemma, we have

$$y(t) \leq y(t_0)\exp\left(-\frac{\sigma}{8}(t - t_0)\right) + \frac{8C_3}{\sigma}. \tag{3.18}$$

Defining ρ_1' by

$$\rho_1'^2 = \frac{8C_3}{\sigma}, \tag{3.19}$$

we see that

$$\limsup_{t\to\infty} y(t) \leq \rho_1'^2, \tag{3.20}$$

and we conclude that

The ball of $\mathcal{H}_1$, $\mathcal{B} \triangleq B_{\mathcal{H}_1}(0,\rho_1)$, centered at $(A^{-1}f, 0)$ of radius $\rho_1 > \rho_1' + \rho_0$, is absorbing in $\mathcal{H}_1$ for the semigroup $S(t), t \geq 0$. □

We now give the property of compactness about the nonlinear operator φ which will be needed in the proof of the condition (C).

Lemma 3.4. *Assume that $\varphi \in C^1(R,R)$ and $\varphi : D(A) \longrightarrow V$ are defined by*

$$((\varphi(u), v)) = \int_\Omega \varphi'(u)\nabla u \nabla v dx, \tag{3.21}$$

for all $u \in D(A), v \in H_0^1(\Omega)$. Then, φ is continuous compact.

Proof. Let $\{u_m\}$ be a bounded sequence in $D(A)$. Without loss of generality, we assume that $\{u_m\}$ weakly converges to u_0 in $D(A)$, since $D(A)$ is reflexive. By the Sobolev embedding theorem, we know that $H^2(\Omega) \hookrightarrow L^\infty \bigcap H^1(\Omega)$ and the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact in R^3. Hence, we have that

$$u_m \longrightarrow u_0 \quad \text{in } H^1. \tag{3.22}$$

Furthermore, there exists a constant C such that

$$\|u_0\|_{L^\infty \bigcap H^1(\Omega)} \leq C, \qquad \|u_m\|_{L^\infty \bigcap H^1(\Omega)} \leq C. \tag{3.23}$$

It is sufficient to prove that $\{\varphi(u_m)\}$ converges to $\{\varphi(u_0)\}$ in V:

$$\begin{aligned} \|\varphi(u_m) - \varphi(u_0)\| &= \left(\int_\Omega \left|\varphi'(u_m)\nabla(u_m) - \varphi'(u_0)\nabla u_0\right|^2 dx\right)^{1/2} \\ &\leq \left(\int_\Omega \left|\varphi'(u_m)\right|^2 |\nabla(u_m) - \nabla u_0|^2 dx\right)^{1/2} + \left(\int_\Omega |\nabla u_0|^2 \left|\varphi'(u_m) - \varphi'(u_0)\right|^2 dx\right)^{1/2}. \end{aligned} \tag{3.24}$$

On the one hand, for the first term in (3.24), combining with (3.23) and the continuity of $\varphi'(\cdot)$, we have

$$\int_\Omega \left|\varphi'(u_m)\right|^2 |\nabla(u_m) - \nabla u_0|^2 dx \leq C_{\|u_m\|_{L^\infty(\Omega)}} \|\nabla(u_m) - \nabla u_0\|^2. \tag{3.25}$$

On the other hand, for the second term in (3.24), using the continuity of $\varphi'(\cdot)$,

$$\int_\Omega |\nabla u_0|^2 \left|\varphi'(u_m) - \varphi'(u_0)\right|^2 dx \longrightarrow 0, \quad \text{as } m \longrightarrow 0, \tag{3.26}$$

follows immediately by dominated convergence theorem.

Also, considering (3.22), passing to the limit in (3.24), we can obtain

$$\lim_{m\to\infty} \|\varphi(u_m) - \varphi(u_0)\| = 0. \tag{3.27}$$

This completes the proof. □

Lemma 3.5. *Suppose the conditions* (1.4), (1.5), (1.6) *hold, the solution semigroup* $\{S(t)\}_{t\geq 0}$ *of the problem* (1.1)–(1.3) *satisfies the condition* (C) *in* $\mathcal{H}_1$.

Proof. Let $\{\omega_i\}$ be an orthonormal basis of $L^2(\Omega)$ which consists of eigenvalues of A. The corresponding eigenvalues are denoted by $\{\lambda_j\}_{j=1}^{\infty}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots, \qquad \lambda_j \longrightarrow \infty \tag{3.28}$$

with

$$A\omega_i = \lambda_i \omega_i, \qquad \forall i \in \mathbb{N}. \tag{3.29}$$

Let $V_m = \operatorname{span}\{\omega_1, \ldots, \omega_m\}$ in V and let $P_m : V \to V_m$ be an orthogonal projector. We write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2. \tag{3.30}$$

Taking the scalar product of (1.1) in H with $Av_2 = Au_{2t} + \sigma A u_2$, we find

$$\frac{1}{2}\frac{d}{dt}\left(|Au_2|^2 + \|v_2\|^2\right) + \sigma|Au_2|^2 + (\alpha - \sigma)\|v_2\|^2 - \sigma(\alpha - \sigma)(Au_2, v_2) + \left(\varphi(u), Av_2\right) = \left(f, Av_2\right). \tag{3.31}$$

Choose $0 < \sigma \leq \sigma_0$, similar to (3.8), we have

$$\sigma|Au_2|^2 + (\alpha - \sigma)\|v_2\|^2 - \sigma(\alpha - \sigma)(Au_2, v_2) \geq \frac{\sigma}{2}|Au_2|^2 + \frac{\alpha}{2}\|v_2\|^2. \tag{3.32}$$

Since $f \in L^2(\Omega)$, $\varphi : D(A) \to V$ is compact by Lemma 3.4, for any $\varepsilon > 0$, there exists some m such that

$$\left|(I - P_m)f\right|_H \leq \varepsilon, \tag{3.33}$$

$$\left\|(I - P_m)\varphi(u)\right\|_V \leq \varepsilon, \qquad \forall u \in B_1(0, \rho_1), \tag{3.34}$$

where ρ_1 is given by Lemma 3.2.

By exploiting the Hölder inequality and Cauchy inequality, we have

$$\left(\varphi(u), Av_2\right) \le \left\|(I-P_m)\varphi\right\| \|v_2\| \le \varepsilon \|v_2\| \le \frac{\alpha}{4}\|v_2\|^2 + \frac{\varepsilon^2}{\alpha}, \tag{3.35}$$

$$\begin{aligned}(f, Av) &= \frac{d}{dt}(f, Au) + \sigma(f, Au) \\ &\le \frac{d}{dt}(f, Au) + \frac{\sigma}{4}|Au|^2 + \sigma\left|f\right|^2.\end{aligned} \tag{3.36}$$

By (3.33) and (3.36), we have

$$\begin{aligned}(f, Av_2) &= \frac{d}{dt}(f_2, Au_2) + \sigma(f, Au_2) \\ &\le \frac{d}{dt}(f_2, Au_2) + \frac{\sigma}{4}|Au_2|^2 + \sigma\left|f_2\right|^2 \\ &\le \frac{d}{dt}(f_2, Au_2) + \frac{\sigma}{4}|Au_2|^2 + \sigma\varepsilon^2,\end{aligned} \tag{3.37}$$

where $f_2 \triangleq (I - P_m)f$.

Hence, combining with (3.32), (3.35), and (3.37), we obtain from (3.31) that

$$\frac{d}{dt}\left(|Au_2|^2 + \|v_2\|^2 - 2(f_2, Au_2)\right) + \frac{\sigma}{2}\left(|Au_2|^2 + \|v_2\|^2\right) \le \left(2\sigma + \frac{2}{\alpha}\right)\varepsilon^2. \tag{3.38}$$

Let $y = |Au_2 - f_2|^2 + \|v_2\|^2$, from the above inequality, similar to (3.17), we can obtain

$$\begin{aligned}\frac{dy}{dt} + \frac{\sigma}{4}y &\le \frac{dy}{dt} + \frac{\sigma}{4}\left(|Au_2|^2 + \|v_2\|^2 + \left|f_2\right|^2\right) + \frac{\sigma}{2}|Au_2|\left|f_2\right| \\ &\le \left(\frac{5\sigma}{2} + \frac{2}{\alpha}\right)\varepsilon^2.\end{aligned} \tag{3.39}$$

Let $C_4 = 5\sigma/2 + 2/\alpha$. By the Gronwall lemma, we have

$$y(t) \le y(t_1)\exp\left(-\frac{\sigma}{4}(t-t_1)\right) + \frac{4C_4\varepsilon^2}{\sigma}. \tag{3.40}$$

Choosing $t_2 = t_1 + (4/\sigma)\ln(\rho_1^2/\varepsilon^2)$, it follows that

$$y(t) \le \left(1 + \frac{4C_4}{\sigma}\right)\varepsilon^2, \qquad t \ge t_2. \tag{3.41}$$

That is,

$$\left|Au_2 - f_2\right|^2 + \|v_2\|^2 \leq \widetilde{C}\varepsilon^2, \tag{3.42}$$

for all $t \geq t_2$, where $\widetilde{C} = (1 + 4C_4/\sigma)$.

Thus we complete the proof. □

We are now in a position to state our main results as follows.

Theorem 3.6. *Under the conditions* (1.4), (1.5), (1.6), *problem* (1.1)–(1.3) *has a global attractor* $\mathcal{A}_1$ *in* $\mathcal{H}_1$; *it attracts all bounded subsets of* $\mathcal{H}_1$ *with respect to the norm of* $\mathcal{H}_1$.

Proof. By Lemmas 3.1, 3.3, and 3.5, the conditions of Theorem 2.5 are satisfied. The proof is complete. □

Corollary 3.7. *The global attractor* $\mathcal{A}_0$ *in* $\mathcal{H}_0$ *is coincides with* $\mathcal{A}_1$ *in* $\mathcal{H}_1$, *that is,* $\mathcal{A}_0 = \mathcal{A}_1$.

Proof. By Theorem 2.6, $\mathcal{A}_0$ is a bounded set of $\mathcal{H}_1$, combining with Theorem 3.6, we can easily get $\mathcal{A}_0 = \mathcal{A}_1$. □

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