Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 463082, 15 pages
doi:10.1155/2012/463082

Research Article

Complicated Asymptotic Behavior of Solutions for Heat Equation in Some Weighted Space

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Received 5 June 2012; Accepted 29 July 2012

Academic Editor: Toka Diagana

We investigate the asymptotic behavior of solutions for the heat equation in the weighted space $Y_0^\sigma(\mathbb{R}^N) \equiv \{\varphi \in C(\mathbb{R}^N) : \lim_{|x|\to\infty}(1+|x|^2)^{-\sigma/2}\varphi(x) = 0\}$. Exactly, we find that the unbounded function space $Y_0^\sigma(\mathbb{R}^N)$ with $0 < \sigma < N$ can provide a setting where complexity occurs in the asymptotic behavior of solutions for the heat equation.

1. Introduction

In this paper, we consider the asymptotic behavior of solutions to the Cauchy problem of the heat equation

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,\end{gathered} \tag{1.1}$$

where $N \geq 1$ and the initial value $u_0 \in Y_0^\sigma(\mathbb{R}^N)$.

Whether complexity occurs in the asymptotic behavior of solutions for some evolution equations or not mainly depends on the work spaces that one selects [1–9]. In the space $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$, the problem (1.1) under consideration is well posed and the asymptotic behavior of the solutions is rather simple, reflecting the simple structure of the heat equation. Considering, for instance, the problem (1.1) with the initial value $u_0 \in L^1(\mathbb{R}^N)$, it is well-known that the solutions $u(x,t)$ converge as $t \to \infty$ toward a multiple of the fundamental solution, the one which has the same integral,

$$u(x,t) = S(t)u_0(x) = G(t) * u_0(x) \longrightarrow MG(x,t), \tag{1.2}$$

where $G(x,t) = (4\pi t)^{-N/2}\exp(-|x|^2/4t)$ and $M = \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x$, see [10, 11].

It was first found in 2002 [12] by Vázquez and Zuazua that the bounded function space $L^\infty(\mathbb{R}^N)$ provides a setting where complicated asymptotic behavior of solutions may take place for the heat equation. In fact, they proved that, for any bounded sequence $\{g_j,\ j = 1,2,\ldots\}$ in $L^\infty(\mathbb{R}^N)$, there exists an initial value $u_0 \in L^\infty(\mathbb{R}^N)$ and a sequence $t_{j_k} \to \infty$ as $k \to \infty$ such that

$$u\left(t_{j_k}^{1/2}x, t_{j_k}\right) = S(t_{j_k})u_0(t_{j_k}x) \longrightarrow S(1)g_j(x) \tag{1.3}$$

uniformly on any compact subset of $\mathbb{R}^N$ as $k \to \infty$. Subsequently, Cazenave et al. showed that, in the bounded continuous function space $C_0(\mathbb{R}^N)$, the solutions of the heat equation may present more complex asymptotic behavior [13–15]. Meanwhile, considerable attention has also been paid to study the complicated asymptotic behavior of solutions for the porous medium equation and other evolution equations in some bounded function spaces such as $C_0(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ (see, e.g., [3, 7, 9, 12, 16] and the references therein).

In this paper we find that, even in the unbounded function space $Y_0^\sigma(\mathbb{R}^N)$ with $0 < \sigma < N$, the complicated asymptotic behavior of solutions for the heat equation can also occur. For this purpose, we need to establish the L^p_σ–L^∞_σ smoothing effect and other estimates for the solutions of the problem (1.1) when the initial value $u_0 \in L^p_\sigma(\mathbb{R}^N) \equiv \{\varphi : (1+|\cdot|^2)^{-\sigma/2}\varphi(\cdot) \in L^p(\mathbb{R}^N)\}$ with $1 < p \le \infty$ or $u_0 \in Y_0^\sigma(\mathbb{R}^N)$.

The rest of this paper is organized as follows. In the next section, we give some definitions and some estimates of the solutions to the problem (1.1). Section 3 is devoted to study the complicated asymptotic behavior of the solutions.

2. Main Estimates

In this section, we investigate some properties of solutions for the problem (1.1) when the initial value u_0 belongs to some weighted spaces. For these purposes, we first introduce the mild solutions $u(x,t)$ of the problem (1.1) which are defined as

$$u(x,t) = S(t)u_0(x) = (4\pi t)^{-N/2}\int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right)u_0(y)\mathrm{d}y. \tag{2.1}$$

Letting $\sigma \ge 0$ and $1 \le p \le \infty$, we define two weighted spaces $Y_0^\sigma(\mathbb{R}^N)$ and $L^p_\sigma(\mathbb{R}^N)$ as follows:

$$\begin{aligned} Y_0^\sigma\left(\mathbb{R}^N\right) &\equiv \left\{\varphi(x) \in C\left(\mathbb{R}^N\right) : \lim_{|x|\to\infty} \varphi(x)\left(1+|x|^2\right)^{-\sigma/2} = 0\right\}, \\ L^p_\sigma\left(\mathbb{R}^N\right) &\equiv \left\{\varphi : \varphi(\cdot)\left(1+|\cdot|^2\right)^{-\sigma/2} \in L^p\left(\mathbb{R}^N\right)\right\}. \end{aligned} \tag{2.2}$$

Endowed with the obvious norms,

$$\|\varphi\|_{Y_0^\sigma(\mathbb{R}^N)} = \left\|\varphi(\cdot)\left(1+|\cdot|^2\right)^{-\sigma/2}\right\|_{L^\infty(\mathbb{R}^N)},$$
$$\|\varphi\|_{L_\sigma^p(\mathbb{R}^N)} = \left\|\varphi(\cdot)\left(1+|\cdot|^2\right)^{-\sigma/2}\right\|_{L^p(\mathbb{R}^N)}, \tag{2.3}$$

the spaces $Y_0^\sigma(\mathbb{R}^N)$ and $L_\sigma^p(\mathbb{R}^N)$ are both Banach spaces. Notice that if $\sigma = 0$, then

$$Y_0^0\left(\mathbb{R}^N\right) = C_0\left(\mathbb{R}^N\right), \qquad L_0^\infty\left(\mathbb{R}^N\right) = L^\infty\left(\mathbb{R}^N\right),$$
$$L_0^p\left(\mathbb{R}^N\right) = L^p\left(\mathbb{R}^N\right). \tag{2.4}$$

Next we give the definition of the ω-limit set $\omega_\sigma^{\mu,\beta}(u_0)$ which is our main study object in this paper .

Definition 2.1. Let $\sigma \geq 0$, μ, $\beta > 0$, and suppose that $u_0 \in Y_0^\sigma(\mathbb{R}^N)$. The ω-limit set $\omega_\sigma^{\mu,\beta}(u_0)$ is given by

$$\omega_\sigma^{\mu,\beta}(u_0) \equiv \left\{ f \in Y_0^\sigma\left(\mathbb{R}^N\right);\ \exists t_n \longrightarrow \infty \text{ s.t. } D_{\sqrt{t_n}}^{\mu,\beta}[S(t_n)u_0] \xrightarrow{n\to\infty} f \text{ in } Y_0^\sigma\left(\mathbb{R}^N\right) \right\}. \tag{2.5}$$

Here $D_\lambda^{\mu,\beta}\varphi(x) \equiv \lambda^\mu \varphi(\lambda^{2\beta}x)$ for $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\lambda > 0$.

In the rest of this section, we will consider the properties of the solutions for the problem (1.1) when the initial value $u_0 \in L_\sigma^p(\mathbb{R}^N)$ or $u_0 \in Y_0^\sigma(\mathbb{R}^N)$.

The following theorem can be seen as some extension of the maximum principle for the problem (1.1).

Theorem 2.2. *Let* $0 \leq \sigma < \infty$. *Suppose that*

$$u_0 \in L_\sigma^\infty\left(\mathbb{R}^N\right) \tag{2.6}$$

and that $u(x,t) = S(t)u_0(x)$ *are the mild solutions of the problem* (1.1). *Then*

$$u(t) = S(t)u_0 \in L_\sigma^\infty\left(\mathbb{R}^N\right) \quad \text{for } t > 0. \tag{2.7}$$

Moreover, if $t > 1$, *then*

$$\|S(t)u_0\|_{L_\sigma^\infty(\mathbb{R}^N)} \leq Ct^\sigma \|u_0\|_{L_\sigma^\infty(\mathbb{R}^N)}, \tag{2.8}$$

or if $0 < t \leq 1$, *then*

$$\|S(t)u_0\|_{L_\sigma^\infty(\mathbb{R}^N)} \leq C\|u_0\|_{L_\sigma^\infty(\mathbb{R}^N)}. \tag{2.9}$$

Remark 2.3. Let $\sigma = 0$. From Theorem 2.2, we can obtain the well-known result (maximum principle) that if $u_0 \in L_0^\infty(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$, then

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leq C\|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{2.10}$$

Proof. To prove this theorem, we need the fact that if

$$\varphi(x) = M\left(1 + |x|^2\right)^{\sigma/2} \quad \text{for some } M > 0, \tag{2.11}$$

then there exists a constant C such that

$$S(t)\varphi(x) \leq C\left(1 + t + |x|^2\right)^{\sigma/2}, \tag{2.12}$$

which proof can be found in [17]; we give the proof here for completeness. Consider the following problem:

$$\begin{gathered} \frac{\partial v}{\partial t} - \Delta v = 0, \quad \text{in } \mathbb{R}^N \times (0, \infty), \\ v(x,0) = v_0(x) = M|x|^\sigma, \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{2.13}$$

For $\lambda > 0$, from (2.1), we can get that

$$D_\lambda^{-\sigma/2,1/2} S(\lambda t)v_0(x) = \lambda^{-\sigma/2}[S(\lambda t)v_0]\left(\lambda^{1/2}x\right) = S(t)\left[D_\lambda^{-\sigma/2,1/2}v_0\right](x) = S(t)v_0(x). \tag{2.14}$$

By the existence and the regularity theories of the solutions, we can obtain that, for $t > 0$,

$$0 < S(t)v_0 \in C^\infty\left((0,\infty) \times \mathbb{R}^N\right), \tag{2.15}$$

see [10, 18]. Now taking $t = 1$, $\lambda = s$ and $g(x) = S(1)v_0(x)$ in the expression (2.14), we have

$$S(s)v_0(x) = s^{\sigma/2} g\left(s^{-1/2}x\right). \tag{2.16}$$

The fact that $S(s)v_0(x) \in C([0,\infty) \times \mathbb{R}^N \setminus (0,0))$ clearly implies that, for $|x| = 1$,

$$s^{\sigma/2} g\left(s^{-1/2}x\right) = S(s)v_0(x) \longrightarrow v_0(x) = M|x|^\sigma = M \tag{2.17}$$

as $s \to 0$. Let

$$y = s^{-1/2}x. \tag{2.18}$$

So,

$$|y| \longrightarrow \infty \quad \text{as } s \longrightarrow 0. \tag{2.19}$$

Therefore,

$$|y|^{-\sigma} g(y) - M \longrightarrow 0 \tag{2.20}$$

as $|y| \to \infty$. So, there exists constant $0 < C < \infty$ such that

$$0 \leq g(x) \leq C\left(1 + |x|^2\right)^{\sigma/2}. \tag{2.21}$$

By (2.16), we thus have

$$S(s)v_0(x) \leq C\left(s + |x|^2\right)^{\sigma/2}. \tag{2.22}$$

Notice that

$$0 \leq \varphi(x) \leq C(1 + v_0(x)). \tag{2.23}$$

Therefore, by comparison principle and (2.22), we can get that, for all $t \geq 0$, there exists constant $C > 0$ such that

$$0 \leq S(t)\varphi(x) \leq C\left(1 + t + |x|^2\right)^{\sigma/2}. \tag{2.24}$$

So we complete the proof of (2.12). For any $t > 0$, from (2.1) and (2.12), we thus obtain that

$$\begin{aligned}
|u(x,t)| &= |S(t)u_0(x)| = \left|(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} u_0(y)\mathrm{d}y\right| \\
&= \left|(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} \left(1 + |y|^2\right)^{\sigma/2} u_0(y) \left(1 + |y|^2\right)^{-\sigma/2} \mathrm{d}y\right| \\
&\leq \|u_0\|_{Y_\infty^\sigma(\mathbb{R}^N)} (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} \left(1 + |y|^2\right)^{\sigma/2} \mathrm{d}y \\
&\leq C\left(1 + t + |x|^2\right)^{\sigma/2} \|u_0\|_{Y_\infty^\sigma(\mathbb{R}^N)}.
\end{aligned} \tag{2.25}$$

Therefore, if $t > 1$, then

$$|u(x,t)| \leq C\left(1 + |x|^2\right)^{\sigma/2} t^{\sigma} \|u_0\|_{Y_\infty^\sigma(\mathbb{R}^N)}. \tag{2.26}$$

This clearly illustrates (2.8). If $0 < t \leq 1$, then

$$|u(x,t)| \leq C\left(1+|x|^2\right)^{\sigma/2} \|u_0\|_{Y_\infty^\sigma(\mathbb{R}^N)}. \tag{2.27}$$

From this, we can get (2.9). So we complete the proof of this theorem. □

Theorem 2.4 (L^p_σ–L^∞_σ smoothing effect). *Let $1 < p < \infty$ and $0 \leq \sigma < \infty$. Suppose $u_0 \in L^p_\sigma(\mathbb{R}^N)$ and that $u(x,t) = S(t)u_0(x)$ are the solutions of the problem* (1.1). *Then*

$$u(t) = S(t)u_0 \in L^\infty_\sigma\left(\mathbb{R}^N\right) \quad \textit{for } t > 0. \tag{2.28}$$

Moreover, if $t > 1$, then

$$\|S(t)u_0\|_{L^\infty_\sigma(\mathbb{R}^N)} \leq Ct^{\sigma - N/2p} \|u_0\|_{L^p_\sigma(\mathbb{R}^N)}, \tag{2.29}$$

or if $0 < t \leq 1$, then

$$\|S(t)u_0\|_{L^\infty_\sigma(\mathbb{R}^N)} \leq Ct^{-N/2p} \|u_0\|_{L^p_\sigma(\mathbb{R}^N)}. \tag{2.30}$$

Remark 2.5. If $\sigma = 0$, then Theorem 2.4 captures the result L^p–L^∞ smoothing effect for the heat equation.

Proof. For any $t > 0$, from (2.1) and Theorem 2.2, we thus obtain that

$$\begin{aligned}
|u(x,t)| = |S(t)u_0(x)| &= \left|(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} u_0(y)\,\mathrm{d}y\right| \\
&= \left|(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} \left(1+|y|^2\right)^{\sigma/2} u_0(y) \left(1+|y|^2\right)^{-\sigma/2} \mathrm{d}y\right| \\
&\leq \left[(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} \left(1+|y|^2\right)^{p'\sigma/2} \mathrm{d}y\right]^{1/p'} \\
&\quad \times \left[(4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} \left(1+|y|^2\right)^{-p\sigma/2} u_0^p(y)\,\mathrm{d}y\right]^{1/p} \\
&\leq C\left(1+t+|x|^2\right)^{\sigma/2} t^{-N/2p} \|u_0\|_{L^p_\sigma(\mathbb{R}^N)}.
\end{aligned} \tag{2.31}$$

Here $1/p + 1/p' = 1$. From this, we can get that, if $t \geq 1$, then

$$\|u(t)\|_{L^\infty_\sigma(\mathbb{R}^N)} \leq Ct^{\sigma/2 - N/2p} \|u_0\|_{L^p_\sigma(\mathbb{R}^N)}, \tag{2.32}$$

or if $0 < t < 1$, then

$$\|u(t)\|_{L^\infty_\sigma(\mathbb{R}^N)} \leq Ct^{-N/2p} \|u_0\|_{L^p_\sigma(\mathbb{R}^N)}. \tag{2.33}$$

So we complete the proof of this theorem. □

In the following theorem, we consider the property of the solutions $u(x,t)$ of (1.1) with the initial data $u_0 \in Y_0^\sigma(\mathbb{R}^N)$.

Theorem 2.6. *Let* $0 \le \sigma < \infty$. *If* $u_0 \in Y_0^\sigma(\mathbb{R}^N)$, *then*

$$S(t)u_0 \in Y_0^\sigma\left(\mathbb{R}^N\right) \quad \text{for } t \ge 0. \tag{2.34}$$

Proof. For $\sigma = 0$, the above theorem is a well-known result. So, in the rest of this proof, we assume that $0 < \sigma < \infty$. From (2.1), we have

$$\begin{aligned}
&\left(1+|x|^2\right)^{-\sigma/2}|u(x,t)| \\
&\quad = \left(1+|x|^2\right)^{-\sigma/2}\left|(4\pi t)^{-N/2}\int_{\mathbb{R}^N} \exp^{(-|x-y|^2/4t)} u_0(y)\,dy\right| \\
&\quad = \left(1+|x|^2\right)^{-\sigma/2}\left|(4\pi t)^{-N/2}\int_{|y|\le M} \exp^{(-|x-y|^2/4t)} u_0(y)\,dy\right| \\
&\qquad + \left(1+|x|^2\right)^{-\sigma/2}\left|(4\pi t)^{-N/2}\int_{|y|\le M} \exp^{(-|x-y|^2/4t)} u_0(y)\,dy\right| \\
&\quad = \mathrm{I}_1(x) + \mathrm{I}_2(x).
\end{aligned} \tag{2.35}$$

For any $\varepsilon > 0$, from $u_0 \in Y_0^\sigma(\mathbb{R}^N)$, we obtain that there exists an $M > 0$ such that if $|y| > M$, then

$$\left|u_0(y)\left(1+|y|^2\right)^{-\sigma/2}\right| < \varepsilon. \tag{2.36}$$

So, from (2.12), we have

$$\begin{aligned}
\mathrm{I}_2(x) &= \left(1+|x|^2\right)^{-\sigma/2}\left|(4\pi t)^{-N/2}\int_{|y|>M} \exp^{(-|x-y|^2/4t)} u_0(y)\left(1+|y|^2\right)^{-\sigma/2}\left(1+|y|^2\right)^{\sigma/2} dy\right| \\
&\le \left(1+|x|^2\right)^{-\sigma/2}\sup_{|y|>M}\left|u_0(y)\left(1+|y|^2\right)^{-\sigma/2}\right|(4\pi t)^{-N/2}\int_{|y|>M} \exp^{(-|x-y|^2/4t)}\left(1+|y|^2\right)^{\sigma/2} dy \\
&\le C\left(1+|x|^2\right)^{-\sigma/2}\left(1+t+|x|^2\right)^{\sigma/2}\varepsilon.
\end{aligned} \tag{2.37}$$

So, if $t > 1$, then

$$\mathrm{I}_2(x) \le C\left(1+|x|^2\right)^{-\sigma/2}\left(1+t+|x|^2\right)^{\sigma/2}\varepsilon \le C t^{\sigma/2}\varepsilon, \tag{2.38}$$

or if $0 \le t \le 1$, then

$$I_2(x) \le C\left(1+|x|^2\right)^{-\sigma/2}\left(1+t+|x|^2\right)^{\sigma/2}\varepsilon \le C\varepsilon. \tag{2.39}$$

Notice also that

$$u_0 \in Y_0^{\sigma}\left(\mathbb{R}^N\right) \subset C\left(\mathbb{R}^N\right). \tag{2.40}$$

So, there exists a constant C such that

$$\sup_{|y|\le M} \left|u_0(y)\right| \le C. \tag{2.41}$$

This means that

$$I_1(x) = \left(1+|x|^2\right)^{-\sigma/2}\left|(4\pi t)^{-N/2}\int_{|y|\le M} \exp^{(-|x-y|^2/4t)} u_0(y)\,dy\right| \le C\left(1+|x|^2\right)^{-\sigma/2}. \tag{2.42}$$

So, there exists an $M_1 > 0$ such that if $|x| > M_1$, then

$$I_1(x) < \varepsilon. \tag{2.43}$$

Combining this with (2.35), (2.38), and (2.39), we can obtain that, for $t \ge 0$,

$$u(t) = S(t)u_0 \in Y_0^{\sigma}\left(\mathbb{R}^N\right). \tag{2.44}$$

So we complete the proof of this theorem. □

3. Complicated Asymptotic Behavior

In this section, we investigate the asymptotic behavior of solutions for the problem (1.1) and give the fact that the weighted space $Y_0^{\sigma}(\mathbb{R}^N)$ with $0 \le \sigma < N$ can provide a setting where complexity occurs in the asymptotic behavior of solutions.

Theorem 3.1. *Let $\mu > 0$, $\sigma \ge 0$, $p > 1$, and $\beta > 1/2$. If*

$$\begin{gathered} \mu + 2\beta\sigma < \frac{N}{p}, \\ f \in Y_0^{\sigma}\left(\mathbb{R}^N\right) \bigcap L_{\sigma}^{p}\left(\mathbb{R}^N\right), \end{gathered} \tag{3.1}$$

then there exists an initial value $u_0 \in Y_0^{\sigma}(\mathbb{R}^N)$ and a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$D_{\sqrt{t_n}}^{\mu,\beta} S(t_n)u_0 \xrightarrow{n\to\infty} f \quad \text{in } Y_0^{\sigma}\left(\mathbb{R}^N\right). \tag{3.2}$$

That is, $f \in \omega_{\sigma}^{\mu,\beta}(u_0)$.

Proof. Suppose that $a > 2$ is a constant. Then let

$$\lambda_1 = a,$$
$$\lambda_j = \max\left(j^{2p/(N-p(\mu+2\beta\sigma))}\lambda_{j-1}^{(2\beta N-p(\mu+2\beta\sigma))/(N-p(\mu+2\beta\sigma))},\ \left(2^j\lambda_{j-1}\right)^{1/\mu}\right) \quad \text{for } j > 1. \tag{3.3}$$

Now we define the initial-value u_0 as

$$u_0(x) = \sum_{j=1}^{\infty}\lambda_j^{-\mu} f\left(\frac{x}{\lambda_j^{2\beta}}\right) = \sum_{j=1}^{\infty} D_{\lambda_j^{-1}}^{\mu,\beta} f(x). \tag{3.4}$$

Let

$$\ell = \max\left(\|f\|_{Y_0^\sigma(\mathbb{R}^N)}, \|f\|_{L_\sigma^p(\mathbb{R}^N)}\right). \tag{3.5}$$

So,

$$\|u_0\|_{Y_0^\sigma(\mathbb{R}^N)} \leq \sum_{j=1}^{\infty}\lambda_j^{-\mu}\left\| f\left(\frac{1}{\lambda_j^{2\beta}}\cdot\right)\right\|_{Y_0^\sigma(\mathbb{R}^N)} \leq \sum_{j=1}^{\infty} 2^{-j}\|f(x)\|_{Y_0^\sigma(\mathbb{R}^N)} \leq \ell. \tag{3.6}$$

Here we have used the fact that if $0 < \lambda \leq 1$, then

$$\begin{aligned}
\|f(\lambda\cdot)\|_{Y_0^\sigma(\mathbb{R}^N)} &= \sup_{x\in\mathbb{R}^N}\left|\left(1+x^2\right)^{-\sigma/2} f(\lambda x)\right| \\
&= \sup_{x\in\mathbb{R}^N}\left| f(\lambda x)\left(1+|\lambda x|^2\right)^{-\sigma/2}\left(\frac{1+|\lambda x|^2}{1+|x|^2}\right)^{\sigma/2}\right| \\
&\leq \sup_{x\in\mathbb{R}^N}\left| f(\lambda x)\left(1+|\lambda x|^2\right)^{-\sigma/2}\right| \\
&= \|f\|_{Y_0^\sigma(\mathbb{R}^N)}.
\end{aligned} \tag{3.7}$$

Therefore, the sequence of (3.4) is convergent in $Y_0^\sigma(\mathbb{R}^N)$. This means that

$$u_0 \in Y_0^\sigma\left(\mathbb{R}^N\right). \tag{3.8}$$

As a result of (2.1), we see that, for $0 < t < T < \infty$,

$$D_{\lambda_n}^{\mu,\beta}\left[S\left(\lambda_n^2 t\right)u_0\right](x) = S\left(t\lambda_n^{2-4\beta}\right)(u_n + v_n + w_n) = S\left(t\lambda_n^{2-4\beta}\right)u_n + S\left(t\lambda_n^{2-4\beta}\right)v_n + S\left(t\lambda_n^{2-4\beta}\right)w_n, \tag{3.9}$$

where

$$\begin{aligned} u_n(x) &= \sum_{j=1}^{n-1} D_{\lambda_n}^{\mu,\beta}\left[D_{\lambda_j^{-1}}^{\mu,\beta} f(x)\right] = \lambda_n^{\mu}\sum_{j=1}^{n-1}\lambda_j^{-\mu} f\left(\frac{x\lambda_n^{2\beta}}{\lambda_j^{2\beta}}\right), \\ v_n(x) &= D_{\lambda_n}^{\mu,\beta}\left[D_{\lambda_n^{-1}}^{\mu,\beta} f(x)\right] = f(x), \\ w_n &= \sum_{j=n+1}^{\infty} D_{\lambda_n}^{\mu,\beta}\left[D_{\lambda_j^{-1}}^{\mu,\beta} f(x)\right] = \lambda_n^{\mu}\sum_{j=n+1}^{\infty}\lambda_j^{-\mu} f\left(\frac{x\lambda_n^{2\beta}}{\lambda_j^{2\beta}}\right). \end{aligned} \tag{3.10}$$

Notice that, if $\lambda \geq 1$, then

$$\begin{aligned} \|f(\lambda\cdot)\|_{Y_p^\sigma(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N}\left|\left(1+|x|^2\right)^{-\sigma/2} f(\lambda x)\right|^p \mathrm{d}x\right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^N}\left|f(\lambda x)\left(1+|\lambda x|^2\right)^{-\sigma/2}\left(\frac{1+|\lambda x|^2}{1+|x|^2}\right)^{\sigma/2}\right|^p \mathrm{d}x\right)^{1/p} \\ &\leq \lambda^{\sigma}\left(\int_{\mathbb{R}^N}\left|f(\lambda x)\left(1+|\lambda x|^2\right)^{-\sigma/2}\right|^p \mathrm{d}x\right)^{1/p} \\ &= \lambda^{\sigma-N/p}\|f\|_{Y_p^\sigma(\mathbb{R}^N)}, \end{aligned} \tag{3.11}$$

we thus obtain that

$$\|u_n\|_{Y_p^\sigma(\mathbb{R}^N)} \leq \sum_{j=1}^{n-1}\left(\frac{\lambda_n}{\lambda_j}\right)^{\mu-2\beta N/p+2\beta\sigma}\|f\|_{Y_p^\sigma(\mathbb{R}^N)} \leq n\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\mu-2\beta N/p+2\beta\sigma}\|f\|_{Y_p^\sigma(\mathbb{R}^N)}. \tag{3.12}$$

Consequently, for any $t > 0$, we can select N large enough to satisfy that if $n > N$, then

$$0 < \lambda_n^{2-4\beta} t < 1. \tag{3.13}$$

Here we have used the hypothesis $\beta > 1/2$. So, (2.30), (3.11), and (3.12) indicate

$$\begin{aligned}&\left\|S\left(\lambda_n^{2-4\beta}t\right)u_n(x)\right\|_{Y_0^\sigma(\mathbb{R}^N)}\\&\quad\leq C\left(\lambda_n^{2-4\beta}t\right)^{-N/2p}\|u_n(x)\|_{L_\sigma^p(\mathbb{R}^N)}\\&\quad\leq C\lambda_{n-1}^{2\beta N/p-2\beta\sigma-\mu}t^{-2N/p}(n\ell)\lambda_n^{\mu-N/p+2\beta\sigma}.\end{aligned}\tag{3.14}$$

From (3.3), we thus obtain that

$$\left\|S\left(\lambda_n^{2-4\beta}t\right)u_n(x)\right\|_{Y_0^\sigma(\mathbb{R}^N)}\leq Ct^{-2N/p}\ell n^{-1}\longrightarrow 0\tag{3.15}$$

as $n \to \infty$. From (3.7) and the definition of $w_n(x)$, we have

$$\|w_n\|_{Y_0^\sigma(\mathbb{R}^N)}\leq\left\|\sum_{j=n+1}^{\infty}\left(\frac{\lambda_n}{\lambda_j}\right)^{\mu}f\left(\left(\frac{\lambda_n}{\lambda_j}\right)^{2\beta}\cdot\right)\right\|_{Y_0^\sigma(\mathbb{R}^N)}\leq\sum_{j=n+1}^{\infty}\left(\frac{\lambda_n}{\lambda_j}\right)^{\mu}\ell.\tag{3.16}$$

Applying (2.9) to $S(t\lambda_n^{2-4\beta})w_n$, we thus have

$$\left\|S\left(t\lambda_n^{2-4\beta}\right)w_n\right\|_{Y_0^\sigma(\mathbb{R}^N)}\leq C\lambda_n^{\mu}\sum_{j=n+1}^{\infty}\lambda_{j+1}{}^{-\mu}\leq C\ell\sum_{j=n+1}^{\infty}2^{-j}\longrightarrow 0\tag{3.17}$$

as $n \to \infty$. Here we have used (3.3) and the fact that $0 < \lambda_n^{2-2\beta}t < 1$ for $n > N$. At present, we want to verify the claim that, for $0 \leq t < \infty$,

$$\left(S\left(\lambda_n^{2-4\beta}t\right)v_n - f\right)_{Y_0^\sigma(\mathbb{R}^N)}\xrightarrow{n\to\infty}0.\tag{3.18}$$

In fact, if $0 < t < \infty$, then

$$\begin{aligned}\left\|S\left(\lambda_n^{2-4\beta}t\right)v_n-f\right\|_{Y_0^\sigma(\mathbb{R}^N)}&\leq\sup_{|x|\leq M}\left(1+|x|^2\right)^{-\sigma/2}\left|S\left(\lambda_n^{2-4\beta}t\right)v_n(x)-f(x)\right|\\&\quad+\sup_{|x|>M}\left(1+|x|^2\right)^{-\sigma/2}\left|S\left(\lambda_n^{2-4\beta}t\right)v_n(x)\right|+\sup_{|x|>M}\left(1+|x|^2\right)^{-\sigma/2}|f(x)|.\end{aligned}\tag{3.19}$$

Notice that

$$v_n = f \in Y_0^\sigma\left(\mathbb{R}^N\right).\tag{3.20}$$

Therefore, from Theorem 2.6, we have

$$S\left(\lambda_n^{2-4\beta}t\right)v_n \in Y_0^\sigma\left(\mathbb{R}^N\right). \tag{3.21}$$

So, for any $\varepsilon > 0$, from (2.35), (2.39), and (2.43), we get that if $n > N$, there exists a constant M independing on n and t such that

$$\begin{gathered}\sup_{|x|>M}\left(1+|x|^2\right)^{-\sigma/2}\left|S\left(\lambda_n^{2-4\beta}t\right)v_n(x)\right| < \frac{\varepsilon}{3},\\ \sup_{|x|>M}\left(1+|x|^2\right)^{-\sigma/2}\left|f(x)\right| < \frac{\varepsilon}{3}.\end{gathered} \tag{3.22}$$

The fact $v_n = f \in C(\mathbb{R}^N)$ means that

$$S(t)v_n \in C\left([0,\infty)\times\mathbb{R}^N\right). \tag{3.23}$$

Therefore, for any $\varepsilon > 0$, there exists an $N_1 > 0$ such that if $n > N_1$, then

$$\sup_{|x|\le M}\left(1+|x|^2\right)^{-\sigma/2}\left|S\left(\lambda_n^{2-4\beta}t\right)v_n(x) - f(x)\right| < \frac{\varepsilon}{3}. \tag{3.24}$$

So, from (3.19), (3.22), and (3.24), we thus obtain that

$$\left\|S\left(\lambda_n^{2-4\beta}t\right)v_n - f\right\|_{Y_0^\sigma(\mathbb{R}^N)} \longrightarrow 0 \tag{3.25}$$

as $n \to \infty$. From (3.9), (3.15), (3.17), and (3.18), we obtain that for any fixed $0 < t < \infty$,

$$D_{\lambda_n}^{\mu,\beta}S\left(\lambda_n^2 t\right)u_0 = S\left(\lambda_n^{2-2\beta}t\right)(u_n+v_n+w_n) \xrightarrow{n\to\infty} f \quad \text{in } Y_0^\sigma\left(\mathbb{R}^N\right). \tag{3.26}$$

Equation (3.2) follows from (3.26) by setting $t = 1$ and $t_n = \lambda_n^2$. The proof of this theorem is complete. □

Theorem 3.2. *Let $\mu > 0$, $\sigma \ge 0$, and $\beta > 1/2$. If*

$$0 < \mu + 2\beta\sigma < N, \tag{3.27}$$

then there exists an initial value $u_0 \in Y_0^\sigma(\mathbb{R}^N)$ such that

$$\omega_\sigma^{\mu,\beta}(u_0) = Y_0^\sigma\left(\mathbb{R}^N\right). \tag{3.28}$$

Remark 3.3. If $\sigma = 0$, the above results had been given by Cazenave et al. (see [15, Corollary 6.3]). So our results capture their results. Here we have used some of their ideas.

Proof. By (3.27), there exists a constant $p > 1$ such that

$$\mu + 2\beta\sigma < \frac{N}{p}. \tag{3.29}$$

Therefore, there exists a countable dense subset F of $Y_0^{\sigma}(\mathbb{R}^N)$ such that

$$F \subset Y_0^{\sigma}\left(\mathbb{R}^N\right) \bigcap L_{\sigma}^{p}\left(\mathbb{R}^N\right). \tag{3.30}$$

So, there exists a sequence $\{\varphi_j\}_{j\geq 1} \subset F$ such that

(i) for any $\phi \in F$, there exists a subsequence $\{\varphi_{j_k}\}_{k\geq 1}$ of the sequence $\{\varphi_j\}_{j\geq 1}$ satisfying

$$\varphi_{j_k}(x) = \phi \quad \forall k \geq 1, \tag{3.31}$$

(ii) there exists a constant $C > 0$ satisfying

$$\max\left(\|\varphi_j\|_{Y_0^{\sigma}(\mathbb{R}^N)}, \|\varphi_j\|_{L_{\sigma}^{p}(\mathbb{R}^N)}\right) \leq Cj \quad \text{for } j \geq 1. \tag{3.32}$$

Now we select a constant

$$a > 2 \tag{3.33}$$

and then take

$$\lambda_j = \begin{cases} a & \text{if } j = 1, \\ \max\left(j^{4p/(N-p(\mu+2\beta\sigma))}\lambda_{j-1}^{(2\beta N-p(\mu+2\beta\sigma))/(N-p(\mu+2\beta\sigma))}, \left(2^j\lambda_{j-1}j\right)^{1/\mu}\right) & \text{if } j > 1. \end{cases} \tag{3.34}$$

The initial-value u_0 is given by

$$u_0(x) = \sum_{j=1}^{\infty}\lambda_j^{-\mu}\varphi_j\left(\frac{x}{\lambda_j^{2\beta}}\right) = \sum_{j=1}^{\infty}D_{\lambda_j^{-1}}^{\mu,\beta}\varphi_j(x). \tag{3.35}$$

From (3.7), (3.32), and (3.34), we have

$$\|u_0\|_{Y_0^{\sigma}(\mathbb{R}^N)} \leq \sum_{j=1}^{\infty}\lambda_j^{-\mu}\|\varphi_j\|_{Y_0^{\sigma}(\mathbb{R}^N)} \leq C\sum_{j=1}^{\infty}2^{-j} \leq C. \tag{3.36}$$

This means that the sequence (3.35) is convergent in $Y_0^{\sigma}(\mathbb{R}^N)$. So,

$$u_0 \in Y_0^{\sigma}\left(\mathbb{R}^N\right). \tag{3.37}$$

Similar to the proof of Theorem 2.2, we can prove that, for any $\phi \in F$ and $0 < t < \infty$, there exists a sequence $\lambda_{j_k} \to \infty$ as $k \to \infty$ such that

$$D^{\mu,\beta}_{\lambda_{j_k}} S\left(\lambda^2_{j_k} t\right) u_0 \to \phi \quad \text{in } Y^{\sigma}_0\left(\mathbb{R}^N\right) \tag{3.38}$$

as $k \to \infty$. Let $t_{n_k} = \lambda^{1/2}_{n_k}$ and $t = 1$ in (2.1); we thus have

$$D^{\mu,\beta}_{\sqrt{t_{j_k}}} S(t_{j_k}) u_0 \longrightarrow \phi \quad \text{in } Y^{\sigma}_0\left(\mathbb{R}^N\right). \tag{3.39}$$

Therefore,

$$F \subset \omega^{\mu,\beta}_{\sigma}(u_0) \subset Y^{\sigma}_0\left(\mathbb{R}^N\right). \tag{3.40}$$

Notice that F is dense in $Y^{\sigma}_0(\mathbb{R}^N)$ and that $\omega^{\mu,\beta}_{\sigma}(u_0)$ is a closed subset of $Y^{\sigma}_0(\mathbb{R}^N)$. We thus obtain from (3.40) that

$$\omega^{\mu,\beta}_{\sigma}(u_0) = Y^{\sigma}_0\left(\mathbb{R}^N\right). \tag{3.41}$$

So we complete the proof of this theorem. □

Remark 3.4. For any $0 \leq \sigma < N$, there exist constants $\mu > 0$ and $\beta > 1/2$ such that

$$0 < \mu + 2\beta\sigma < N. \tag{3.42}$$

Therefore, from Theorem 3.2, we can get that the weighted space $Y^{\sigma}_0(\mathbb{R}^N)$ provides a setting where complexity occurs in the asymptotic behavior of solutions for the problem (1.1).

Acknowledgment

This work is supported by the National Natural Science Foundation of China, the Specialized Research Fund for the Doctoral Program of Higher Education of China, the Scientific and Technological Projects of Chongqing Municipal Commission of Education.

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