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Research Article

Necessary Optimality Conditions for a Class of Impulsive and Switching Systems

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An optimal control problem for a class of hybrid impulsive and switching systems is considered. By defining switching times as part of extended state, we get the necessary optimality conditions for this problem. It is shown that the adjoint variables satisfy certain jump conditions and the Hamiltonian are continuous at switching instants. In addition, necessary optimality conditions of Fréchet subdifferential form are presented in this paper.

1. Introduction

In a broad sense, hybrid systems are control systems involving both continuous and discrete variables. Much attention has been paid on the control of hybrid systems in recent years, see [1–27] and references therein. Impulsive and switching systems are a particular class of hybrid systems characterized by switches of states and abrupt changes at the switching instants. Many real-world processes such as evolutionary processes, flying object motions, signal processing systems, and so forth, can be modeled as impulsive and switching systems. Therefore, it is important to study the properties of this kind of systems. In [1], by using switched Lyapunov functions, Guan et al. gave some general criteria for exponential stability and asymptotic stability of some impulsive and switching models. The robust stability conditions for several types of impulsive switching systems were presented in [2–5]. Sufficient conditions for exponential stability and input-to-state stability of nonlinear impulsive switched systems were established in [6, 7]. All of these papers concentrated on the stability of impulsive and switching systems.

Over the last few decades, there have been a large number of researches on optimal control of hybrid systems. Different approaches have been proposed in this fields, such as the viscosity solution technique [8], dynamic programming methods [9–11], the embedding approach [12], direct differentiation of the cost function [13–15], necessary optimality conditions [16–24] and the method of smoothed approximation [25].

In [26], Gao et al. investigated optimal control problem for a class of impulsive and switching systems, where the impulses and switches occurred at fixed instants. Taking advantage of Ekeland's variational principle, they obtained the necessary optimality conditions for the continuous parts. In this paper, we consider similar problems with unfixed impulsive and switching instants. By introducing a new time variable, we get necessary optimality conditions for the continuous parts and the switching instants. Compared with the continuous parts, the necessary conditions at the switching times is more challenging and difficult to obtain. Furthermore, by applying Fréchet subdifferential, we give necessary optimality conditions with nonsmooth cost functional. Usually, the advantage of hybrid system lies in its nonsmooth trajectory. Therefore, the use of nonsmooth objective functional can reflect this advantage better.

The rest of the paper is organized as follows. In Section 2, the problem is formulated. In Section 3, necessary optimality conditions with smooth and nonsmooth cost functional are presented. Section 4 concludes the work.

2. Problem Formulation

Consider the following controlled nonlinear systems in fixed time interval $[t_0, t_N]$:

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) + v(t, x), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \Omega$ is a piecewise continuous control input, Ω is a bounded and convex set in $\mathbb{R}^m$, f is twice continuous differentiable with their variables. $v(t, x)$ is a hybrid impulsive and switching control, which can be described as follows:

$$\begin{aligned} v(t,x) &= v_1(t,x) + v_2(t,x) \\ &= \sum_{i=1}^{N-1} B_{1i} x(t) l_i(t) + \sum_{i=1}^{N-1} B_{2i} x(t) \delta(t - t_i), \end{aligned} \tag{2.2}$$

where B_{1i}, B_{2i} are $n \times n$ constant matrices, $\delta(\cdot)$ is the Dirac impulse and

$$l_i(t) = \begin{cases} 1, & t \in (t_{i-1}, t_i],\ i = 1, \ldots, N \\ 0, & \text{others}. \end{cases} \tag{2.3}$$

Here we suppose $t_1 < t_2 < \cdots < t_{N-1}$ are unfixed points.

Let the state of (2.1) is left continuous, $x(t_i) = x(t_i^-) = \lim_{t \to t_i^-} x(t)$. Denote $x(t_i^+) = \lim_{t \to t_i^+} x(t)$, $A_i = A + B_{1i}$ and $B_i = B_{2i}$, then under the control of (2.2), the nonlinear system (2.1) can be expressed as follows:

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + f(x(t), u(t)), \quad t \in (t_{i-1}, t_i],\ i = 1, \ldots, N, \\ x(t_i^+) &= (I_n + B_i) x(t_i), \quad i = 1, \ldots, N-1. \end{aligned} \tag{2.4}$$

In addition, we suppose that the system (2.4) satisfies $x(t_0^+) = x_0$ and $g(x(t_N)) = 0$.

Now we give the following optimal control problem (P): for impulsive switching systems (2.4), find impulsive switching instants t_i, $i = 1, \ldots, N-1$, and a piecewise continuous input $u(t)$, such that the cost functional

$$J = \sum_{i=1}^{N} \varphi_i(x(t_i)) + \int_{t_0}^{t_N} L(x(t), u(t))dt \tag{2.5}$$

is minimized, where φ_i denote the switching cost, $i = 1, \ldots, N-1$, L describes the operating cost of the continuous parts and φ_N denotes the cost relative to the terminal state. All of these functions are twice continuous differentiable with their variables.

Optimal control problem for (2.4) was considered in [26], where the switching instants were fixed. In this paper, we consider such a problem with unfixed switching times, which is more flexible in practical applications.

To this end, let (x^0, u^0) be a solution of problem (P) with a piecewise continuous control function u^0.

3. Necessary Conditions of Optimality with Smooth Cost Functional

Now we give a necessary condition for problem (P).

Theorem 3.1. *There exist a piecewise continuously differential variable $\lambda(t) : [t_0, t_N] \rightarrow \mathbb{R}^n$, multipliers $\mu \in \mathbb{R}$ and $\lambda_0 \in \mathbb{R}$, such that (x^0, u^0) satisfies*

$$\dot{\lambda}(t) = -\lambda_0 \frac{\partial L}{\partial x} - \left(A_i + \frac{\partial f}{\partial x}\right)^T \lambda(t), \quad t \in [t_0, t_N], \tag{3.1}$$

$$\lambda(t_N) = \lambda_0 \frac{\partial \varphi_N}{\partial x(t_N)} + \mu \frac{\partial g}{\partial x(t_N)}, \tag{3.2}$$

$$u^0(t) = \operatorname{argmin}\left\{ H\left(x^0(t), u, \lambda_0, \lambda(t)\right) \mid u \in \Omega \right\}, \quad t \in [t_0, t_N], \tag{3.3}$$

$$H\left[t_i^{0+}\right] = H\left[t_i^{0-}\right], \tag{3.4}$$

$$\lambda\left(t_i^{0-}\right) = \lambda_0 \frac{\partial\left[\varphi_i\left(x^0\left(t_i^0\right)\right)\right]}{\partial x(t_i)} + (I_n + B_i)^T \lambda\left(t_i^{0+}\right), \quad i = 1, \ldots, N-1, \tag{3.5}$$

where $H(x, u, \lambda) = \lambda_0 L(x, u) + \lambda^T [A_i x + f(x, u)]$.

Proof. For $i = 1, \ldots, N$, define

$$\begin{aligned} x_i(s) &= x(t_{i-1} + s(t_i - t_{i-1})), \\ u_i(s) &= u(t_{i-1} + s(t_i - t_{i-1})), \end{aligned} \tag{3.6}$$

with a new time variables $s \in [0,1]$, then problem (P) can be reformulated as the following classical problem (Q): minimize

$$\sum_{i=1}^{N} \varphi_i(x_i(1)) + \sum_{i=1}^{N} \int_0^1 [(t_i(s) - t_{i-1}(s)) L(x_i(s), u_i(s))] ds, \tag{3.7}$$

subject to

$$\begin{gathered}
\dot{x}_i(s) = (t_i(s) - t_{i-1}(s)) \left[A_i x(s) + f(x_i(s), u_i(s)) \right], \quad s \in [0,1],\ i = 1, \ldots, N, \\
\dot{t}_i(s) = 0, \quad s \in [0,1],\ i = 1, \ldots, N-1, \\
x_{i+1}(0) = (I_n + B_i) x_i(1), \quad s \in [0,1],\ i = 1, \ldots, N-1, \\
x_1(0) = x_0, \\
g(x_N(1)) = 0.
\end{gathered} \tag{3.8}$$

The Hamiltonian of problem (Q) is denoted by $\widehat{H}(x, u, \lambda) = \sum_{i=1}^{N} \{ (t_i - t_{i-1}) [\lambda_0 L(x_i, u_i) + \lambda_i^T (A_i x_i + f(x_i, u_i))] \}$.

Applying classical necessary optimality conditions to problem (R), there exist $\lambda_i(s) \in \mathbb{R}^n$, $i = 1, \ldots, N$, $\lambda_i^*(s) \in \mathbb{R}$, $i = 1, \ldots, N-1$, $\xi_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}^n$, $i = 1, \ldots, N-1$, such that for

$$\phi(x, u) = \lambda_0 \left[\sum_{i=1}^{N} \varphi_i(x_i(1)) \right] + \sum_{i=1}^{N-1} \omega_i^T [x_{i+1}(0) - (I_n + B_i) x_i(1)] + \mu g(x_N(1)), \tag{3.9}$$

we have

$$\dot{\lambda}_i(s) = -\frac{\partial \widehat{H}(x^0, u^0, \lambda_i)}{\partial x_i} = -(t_i - t_{i-1}) \left[\lambda_0 \frac{\partial L}{\partial x_i} - \left(A_i + \frac{\partial f}{\partial x_i} \right)^T \lambda_i(t) \right], \quad i = 1, \ldots, N, \tag{3.10}$$

$$\dot{\lambda}_i^*(s) = -\frac{\partial \widehat{H}(x^0, u^0, \lambda_i)}{\partial t_i} = -H_i\left(x_i^0, u_i^0, \lambda_i\right), \quad i = 1, \ldots, N-1, \tag{3.11}$$

$$\lambda_{i+1}(0) = -\frac{\partial \phi(x^0, u^0)}{\partial x_{i+1}(0)} = -\omega_i, \quad i = 1, \ldots, N-1, \tag{3.12}$$

$$\lambda_i(1) = \frac{\partial \phi(x^0, u^0)}{\partial x_i(1)} = \lambda_0 \frac{\partial \phi_i(x_i(1))}{\partial x_i(1)} - (I_n + B_i)^T \omega_i, \quad i = 1, \ldots, N-1, \tag{3.13}$$

$$\lambda_N(1) = \frac{\partial \phi(x^0, u^0)}{\partial x_N(1)} = \lambda_0 \frac{\partial \phi_N}{\partial x_N(1)} + \mu \frac{\partial g}{\partial x_N(1)}, \tag{3.14}$$

$$\lambda_i^*(0) = -\frac{\partial \phi(x^0, u^0)}{\partial t_i(0)} = 0, \quad i = 1, \ldots, N-1, \tag{3.15}$$

$$\lambda_i^*(1) = \frac{\partial \phi(x^0, u^0)}{\partial t_i(1)} = 0, \quad i = 1, \ldots, N-1. \tag{3.16}$$

$$u^0(s) = \operatorname{argmin}\left\{ \widehat{H}\left(x^0(s), u(s), \lambda_0, \lambda(s)\right) \mid u \in \Omega \right\}. \tag{3.17}$$

For autonomous system, the Hamiltonian is constant along the optimal trajectory, so the right hand of (3.11) is constant on $[0, 1]$. Combining (3.11) with the boundary conditions (3.15) and (3.16), We get the continuity condition (3.4).

Recombine the adjoint variable

$$\lambda(t) = \lambda_i\left(\frac{t - t_{i-1}^0}{t_i^0 - t_{i-1}^0}\right), \quad t \in \left[t_{i-1}^0, t_i^0\right], \, i = 1, \ldots, N, \tag{3.18}$$

we get (3.1) from (3.10). (3.2) and (3.3) come from (3.14) and (3.20), respectively. (3.12) and (3.13) result in the jump condition (3.5) by eliminating ω_i. The proof of the theorem is completed. □

Remark 3.2. Besides the necessary optimality conditions for the continuous parts and the terminal costate, which were derived in [26], Theorem 3.1 also give the necessary optimality conditions for the switching instants. Therefore, Theorem 3.1 is an important improvement and generalization of the main results in [26].

Example 3.3. Minimize the cost functional

$$J = x^2(t_1) + \int_0^2 (x(t) - 2u(t))dt, \tag{3.19}$$

subject to

$$\begin{gathered} \dot{x}(t) = -x(t) + u(t), \quad t \in [0, t_1], \\ x\left(t_1^+\right) = 2x\left(t_1^-\right), \\ \dot{x}(t) = x(t) + u(t), \quad t \in (t_1, 2], \\ x(0) = 2, \\ x(2) = 6, \end{gathered} \tag{3.20}$$

where $0 \le u \le 1$.

For $t \in [0, t_1]$, we deduce that $\lambda(t) = c_1 e^t + 1$ by (3.1). Denote $H = x(t) - 2u(t) + \lambda(t)(-x(t) + u(t))$, taking advantage of (3.3), we get that

$$u(t) = \begin{cases} 0, & \lambda - 2 > 0 \\ 1, & \lambda - 2 > 0 \end{cases} = \begin{cases} 0, & c_1 e^t > 1, \\ 1, & c_1 e^t < 1, \end{cases} \tag{3.21}$$

where c_1 is a constant in $\mathbb{R}$.

In addition, by $\dot{x}(t) = -x(t) + u(t)$ and $x(0) = 2$, we obtain that

$$x(t) = \begin{cases} e^{-t} + 1, & t \in [0, t_s], \\ (e^{t_1} + 1)e^{-t}, & t \in (t_s, t_1], \end{cases} \tag{3.22}$$

where t_s is the root of $c_1 e^t = 1$.

Similarly, we can get the equations for the control, adjoint and state variable for $t \in (t_1, 2]$. By (3.4), we get that

$$\lambda(t_1^-) = 2x(t_1) + \lambda(t_1^+). \tag{3.23}$$

After some calculation, we obtain four equations by (3.5), (3.23), $c_1 e^{t_s} = 1$, and $x(t_1^+) = 2x(t_1^-)$. By MatLab procedure and Newton methods of solving nonlinear equations, we get that the impulsive and switching time is $t_1 = 1.1595$ and the control variables is

$$u(t) = \begin{cases} 1, & t \in [0, 0.8041], \\ 0, & t \in (0.8041, 1.1595], \\ 1, & t \in (1.1595, 2]. \end{cases} \tag{3.24}$$

The behavior of control, adjoint and state variable is shown in Figure 1.

4. Necessary Optimality Conditions with Nonsmooth Cost Functional

To establish superdifferential form of necessary optimality conditions, we shall introduce some basic knowledge of nonsmooth analysis. For more knowledge, the reader can refer to [18, 28].

Given a nonempty set $\Omega \in \mathbb{R}^n$ and $\overline{x} \in \Omega$, the prenormal cone to Ω at $\overline{x}$ is defined by the following:

$$\widehat{N}(\overline{x}; \Omega) := \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \to \overline{x}} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq 0 \right\}. \tag{4.1}$$

If $\overline{x} \notin \Omega$, put $\widehat{N}(\overline{x}; \Omega) := \emptyset$.

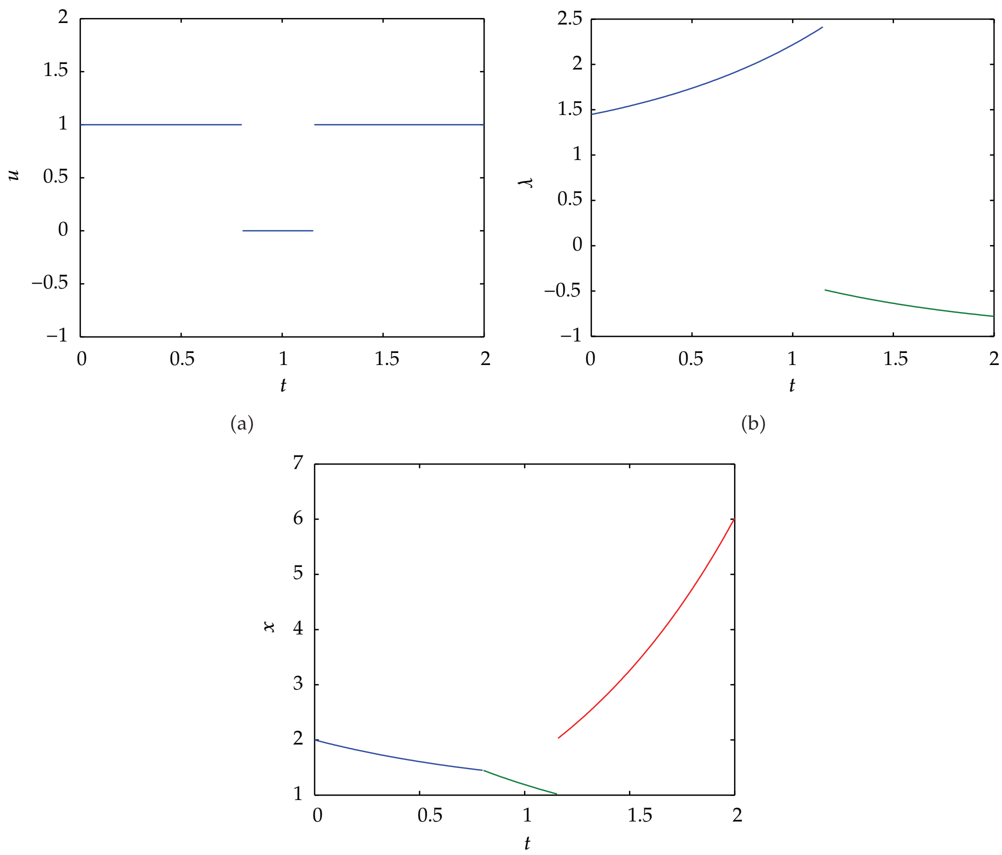

Figure 1: Control, adjoint, and state variable in the Example 3.3.

Let $\operatorname{epi}\varphi = \{(x,\alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq \varphi(x)\}$, if φ is lower semicontinuous, the Fréchet subdifferential of φ at $\overline{x}$ is expressed in geometric form

$$\widehat{\partial}\varphi(\overline{x}) := \left\{x^* \in \mathbb{R}^n \mid (x^*,-1) \in \widehat{N}\big((\overline{x},\varphi(\overline{x}));\operatorname{epi}\varphi\big)\right\}. \tag{4.2}$$

Similarly, if φ is supper semicontinuous, we can define the Fréchet supperdifferential of φ at $\overline{x}$ by $\widehat{\partial}^+\varphi(\overline{x}) := -\widehat{\partial}(-\varphi)(\overline{x})$. For example, let $\varphi(x) = |x|$, then $\widehat{\partial}\varphi(0) = [-1,1]$ and $\widehat{\partial}\varphi^+(0) = \emptyset$.

If φ is continuous, the Fréchet differential of φ at $\overline{x}$ is defined by

$$\nabla\varphi(\overline{x}) := \left\{x^* \in \mathbb{R}^n \mid \lim_{x\to\overline{x}} \frac{\varphi(x)-\varphi(\overline{x})-\langle x^*, x-\overline{x}\rangle}{\|x-\overline{x}\|} = 0\right\}. \tag{4.3}$$

Lemma 4.1 (see [18, 28]). *Let $\varphi : X \to \mathbb{R}$, $|\varphi(\overline{x})| < \infty$. Then for any $x^* \in \widehat{\partial}\varphi(\overline{x})$, there exists a function $s : X \to \mathbb{R}$ with $s(\overline{x}) = \varphi(\overline{x})$ and $s(x) \leq \varphi(x)$ whenever $x \in X$, such that $s(\cdot)$ is Fréchet differentiable at $\overline{x}$ with $\nabla s(\overline{x}) = x^*$.*

Theorem 4.2. *Let (x^0, u^0) be a weak local minimum of problem (P), φ_i is Fréchet subdifferentiable at $x^0(t_i)$, then for every $x^*(t_i) \in \widehat{\partial}\varphi_i(x^0(t_i))$, $i = 1, 2, \ldots, N$, the results of Theorem 3.1 hold except that* (3.1) *and* (3.4) *are replaced by the following:*

$$\lambda(t_N) = \lambda_0 x^*(b) + \mu \frac{\partial g}{\partial x(t_N)}, \tag{4.4}$$

$$\lambda\left(t_i^{0-}\right) = \lambda_0 x^*\left(t_i^0\right) + (I_n + B_i)^T \lambda\left(t_i^{0+}\right), \quad i = 1, \ldots, N-1. \tag{4.5}$$

Proof. For any $x^*(t_i^0) \in \widehat{\partial}\varphi_i(x^0(t_i))$, taking advantage of Lemma 4.1, there exist functions s_i, such that they are Fréchet differentiable at $x^0(t_i^0)$ and the following conditions are satisfied: $s_i(x^0(t_i^0)) = \varphi_i(x^0(t_i^0))$, $\nabla s_i(x^0(t_i^0)) = x^*(t_i^0)$, and $s_i(x(t_i)) = \varphi_i(x(t_i))$ in some neighborhood of $s_i(x^0(t_i^0))$, $i = 1, \ldots, N$. Therefore (τ^0, x^0, u^0) is a weak local minimum of problem (P3): Minimize the functional

$$J = \sum_{i=1}^{N} s_i(x(t_i)) + \int_a^b L(x(t), u(t))dt, \tag{4.6}$$

subject to (2.4). Combining the results of Lemma 4.1 and Theorem 3.1, we complete the proof of the theorem. □

Example 4.3. Minimize the cost functional

$$J = |x(t_1)| + \int_0^2 (x(t) - u(t))dt, \tag{4.7}$$

subject to

$$\begin{gathered} \dot{x}(t) = u(t), \quad t \in [0, t_1], \\ x\left(t_1^+\right) = \frac{1}{2} x\left(t_1^-\right), \\ \dot{x}(t) = -x(t) + u(t), \quad t \in (t_1, 2], \\ x(0) = 1, \\ x(2) = 0, \end{gathered} \tag{4.8}$$

where $-1 \leq u \leq 1$.

By (4.5), we have

$$\lambda\left(t_1^-\right) = d + \lambda\left(t_1^+\right), \tag{4.9}$$

where $d \in [-1, 1]$ is Fréchet subdifferential at the switching point.

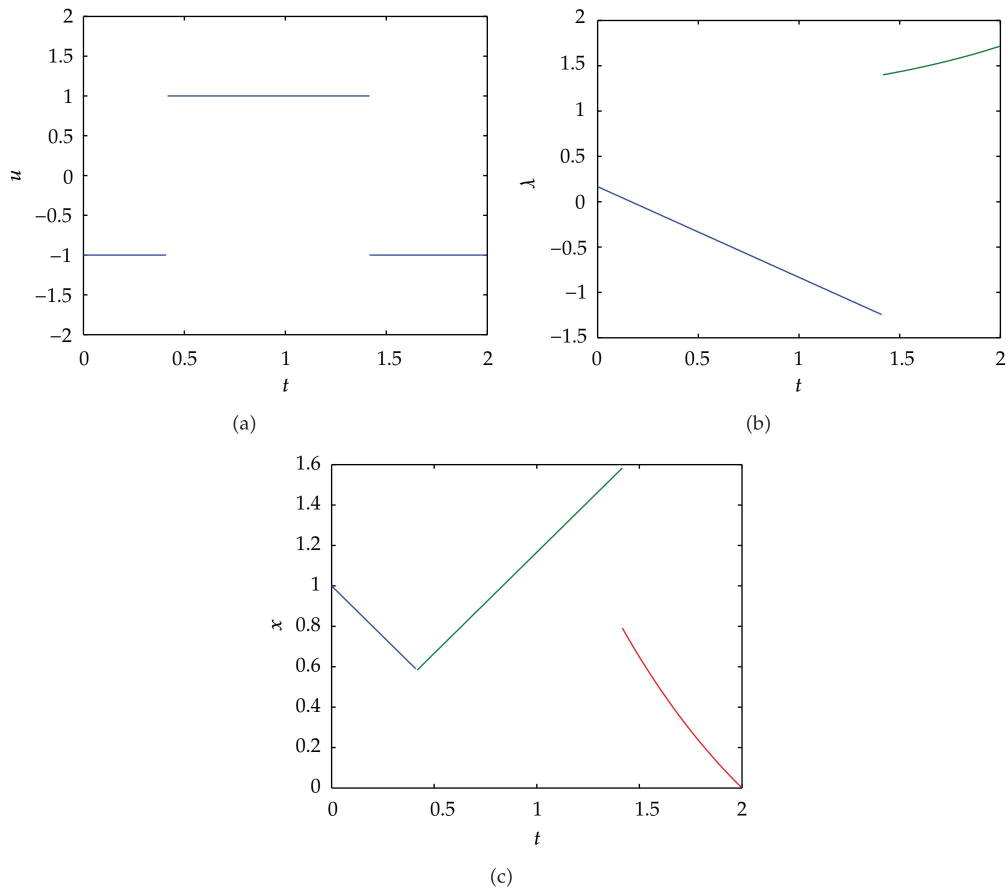

Figure 2: Control, adjoint, and state variable in the Example 4.3.

Taking advantage of similar method in Example 3.3, we get that the optimal impulsive switching time is 1.4169 and the control variables is

$$u(t) = \begin{cases} -1, & t \in [0, 0.4169], \\ 1, & t \in (0.4169, 1.4169], \\ -1, & t \in (1.4169, 2]. \end{cases} \tag{4.10}$$

The behavior of control, adjoint and state variable is shown in Figure 2.

Remark 4.4. As shown in Example 4.3, the necessary optimality condition of Fréchet subdifferential form allows us to get the optimal switching point for the problem with nonsmooth cost functional, but the usual necessary optimality conditions fails to do this.

5. Conclusions

In this paper, we have investigated optimal control problems for a class of impulsive and switching systems, where the switching transitions are unfixed. By defining the switching instants as part of extended state and taking advantage of the knowledge of nonsmooth analysis, the necessary optimality conditions with both smooth cost functional and nonsmooth cost functional are derived, which are the substantial extension and generalization of some known results in the literature.

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