Research Article

Solving Linear and Nonlinear Fractional Differential Equations Using Spline Functions

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Suitable spline functions of polynomial form are derived and used to solve linear and nonlinear fractional differential equations. The proposed method is applicable for $0 < \alpha \leq 1$ and $\alpha \geq 1$, where α denotes the order of the fractional derivative in the Caputo sense. The results obtained are in good agreement with the exact analytical solutions and the numerical results presented elsewhere. Results also show that the technique introduced here is robust and easy to apply.

1. Introduction

Mathematical modelling of complex processes is a major challenge for contemporary scientists. In contrast to simple classical systems, where the theory of integer-order differential equations is sufficient to describe their dynamics, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various complex materials and systems. Therefore, the number of scientific and engineering problems involving fractional derivatives is already very large and still growing and perhaps the fractional calculus (i.e., derivatives and integrals of any real or complex order) will be the calculus of the twenty-first century [1–6].

The analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors (see, e.g., [7, 8]). During the last decades, several numerical and analytical methods have been proposed in the literature to solve fractional differential equations. The most commonly used ones are fractional difference method [9, 10], Adomian decomposition method [11], variational iteration method

[12, 13], and Adams-Bashforth-Moulton method [14–16]. Recently, Lagrange multiplier method and the homotopy perturbation method are used to numerically solve multiorder fractional differential equations, see [17].

In view of successful application of spline functions of polynomial form in system analysis [18], delay differential equations [19], and delay differential equations of fractional order [20], we hold that it should be applicable to solve linear and nonlinear fractional-order systems.

In the present paper, we intend to extend the application of the spline functions of polynomial form to solve the fractional differential equations:

$$y^\alpha(x) = f(x, y(x)), \quad y(0) = y_0, \tag{1.1}$$

where α denotes the order of the fractional derivative in the Caputo sense.

2. The Spline Function Method of Polynomial Form

Recently, many studies were devoted to the problems of approximate solutions of system ordinary as well as delay differential equations by spline functions [18–21]. Micula et al. [18] considered the following system:

$$\begin{aligned}
y'(x) &= f_1(x, y, z), \quad y(x_0) = y_0, \\
z'(x) &= f_2(x, y, z), \quad z(x_0) = z_0,
\end{aligned} \tag{2.1}$$

where $f_1, f_2 \in C^r[0, 1] \times \mathbb{R} \times \mathbb{R}$, $(x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$. They assumed that the functions $f_i^{(q)}$, $i = 1, 2$, and $q = 0, 1, \ldots, r$ satisfy the Lipschitz condition of the form

$$\left| f_i^{(q)}(x, y_1, z_1) - f_i^{(q)}(x, y_2, z_2) \right| \le L_i \{ |y_1 - y_2| + |z_1 - z_2| \}, \tag{2.2}$$

with constant L_i for all $(x, y_i, z_i) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$. In their paper, they presented investigations to the extension of the spline functions form for approximating the solution of the system of initial value problems (2.1), with unique solutions $y = y(x)$ and $z = z(x)$.

The spline functions S_Δ and $\widetilde{S}_\Delta$ to approximate $y = y(x)$ and $z = z(x)$, respectively, are defined in polynomial form as

$$\begin{aligned}
S_\Delta(x) &= S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} f_1^{(i)}\left(x_k, S_{k-1}(x_k), \widetilde{S}_{k-1}(x_k)\right) \frac{(x - x_k)^{i+1}}{(i+1)!}, \\
\widetilde{S}_\Delta(x) &= \widetilde{S}_k(x) = \widetilde{S}_{k-1}(x_k) + \sum_{i=0}^{r} f_2^{(i)}\left(x_k, S_{k-1}(x_k), \widetilde{S}_{k-1}(x_k)\right) \frac{(x - x_k)^{i+1}}{(i+1)!},
\end{aligned} \tag{2.3}$$

for $x \in [x_k, x_{k+1}]$, $k = 0, 1, \ldots, n-1$, $S_{-1}(x_0) = y_0, \widetilde{S}_{-1}(x_0) = z_0$.

Ramadan introduced in [19] the solution of the first-order delay differential equation of the form:

$$y'(x) = f(x, y(x), y(g(x))), \quad a \le x \le b,$$
$$y(a) = y_0, \quad y(x) = \Phi(x), \quad x \in [a^*, a], \ a^* < 0, \ a^* = \inf\{g(x) : x \in [a, b]\}, \tag{2.4}$$

using the spline functions of the polynomial form, defined as

$$S_\Delta(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} M_k^{(i)} \frac{(x - x_k)^{i+1}}{(i+1)!}, \tag{2.5}$$

where $M_k^{(i)} = f^{(i)}(x_k, S_{k-1}(x_k), S_{k-1}(g(x_k)))$, with $S_{-1}(x_0) = y_0$, $S_{-1}(g(x_0)) = \Phi(g(x_0))$.

Ramadan in [21] considered the system of the initial value problem

$$y'''(x) = f_1(x, y, y', z, z'), \quad y^{(i)}(x_0) = y_0^{(i)},$$
$$z'''(x) = f_2(x, y, y', z, z'), \quad z^{(i)}(x_0) = z_0^{(i)}, \tag{2.6}$$

where $f_1, f_2 \in C^r([0, 1] \times \mathbb{R}^{(4)})$, $i = 0, 1, 2$. The method in their work is based on polynomial splines, to approximate the solutions of the system.

Ramadan et al. [20] presented an extension and generalization of the polynomial spline functions used in the case of [19], to approximate the solution of the first delay differential equation and to investigate the solution of the fractional ordinary differential equation given by

$$y^{(\alpha)}(x) = f(x, y(x)), \quad a \le x \le b,$$
$$y(a) = y_0, \quad \alpha > 1. \tag{2.7}$$

The formulation of the method presented in [20] is based on the fractional generalization of Taylor's theorem [22], which is presented in the following theorem

Theorem 2.1. *Let $\alpha > 0$, $n \in \mathbb{Z}^+$, and $f(x) \in C^{[\alpha]+n+1}([a, b])$. Then*

$$f(x) = \sum_{k=-n}^{n-1} \frac{D_{a+}^{\alpha+k} f(x_0)}{\Gamma(\alpha + k + 1)} (x - x_0)^{\alpha+k} + R_n(x), \tag{2.8}$$

for all $a \le x_0 < x \le b$, where

$$R_n(x) = J_{a+}^{\alpha+n} D_{a+}^{\alpha+n} f(x) \tag{2.9}$$

is the remainder.

Error estimation and convergence analysis of the proposed method was presented by Ramadan et al. [20]. In their work, they used $\alpha > 1$ and applied the method to solve two linear cases.

The present paper is a sequel to this work [20], and we extend the application of spline functions method of polynomial form to a more general case incorporating nonlinearities as well. And we show that this analysis is also applicable in the case $0 < \alpha \leq 1$, namely,

$$y^{(\alpha)}(x) = f(x, y(x)), \quad a \leq x \leq b,$$
$$y(a) = y_0, \quad \alpha > 0. \tag{2.10}$$

Let Δ be a uniform partition to the interval $[a, b]$, defined by the nodes $\Delta : a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n = b$, where $x_k = x_0 + kh$, $k = 0, 1, \ldots, n$, and $h = (b - a)/n$.

Define the form of fractional spline function $S(x)$ of polynomial form approximating the exact solution y by

$$S_\Delta(x) = S_k(x) = S_{k-1}^{(\alpha-1)}(x_k)\frac{(x - x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)}\frac{(x - x_k)^{\alpha+i}}{\Gamma(\alpha + i + 1)}, \quad n \in \mathbb{Z}^+, \tag{2.11}$$

where $M_k^{(\alpha)} = f^{(\alpha)}(x_k, S_{k-1}(x_k))$, with $S_{-1}(x_0) = y_0$, for $x \in [x_k, x_{k+1}]$.

Definition 2.2. Let $\alpha \in \mathbb{R}_+$. The operator J_a^α, defined on $L_1[a, b]$ by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x - t)^{1-\alpha}} dt \tag{2.12}$$

for $a \leq x \leq b$, is called the Riemann-Lioville fractional integral operator of order α.

For $\alpha = 0$, we set $J_a^0 := I$, the identity operator.

Definition 2.3. The Caputo fractional derivative of $f(x)$, of order $\alpha > 0$ with $a \geq 0$, is defined as

$$(D_a^\alpha f)(x) = \left(J_a^{m-\alpha} f^{(m)}\right)(x) := \frac{1}{\Gamma(m - \alpha)} \int_a^x \frac{f^{(m)}(t)}{(x - t)^{\alpha+1-m}} dt \tag{2.13}$$

for $m - 1 \leq \alpha \leq m$, $m \in \mathbb{N}$, $x \geq a$, and $f(x) \in C_{-1}^m$.

The Riemann-Liouville fractional operators may be extended to hold for large values of α, so we denote $\alpha = \lfloor \alpha \rfloor + \beta$, where $\lfloor \alpha \rfloor$ is the integer part of α, and $\beta = \alpha - \lfloor \alpha \rfloor$, thus we give the following definition.

Definition 2.4. If $\alpha > 0$ and $\alpha \notin \mathbb{N}$, then we define

$$D_a^\alpha f = \frac{d^{\lfloor \alpha \rfloor}}{dx^{\lfloor \alpha \rfloor}} D_a^\beta f = \frac{d^{\lfloor \alpha \rfloor+1}}{dx^{\lfloor \alpha \rfloor+1}} J_a^{1-\beta} f, \tag{2.14}$$

thus

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x - t)^{\alpha+1-m}} dt \tag{2.15}$$

Table 1: Solution of Example 3.1 using $\alpha = 0.5$ and $\alpha = 0.75$.

x	$\alpha = 0.5$		$\alpha = 0.75$	
	Appr. solution	Absolute error	Appr. solution	Absolute error
0.0	0.0	0.0	0.0	0.0
0.1	0.0	4.0000×10^{-4}	0.0	4.0000×10^{-4}
0.2	-3.5668×10^{-3}	1.1668×10^{-3}	-2.8054×10^{-3}	4.0543×10^{-4}
0.3	-5.6930×10^{-3}	2.9299×10^{-4}	-5.8753×10^{-3}	4.7533×10^{-4}
0.4	-6.8508×10^{-3}	4.5080×10^{-4}	-6.7877×10^{-3}	3.8767×10^{-4}
0.5	2.2930×10^{-3}	2.2930×10^{-3}	9.7776×10^{-4}	9.7776×10^{-4}
0.6	2.8146×10^{-2}	6.5464×10^{-3}	2.6460×10^{-2}	4.8601×10^{-3}
0.7	8.3911×10^{-2}	1.5311×10^{-2}	8.1759×10^{-2}	1.3159×10^{-2}
0.8	1.8149×10^{-1}	2.7885×10^{-2}	1.8163×10^{-1}	2.8034×10^{-2}
0.9	3.3755×10^{-1}	4.5954×10^{-2}	3.4348×10^{-1}	5.1882×10^{-2}
1.0	5.6863×10^{-1}	6.8632×10^{-2}	5.8718×10^{-1}	8.7178×10^{-2}

for any $f \in C^m([a,b])$, where $m = \lfloor \alpha \rfloor + 1$. If, on the other hand, $\alpha < 0$, then the definition becomes

$$D_a^\alpha f = J_a^{-\alpha} f. \tag{2.16}$$

3. Numerical Examples

To demonstrate the effectiveness of this scheme, we consider two kinds of systems: one is linear and the other is nonlinear. These examples are considered because closed form solutions are available for them, or they have also been solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes.

Example 3.1. Consider the fractional differential equation

$$D^\alpha y(x) = -y(x) + x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}x^{4-\alpha}, \quad 0 < \alpha \le 1, \tag{3.1}$$

with initial condition $y(0) = 0$.

The exact solution is

$$y(x) = x^4 - \frac{1}{2}x^3. \tag{3.2}$$

The numerical results obtained, for different values of α, and for $0 \le x \le 1$, are shown in Table 1 and Table 2, together with absolute errors, to illustrate the accuracy of the spline method of polynomial form. Figure 1(a) for $\alpha = 0.5$, Figure 1(b) for $\alpha = 0.75$, Figure 1(c) for $\alpha = 0.9$, and Figure 1(d) for $\alpha = 1.0$ show the approximate solutions compared to the exact solution, to better illustrate the accuracy of the method.

Table 2: Solution of Example 3.1 using $\alpha = 0.9$, and $\alpha = 1$.

x	$\alpha = 0.9$		$\alpha = 1.0$	
	Appr. solution	Absolute error	Appr. solution	Absolute error
0.0	0.0	0.0	0	0.0
0.1	0.0	4.0000×10^{-4}	-1.0000×10^{-3}	6.0000×10^{-4}
0.2	-2.4593×10^{-3}	5.9337×10^{-5}	-2.9700×10^{-3}	5.7000×10^{-4}
0.3	-5.4032×10^{-3}	3.1688×10^{-6}	-5.6338×10^{-3}	2.3383×10^{-4}
0.4	-6.0963×10^{-3}	3.0367×10^{-4}	-5.9088×10^{-3}	4.9117×10^{-4}
0.5	1.3284×10^{-3}	1.3284×10^{-3}	1.5404×10^{-3}	1.5404×10^{-3}
0.6	2.5395×10^{-2}	3.7951×10^{-3}	2.4489×10^{-2}	2.8891×10^{-3}
0.7	7.7426×10^{-2}	8.8255×10^{-3}	7.3107×10^{-2}	4.5075×10^{-3}
0.8	1.7148×10^{-1}	1.7883×10^{-2}	1.5997×10^{-1}	6.3700×10^{-3}
0.9	3.2435×10^{-1}	3.2755×10^{-2}	3.0005×10^{-1}	8.4533×10^{-3}
1.0	5.5552×10^{-1}	5.5518×10^{-2}	5.1074×10^{-1}	1.0736×10^{-2}

(a) Solution at $\alpha = 0.5$

(b) Solution at $\alpha = 0.75$

(c) Solution at $\alpha = 0.9$
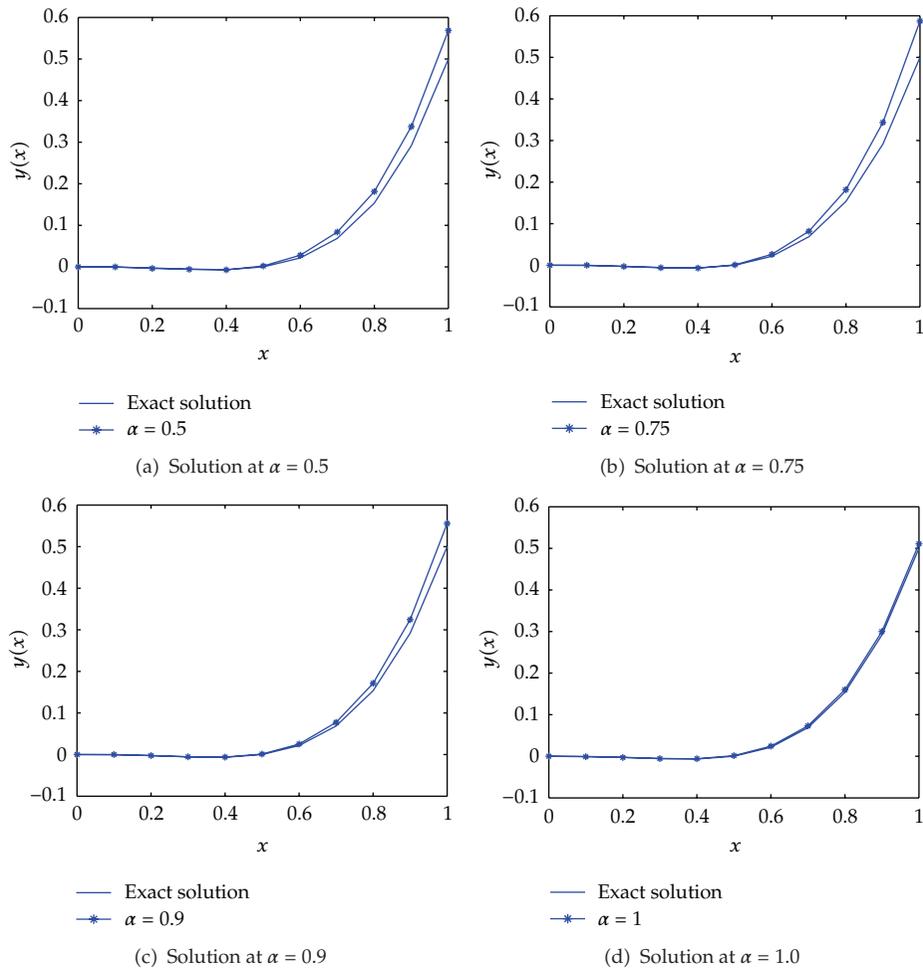
(d) Solution at $\alpha = 1.0$

Figure 1: Approximate solutions for Example 3.1.

Table 3: Solution of Example 3.2 using $\alpha = 1.3$, $\alpha = 1.1$, and $\alpha = 1$.

	$\alpha = 1.3$	$\alpha = 1.1$	$\alpha = 1$	
x	Appr. solution	Appr. solution	Appr. solution	Absolute error
0.0	0.00000	0.00000	0.00000	0.0
0.5	0.32538	0.41045	0.45833	3.7838×10^{-3}
1.0	0.52950	0.67714	0.76095	6.4768×10^{-4}
1.5	0.63709	0.81882	0.90603	8.7977×10^{-4}
2.0	0.68987	0.89089	0.96437	3.3837×10^{-4}
2.5	0.71508	0.92755	0.98660	1.5476×10^{-5}
3.0	0.72699	0.94629	0.99497	8.4583×10^{-5}
3.5	0.73259	0.95590	0.99811	6.4645×10^{-5}
4.0	0.73522	0.96084	0.99929	3.6906×10^{-5}
4.5	0.73646	0.96339	0.99973	1.8574×10^{-5}
5.0	0.73703	0.96470	0.99990	8.7168×10^{-6}

Example 3.2. Consider the following fractional Riccati equation:

$$D^{\alpha}y(x) = -y(x) + 1, \quad 1 \le \alpha < 2, \tag{3.3}$$

subject to the initial condition $y(0) = 0$.

The exact solution, when $\alpha = 1$, is

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{3.4}$$

The numerical results obtained, for different values of α, and for $0 \le x \le 5$, are shown in Table 3; the approximate solution for $\alpha = 1$ is compared to the exact solution, to illustrate the accuracy of the spline method of polynomial form.

Example 3.3. Consider the following nonlinear fractional ordinary differential equation:

$$D^{\alpha}y(t) = A(1 - y)^4, \quad A \in \mathbb{R}^+, \ 0 < \alpha \le 1, \tag{3.5}$$

subject to the initial condition $y(0) = \beta$, where β is a real constant.

This equation describes the cooling of a semi-infinite body by radiation, and the initial value problem has been solved numerically using the fractional difference method and Adomian decomposition method in [23].

The exact solution, when $\alpha = 1$, $\beta = 0$, and $A = 1$, is

$$y(t) = \frac{1 + 3t - (1 + 6t + 9t^2)^{1/3}}{1 + 3t}. \tag{3.6}$$

Table 4: Solution of Example 3.3 using $\alpha = 0.75$, $\alpha = 0.9$, and $\alpha = 1$.

t	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$	
	Appr. solution	Appr. solution	Appr. solution	Absolute error
0.0	0.00000	0.00000	0.00000	0.0
0.1	0.14054	0.10265	0.08383	9.3660×10^{-5}
0.2	0.21290	0.17219	0.14513	1.1397×10^{-4}
0.3	0.25536	0.22294	0.19272	1.1162×10^{-4}
0.4	0.28215	0.26188	0.23122	1.0297×10^{-4}
0.5	0.29977	0.29284	0.26329	9.3223×10^{-5}
0.6	0.31165	0.31809	0.29059	8.4033×10^{-5}
0.7	0.31980	0.33910	0.31425	7.5850×10^{-5}
0.8	0.32544	0.35683	0.33504	6.8710×10^{-5}
0.9	0.32939	0.37199	0.35352	6.2520×10^{-5}
1.0	0.33215	0.38508	0.37010	5.7152×10^{-5}

In this paper, we use the spline function method of polynomial form, to solve the fractional differential equation (3.5), together with $y(0) = \beta = 0$ and $A = 1$. Table 4 shows the approximate solutions for (3.5) obtained for different values of α using the spline method of polynomial form. From the numerical results in Table 4, it is clear that the approximate solutions are in high agreement with the exact solutions, when $\alpha = 1$, and with those obtained in [23] using the fractional difference method and Adomian decomposition method.

4. Conclusions

In this paper, we investigated the possibility of extending and generalizing the spline functions of polynomial fractional form given in Ramadan et al. [20] to be applicable for $\alpha \geq 1$ as well as the case $0 < \alpha \leq 1$. The method is tested by considering three test problems for three fractional ordinary differential equations. Two examples are of fractional order α, $0 < \alpha \leq 1$ (Example 3.1 and 3), while Example 3.2 is of fractional order α where $1 \leq \alpha < 2$. The obtained numerical results are in good agreement with the exact analytical solutions.

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