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Research Article

Existence of Solutions for Nonhomogeneous *A*-Harmonic Equations with Variable Growth

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We study the following nonhomogeneous A-harmonic equations: $d^*A(x, du(x)) + B(x, u(x)) = 0$, $x \in \Omega$, $u(x) = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded and convex Lipschitz domain, $A(x, du(x))$ and $B(x, u(x))$ satisfy some $p(x)$-growth conditions, respectively. We obtain the existence of weak solutions for the above equations in subspace $\mathfrak{K}_0^{1,p(x)}(\Omega, \Lambda^{l-1})$ of $W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$.

1. Introduction

Spaces of differential forms have been discussed in great details (see [1, 2] and the references therein). The theory of differential forms is an approach to multivariable calculus that is independent of coordinates and provides a better definition for integrals. Differential forms have played an important role in physical laws of thermodynamics, analytical mechanics, and physical theories, in particular Maxwell's theory, and the Yang-Mills theory, the theory of relativity, see for example [3–6].

In recent years, the study of A-harmonic equations for differential forms has developed rapidly. Many interesting results concerning A-harmonic equation have been established recently (see [7–11] and the references therein). In [12], spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are first introduced, and they used them to study the solutions of nonlinear Dirichlet boundary value problems with $p(x)$-growth conditions. In [13], spaces $L^{p(x)}(\Omega, \Lambda^l, \omega)$ and $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$ are first introduced and used to study the weak solutions of obstacle problems of A-harmonic equations with variable growth for differential forms.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex Lipschitz domain. It is our purpose to study the following systems:

$$d^*A(x,du(x)) + B(x,u(x)) = 0, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where $u \in \Lambda^{l-1}(\Omega)$, $l = 1,2,\ldots,n$, and $A : \Omega \times \Lambda^l(\Omega) \to \Lambda^l(\Omega)$, $B : \Omega \times \Lambda^{l-1}(\Omega) \to \Lambda^{l-1}(\Omega)$ satisfy the following conditions.

(H1) $A(x,\xi)$ and $B(x,\varsigma)$ are measurable with respect to x for all ξ,ς and continuous with respect to ξ,ς, respectively, for a.e. $x \in \Omega$.

(H2) $|A(x,\xi)|+|B(x,\varsigma)| \le C_1|\xi|^{p(x)-1}+C_2|\varsigma|^{p(x)-1}+G(x)$, where $G \in L^{p'(x)}(\Omega)$ and $C_1, C_2 \ge 0$ are constants.

(H3) $\langle A(x,\xi),\xi\rangle \ge a|\xi|^{p(x)} - |h(x)|$, where $a > 0$ is a constant and $h \in L^1(\Omega)$.

(H4) $\langle B(x,\varsigma),\varsigma\rangle \ge \overline{a}|\varsigma|^{p(x)} - |\overline{h}(x)|$, where $\overline{a} \ge 0$ is a constant and $\overline{h} \in L^1(\Omega)$.

(H5) For a.e. $x_0 \in \Omega$, the mapping $\xi \to A(x_0,\xi)$ satisfies

$$\int_D \langle A(x_0,\xi_0 + dv(x)), dv(x)\rangle dx \ge \gamma \int_D |dv(x)|^{p(x)} dx, \tag{1.2}$$

for each $\xi_0 \in \Lambda^l(\Omega)$, $D \subset \Omega$ and $v \in C_0^1(\Omega,\Lambda^{l-1})$, where $\gamma > 0$ is a constant. Here p' is the conjugate function of p. Throughout this paper we suppose (unless declare specially)

$$p \in \mathcal{P}^{\log}(\Omega), \quad 1 < p_* = \operatorname{essinf}_\Omega p(x) \le p(x) \le \operatorname{esssup}_\Omega p(x) = p^* < \infty. \tag{1.3}$$

2. Preliminaries

Let $e_1, e_2, \ldots, e_n$ be the standard orthogonal basis of $\mathbb{R}^n$. The space of all l-forms in $\mathbb{R}^n$ is denoted by $\Lambda^l(\mathbb{R}^n)$. The dual basis to $e_1, e_2, \ldots, e_n$ is denoted by $e^1, e^2, \ldots, e^n$ and referred to as the standard basis for 1-form $\Lambda^1(\mathbb{R}^n)$. The Grassman algebra $\Lambda(\mathbb{R}^n) = \oplus\Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. The standard ordered basis for $\Lambda(\mathbb{R}^n)$ consists of the forms

$$1, e^1, e^2, \ldots, e^n, e^1 \wedge e^2, \ldots, e^{n-1} \wedge e^n, \ldots, e^1 \wedge e^2 \ldots \wedge e^n. \tag{2.1}$$

For $\alpha(x) = \sum \alpha_I(x)e^I \in \Lambda^l(\mathbb{R}^n)$ and $\beta(x) = \sum \beta_I(x)e^I \in \Lambda^l(\mathbb{R}^n)$, the inner product is obtained by $\langle\alpha,\beta\rangle = \sum \alpha_I(x)\beta_I(x)$ with summation over all l–tuples $I = (i_1, \ldots i_l)$ and all integers $l = 0,1,\ldots,n$. The Hodge star operator (see [14]) $\star : \Lambda(\mathbb{R}^n) \to \Lambda(\mathbb{R}^n)$ is defined by the formulas

$$\star 1 = e^1 \wedge e^2 \cdots \wedge e^n, \qquad \alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle\alpha,\beta\rangle\, e^1 \wedge e^2 \cdots \wedge e^n. \tag{2.2}$$

Hence, the norm of α is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) = \sum \alpha_I(x)\alpha_I(x) \in \Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Notice, the Hodge star operator is an isometric isomorphism operator on $\Lambda(\mathbb{R}^n)$. Moreover,

$$\star : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^{n-l}(\mathbb{R}^n), \qquad \star\star = (-\mathrm{I})^{l(n-l)} : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^l(\mathbb{R}^n), \tag{2.3}$$

where I is the identity map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The coordinate functions $x_1, x_2, \ldots, x_n$ in Ω are considered to be differential forms of degree 0. The 1-forms $dx_1, dx_2, \ldots, dx_n$ are constant functions from Ω into $\Lambda^l(\mathbb{R}^n)$. The value of dx_i is simply e^i, $i = 1, 2, \ldots, n$. Therefore, every l-form $u : \Omega \to \Lambda^l(\mathbb{R}^n)$ may be written uniquely as

$$u(x) = \sum_I u_I(x) dx_I = \sum_{1 \le i_1 < \cdots < i_l \le n} u_{i_1, \ldots i_l}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_l}, \tag{2.4}$$

where the coefficients $u_{i_1, \ldots i_l}(x)$ are distributions from $\mathfrak{D}'(\Omega)$, dual to the space of smooth functions with compact support on Ω.

We use $\mathfrak{D}'(\Omega, \Lambda^l)$ to denote the space of all differential l-forms. For each form $u(x) \in \mathfrak{D}'(\Omega, \Lambda^l)$, the exterior differential $d : \mathfrak{D}'(\Omega, \Lambda^l) \to \mathfrak{D}'(\Omega, \Lambda^{l+1})$ is expressed by

$$du(x) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial u_{i_1, \ldots i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}. \tag{2.5}$$

For $u \in \mathfrak{D}'(\Omega, \Lambda^l)$, the vector-valued differential form

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right) \tag{2.6}$$

consists of differential forms $\partial u / \partial x_i \in \mathfrak{D}'(\Omega, \Lambda^l)$, where the partial differentiation is applied to the coefficients of u.

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^\star = (-1)^{nl-1} \star d \star : \mathfrak{D}'\left(\Omega, \Lambda^{l+1}\right) \longrightarrow \mathfrak{D}'\left(\Omega, \Lambda^l\right). \tag{2.7}$$

By $C^\infty(\Omega, \Lambda^l)$ denote the space of infinitely differentiable l-forms on Ω and by $C_0^\infty(\Omega, \Lambda^l)$ denote the subspace of $C^\infty(\Omega, \Lambda^l)$ with compact support on Ω.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. For $p \in \mathcal{P}(\Omega)$, we put $p_* = \operatorname{essinf}_\Omega p(x)$ and $p^* = \operatorname{esssup}_\Omega p(x)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p' \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad \forall x \in \Omega. \tag{2.8}$$

Definition 2.1 (see [15]). A Lebesgue measurable function $p : \Omega \to \mathbb{R}$ is called globally log-Hölder continuous in Ω if there exist $p_\infty \in \mathbb{R}$ and a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x-y|)}, \qquad |p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \tag{2.9}$$

hold for all $x, y \in \Omega$. $P^{\log}(\Omega)$ is defined by

$$\mathcal{P}^{\log}(\Omega) = \left\{ p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous} \right\}. \tag{2.10}$$

For a differential l-form $u(x)$ on Ω, $l = 0, 1, \ldots, n$, define the functional $\rho_{p(x)}$ by

$$\rho_{p(x),\Lambda^l}(u) = \int_\Omega |u(x)|^{p(x)} dx. \tag{2.11}$$

The space $L^{p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : \exists \lambda > 0, \rho_{p(x),\Lambda^l}(\lambda u) < \infty\}$ is a reflexive Banach space endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega,\Lambda^l)} = \inf\left\{ \lambda > 0 : \rho_{p(x),\Lambda^l}\left(\frac{u}{\lambda}\right) \leq 1 \right\}. \tag{2.12}$$

The space $W^{1,p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : u \in L^{p(x)}(\Omega, \Lambda^l) \text{ and } du \in L^{p(x)}(\Omega, \Lambda^{l+1})\}$ is a reflexive Banach space endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\Lambda^l)} = \|u\|_{L^{p(x)}(\Omega,\Lambda^l)} + \|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1})}. \tag{2.13}$$

Note that $L^{p(m)}(\Omega, \Lambda^0)$ and $W^{1,p(m)}(\Omega, \Lambda^0)$ are spaces of functions on Ω. In this paper, we denote them by $L^{p(m)}(\Omega)$ and $W^{1,p(m)}(\Omega)$.

Iwaniec and Lutoborski proved the following results in [2].

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex domain. If $u(x) \in \Lambda^l(\mathbb{R}^n)$ is defined for some $x \in \Omega$, then the value of $u(x)$ at the vectors $\xi_1, \ldots, \xi_l \in \mathbb{R}^n$ is denoted by $u(x)(\xi_1, \ldots, \xi_l)$. Then to each $y \in \Omega$, there corresponds a linear operator $K_y : L^1_{\text{loc}}(\Omega, \wedge^l) \to L^1_{\text{loc}}(\Omega, \wedge^{l-1})$ defined by

$$K_y u(x)(\xi_1, \xi_2, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty)(x - y, \xi_1, \xi_2, \ldots, \xi_{l-1}) dt. \tag{2.14}$$

The homotopy operator $T : L^1_{\text{loc}}(\Omega, \wedge^l) \to L^1_{\text{loc}}(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$

$$Tu(x) = \int_\Omega \varphi(y) K_y u(x) dy, \tag{2.15}$$

where $\varphi \in C_0^\infty(\Omega)$ is normalized so that $\int_\Omega \varphi(y)dy = 1$. Then we have a pointwise estimate

$$|Tu(x)| \le 2^n \mu(\Omega) \int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy, \quad \forall x \in \Omega, \tag{2.16}$$

where

$$\mu(\Omega) = (\operatorname{diam}\Omega)^{n+1} \inf\left\{ \frac{\|\nabla\varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^1(\Omega)}} : \varphi \in C_0^\infty(\Omega) \right\}, \tag{2.17}$$

further infimum is attained at $\varphi(x) = \operatorname{diam}(x, \partial\Omega)$, and the decomposition

$$u = dTu + Tdu \tag{2.18}$$

holds for $u \in L^1_{\text{loc}}(\Omega, \Lambda^l)$.

Definition 2.2. For $u \in L^1_{\text{loc}}(\Omega, \Lambda^l)$, define the l-form $u_\Omega \in \mathfrak{D}'(\Omega, \Lambda^l)$ by

$$u_\Omega = \begin{cases} \dfrac{1}{\operatorname{meas}(\Omega)} \displaystyle\int_\Omega u(x)dx, & \text{for } l = 0, \\ dTu, & \text{for } l = 1, 2, \cdots, n, \end{cases} \tag{2.19}$$

and the Maximal operator is defined by

$$(Mu)(x) = \sup_{r>0} \frac{1}{\operatorname{meas}(B_r(x))} \int_{B_r(x)} |u(y)| dy, \tag{2.20}$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$.

Lemma 2.3 (see [15]). *Let $p(x)$ satisfies* (1.3). *Then the inequality*

$$\|(Mu)(x)\|_{L^{p(x)}(\mathbb{R}^n)} \le C(n, p) \|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \tag{2.21}$$

holds for every $u \in L^{p(x)}(\mathbb{R}^n)$.

Lemma 2.4 (see [15]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, $x \in \Omega$ and $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy \le C(n)(\operatorname{diam}\Omega)(Mu)(x). \tag{2.22}$$

Lemma 2.5 (see [15]). *Let Ψ be a Calderón-Zygmund operator with Calderón-Zygmund kernel K on $\mathbb{R}^n \times \mathbb{R}^n$. Then Ψ is bounded on $L^{p(x)}(\mathbb{R}^n)$. Further there exists a constant $C = C(n,p)$ such that*

$$\|\Psi u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \le C(n,p)\|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \tag{2.23}$$

holds for every $u \in L^{p(x)}(\mathbb{R}^n)$.

Lemma 2.6. *If $u \in L^{p(x)}(\Omega, \Lambda^l)$, then*

$$\|Tu\|_{L^{p(x)}(\Omega,\Lambda^{l-1})} \le C(n,p)\mu(\Omega)(\operatorname{diam}\Omega)\|u\|_{L^{p(x)}(\Omega,\Lambda^l)}. \tag{2.24}$$

Moreover, if $u \in W^{1,p(x)}(\Omega, \Lambda^l)$, then

$$\|u_\Omega\|_{L^{p(x)}(\Omega,\Lambda^l)} \le C(p)\|u\|_{L^{p(x)}(\Omega,\Lambda^l)} + C(n,p)\mu(\Omega)(\operatorname{diam}\Omega)\|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1})}. \tag{2.25}$$

Proof. First define $u(x) = 0$ if $x \in \mathbb{R}^n \setminus \Omega$. From pointwise estimate (2.16) and Lemma 2.4,

$$|Tu(x)| \le C(n)\mu(\Omega)(\operatorname{diam}\Omega)M(|u|)(x), \quad \forall x \in \Omega. \tag{2.26}$$

In view of Lemma 2.3, we have

$$\||Tu|\|_{L^{p(x)}(\Omega)} \le C(n,p)\mu(\Omega)(\operatorname{diam}\Omega)\||u|\|_{L^{p(x)}(\Omega)}, \tag{2.27}$$

that is to say, (2.24) holds.

From the definition of u_Ω and (2.18), we have $u_\Omega = u - Tdu$. Therefore,

$$\|u_\Omega\|_{L^{p(x)}(\Omega,\Lambda^l)} \le C(p)\|u\|_{L^{p(x)}(\Omega,\Lambda^l)} + C(n,p)\|Tdu\|_{L^{p(x)}(\Omega,\Lambda^l)}. \tag{2.28}$$

Now in (2.24) replace u with du, we obtain (2.25). □

Lemma 2.7. *Let $p(x)$ satisfies* (1.3).

(1) $C_0^\infty(\Omega, \Lambda^l)$ *is dense in* $L^{p(x)}(\Omega, \Lambda^l)$,

(2) $L^{p(x)}(\Omega, \Lambda^l)$ *is separable.*

Proof. (1) For any $u(x) = \sum_I u_I(x)dx_I \in L^{p(x)}(\Omega, \Lambda^l)$, since $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$ and $u_I(x) \in L^{p(x)}(\Omega)$ for all I, we can find a sequence $\{u_{Ik}\}_{k=1}^\infty \subset C_0^\infty(\Omega)$ which converges to $u_I(x)$

in $L^{p(x)}(\Omega)$ for each I. Now let $u_k(x) = \sum_I u_{Ik} dx_I$, then the sequence $\{u_k(x)\} \subset C_0^\infty(\Omega, \Lambda^l)$ converges to $u(x)$ in $L^{p(x)}(\Omega, \Lambda^l)$, since

$$\begin{aligned}\int_\Omega |u(x) - u_k(x)|^{p(x)} dx &= \int_\Omega \left(\left(\sum_I |u_I(x) - u_{Ik}(x)|^2 \right)^{1/2} \right)^{p(x)} dx \\ &\le \int_\Omega \left(\sum_I |u_I(x) - u_{Ik}(x)| \right)^{p(x)} dx \\ &\le 2^{p^*} \sum_I \int_\Omega |u_I(x) - u_{Ik}(x)|^{p(x)} dx.\end{aligned} \tag{2.29}$$

That is to say, $C_0^\infty(\Omega, \Lambda^l)$ is dense in $L^{p(x)}(\Omega, \Lambda^l)$.

(2) Let $u(x) = \sum_I u_I(x) dx_I \in L^{p(x)}(\Omega, \Lambda^l)$. Since $L^{p(x)}(\Omega)$ is separable, there exists a countable dense subset K of $L^{p(x)}(\Omega)$. Then for any $u_I(x)$ above we can extract a sequence $\{u_{Ik}(x)\}$ in K which converges to $u_I(x)$ in $L^{p(x)}(\Omega)$. Similar to (1), the sequence $\{u_k : u_k(x) = \sum_I u_{Ik}(x) dx_I\}$ converges to $u(x)$ in $L^{p(x)}(\Omega, \Lambda^l)$. That is to say, $L^{p(x)}(\Omega, \Lambda^l)$ is separable. □

Let $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l) = \{u(x) = \vartheta(x) - \vartheta_\Omega(x) : \vartheta \in W^{1,p(x)}(\Omega, \Lambda^l)\}$. Note that $u \in \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ if and only if $u_\Omega = 0$.

Lemma 2.8. *Let $p(x)$ satisfies* (1.3). *Then $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ is a closed subspace of $W^{1,p(x)}(\Omega, \Lambda^l)$. In particular, it is a reflexive Banach space.*

Proof. Set a sequence $\{u_k(x)\} \subset \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ convergent to $u(x)$ in $W^{1,p(x)}(\Omega, \Lambda^l)$, then $(u_k)_\Omega = 0$. By Lemma 2.6, the operator T is continuous on $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$. Therefore, $u_\Omega = 0$, we have $u(x) \in \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$. That is to say, $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ is a closed subspace of $W^{1,p(x)}(\Omega, \Lambda^l)$. □

In [2], Iwaniec and Lutoborski obtained

$$\frac{\partial}{\partial x_i}(Tu) = A_i u + S_i u, \tag{2.30}$$

where

$$|A_i u(x)| \le \frac{2^n \mu(\Omega)}{\operatorname{diam}(\Omega)} \int_\Omega \frac{|u(z)|}{|x - z|^{n-1}} dz, \tag{2.31}$$

$$S_i u(x)(\xi) = \int_\Omega u(z)(K_i(z, x - z), \xi) dz, \tag{2.32}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_{l-1})$ and

$$K_i(z, x - z) = \frac{e_i}{|x - z|^n} \int_0^\infty s^{n-1} \varphi\left(z - s\frac{x - z}{|x - z|}\right) ds - \frac{x - z}{|x - z|^{n+1}} \int_0^\infty s^n \varphi_i\left(z - s\frac{x - z}{|x - z|}\right) ds. \tag{2.33}$$

Further for each $z \in \Omega$ and $h \in \mathbb{R}^n - \{0\}$, $K_i(z, h)$ satisfies the following properties:

(i) $K_i(z, h) \le \mu(\Omega)|h|^{-n}$,

(ii) $K_i(z, sh) = s^{-n} K_i(z, h), s > 0$,

(iii) $\int_{|h|=1} K_i(z, h) = 0$ for all $z \in \Omega$.

Let $K_i(z, h) = (K_{i1}, K_{i2}, \ldots, K_{in})$. Then $K_{i\alpha}$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ for each $\alpha = 1, 2, \ldots, n$.

Lemma 2.9. *Let $u \in L^{p(x)}(\Omega, \Lambda^l)$. Then*

$$\| |\nabla T u| \|_{L^{p(x)}(\Omega)} \le C(n, p, \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \tag{2.34}$$

Proof. By Lemmas 2.3 and 2.4, and (2.31),

$$\|A_i u\|_{L^{p(x)}(\Omega, \Lambda^l)} \le C(n, p) \mu(\Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \tag{2.35}$$

Let

$$S_i u(x) = \sum_{1 \le j_1 < j_2 < \cdots < j_{l-1} \le n} \omega_{j_1, j_2, \ldots, j_{l-1}} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{l-1}}, \tag{2.36}$$

we can write $u(x)$ as

$$u(x) = \sum_{1 \le \alpha \le n, \alpha \ne j_1, j_2, \ldots, j_{l-1}} \sum_{1 \le j_1 < j_2 < \cdots < j_{l-1} \le n} u_{\alpha, j_1, j_2, \ldots, j_{l-1}} dx_\alpha \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{l-1}}. \tag{2.37}$$

Hence,

$$\omega_{j_1, j_2, \ldots, j_{l-1}}(x) = S_i u(x)(e_{j_1}, e_{j_2}, \ldots, e_{j_{l-1}}). \tag{2.38}$$

Taking $\xi = (e_{j_1}, e_{j_2}, \ldots, e_{j_{l-1}})$ in (2.32), we obtain

$$\omega_{j_1, j_2, \ldots, j_{l-1}}(x) = \int_\Omega \sum_{1 \le \alpha \le n, \alpha \ne j_1, \ldots, j_{l-1}} K_{i\alpha}(z, x - z) u_{\alpha, j_1, j_2, \ldots, j_{l-1}}(z) dz. \tag{2.39}$$

Now define $u(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^n \setminus \Omega$. Since $K_{i\alpha}$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ for each α, in view of Lemma 2.5,

$$\left\|\omega_{j_1,j_2,\ldots,j_{l-1}}\right\|_{L^{p(x)}(\Omega)} \le C(n,p) \sum_{1\le\alpha\le n,\alpha\ne j_1,\ldots,j_{l-1}} \left\|u_{\alpha,j_1,j_2,\ldots,j_{l-1}}\right\|_{L^{p(x)}(\Omega)}. \tag{2.40}$$

So that

$$\|S_i u\|_{L^{p(x)}(\Omega,\Lambda^l)} \le C(n,p)\|u\|_{L^{p(x)}(\Omega,\Lambda^l)}. \tag{2.41}$$

By (2.30), (2.35), and (2.41), we have

$$\||\nabla T u|\|_{L^{p(x)}(\Omega)} \le C(n,p,\Omega)\|u\|_{L^{p(x)}(\Omega,\Lambda^l)}. \tag{2.42}$$

□

Now define another norm

$$\||\omega|\|_{\mathfrak{K}^{1,p(x)}(\Omega,\Lambda^l)} = \|\omega\|_{L^{p(x)}(\Omega,\Lambda^l)} + \||\nabla\omega|\|_{L^{p(x)}(\Omega)}. \tag{2.43}$$

Remark 2.10. Replacing u with du in (2.34), we get by the definition of u_Ω

$$\begin{aligned}\||\nabla(u-u_\Omega)|\|_{L^{p(x)}(\Omega)} &= \||\nabla T du|\|_{L^{p(x)}(\Omega)} \\ &\le C(n,p)\mu(\Omega)\|du\|_{L^{p(x)}(\Omega,\Lambda^l)} \\ &= C(n,p)\mu(\Omega)\|d(u-u_\Omega)\|_{L^{p(x)}(\Omega,\Lambda^l)}.\end{aligned} \tag{2.44}$$

Therefore $\||\cdot|\|_{\mathfrak{K}^{1,p(x)}(\Omega,\Lambda^l)}$ is equivalent to $\|\cdot\|_{\mathfrak{K}^{1,p(x)}(\Omega,\Lambda^l)}$.

In this paper we also need the following two lemmas.

Lemma 2.11 (see[15]). *Let $p(x)$ satisfies (1.3). Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact.*

Lemma 2.12 (see[15]). *Suppose that $p \in L^\infty(\Omega)$. Let $\{u_k\}_{k=1}^\infty$ be bounded in $L^{p(x)}(\Omega)$. If $u_k \to u$ a.e. on Ω, then $u_k \rightharpoonup u$ weakly in $L^{p(x)}(\Omega)$.*

Remark 2.13. Let $\mathfrak{K}_0^{1,p(x)}(\Omega,\Lambda^l)$ be the completion of $C_0^\infty(\Omega,\Lambda^l)$ in $\mathfrak{K}^{1,p(x)}(\Omega,\Lambda^l)$. Then from Remark 2.10 and Lemma 2.11, the embedding $\mathfrak{K}_0^{1,p(x)}(\Omega,\Lambda^l) \hookrightarrow L^{p(x)}(\Omega,\Lambda^l)$ is compact.

Remark 2.14. Suppose $p(x)$ satisfies (1.3), Lemma 2.12 also holds on space $L^{p(x)}(\Omega,\wedge^l)$.

3. Weak Solutions of Dirichlet Problems for the A-Harmonic Equations with Variable Growth

Theorem 3.1. *Under conditions (H1)–(H5), the Dirichlet problem* (1.1) *has at least one weak solution in* $\mathfrak{K}_0^{1,p(x)}(\Omega,\Lambda^l)$*, that is to say, there exists at least one* $u = \vartheta - \vartheta_\Omega \in \mathfrak{K}_0^{1,p(x)}(\Omega,\Lambda^l)$ *satisfying*

$$\int_\Omega \langle A(x,du(x)),d\varphi(x)\rangle + \langle B(x,u(x)),\varphi(x)\rangle dx = 0, \tag{3.1}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega,\Lambda^{l-1})$*. Here,* $\vartheta \in W^{1,p(x)}(\Omega,\Lambda^{l-1})$ *and* $p(x)$ *satisfies* (1.3).

Let $V = W_0^{1,p(x)}(\Omega,\Lambda^{l-1})$ and $\mathfrak{K}_0 = \mathfrak{K}_0^{1,p(x)}(\Omega,\Lambda^l)$. For $u \in V$, define $\mathfrak{A} : V \to V^*$ in the following way: for each $\varphi \in V$

$$(\mathfrak{A}u,\varphi) = \int_\Omega \langle A(x,du(x)),d\varphi(x)\rangle + \langle B(x,u(x)),\varphi(x)\rangle dx. \tag{3.2}$$

Now we need only to show that there exists $u \in \mathfrak{K}_0$ such that $(\mathfrak{A}u,\varphi) = 0$ for all $\varphi = \sum \varphi_I(x)dx_I \in V$.

Lemma 3.2. $\mathfrak{A}$ *is strong-weakly continuous on* V*.*

Proof. Let $\{u_k : u_k(x) = \sum_I u_{kI}(x)dx_I\} \subset V$ be a sequence strongly convergent to an element $u(x) = \sum u_I(x)dx_I \in V$ in V. Let $du_k(x) = \sum_J \omega_{kJ}(x)dx_J$ and $du(x) = \sum_J \omega_J(x)dx_J$. Then

(h_1) $\|u_k\|_V \le C$ for some constant C,

(h_2) $\{\omega_{kJ}(x)\}$ is a sequence strongly convergent to $\omega_J(x)$ in $L^{p(x)}(\Omega)$ for each J.

In view of (H2) and (h_1), we know that $A(x,du_k) = \sum A_{kJ}(x)dx_J$ and $B(x,u_k) = \sum B_{kI}(x)dx_I$ are uniformly bounded in $L^{p'(x)}(\Omega,\Lambda^l)$ and $L^{p'(x)}(\Omega,\Lambda^{l-1})$, respectively. Hence, $A_{kJ}(x)$ and $B_{kI}(x)$ are uniformly bounded in $L^{p'(x)}(\Omega)$. On the other hand, by (h_2), there exists a subsequence of $\{\omega_{kJ}(x)\}$ (still denoted by $\{\omega_{kJ}(x)\}$) such that

$$\lim_{k\to\infty}\omega_{kJ}(x) = \omega_J(x), \quad \text{a.e. } x \in \Omega, \text{for each } J. \tag{3.3}$$

Then there exists a subsequence of $\{u_k(x)\}$ (still denoted by $\{u_k(x)\}$) such that

$$\lim_{k\to\infty}u_k(x) = u(x), \quad \lim_{k\to\infty}du_k(x) = du(x), \quad \text{a.e. } x \in \Omega. \tag{3.4}$$

In view of (H1), we obtain

$$\begin{aligned}\lim_{k\to\infty}A(x,du_k) &= A(x,du), \quad \text{a.e. } x \in \Omega,\\ \lim_{k\to\infty}B(x,u_k) &= B(x,u), \quad \text{a.e. } x \in \Omega.\end{aligned} \tag{3.5}$$

Let $d\varphi(x) = \sum \varphi_J(x)dx_J$, $A(x,du) = \sum A_J(x)dx_J$, and $B(x,u) = \sum B_I(x)dx_I$, then $\varphi_J(x) \in L^{p(x)}(\Omega)$, in the meantime

$$\lim_{k\to\infty} A_{kJ}(x) = A_J(x), \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

$$\lim_{k\to\infty} B_{kI}(x) = B_I(x), \quad \text{a.e. } x \in \Omega, \tag{3.7}$$

for each J and I.

Now by Lemma 2.12, we can show that $\int_\Omega A_{kJ}(x)\varphi_J(x)dx \to \int_\Omega A_J(x)\varphi_J(x)dx$ and $\int_\Omega B_{kI}(x)\varphi_I(x)dx \to \int_\Omega B_I(x)\varphi_I(x)dx$ as $k \to \infty$. Therefore,

$$\begin{aligned}(\mathfrak{A}u_k,\varphi) &= \int_\Omega \langle A(x,du_k),d\varphi\rangle + \langle B(x,u_k),\varphi\rangle dx \\ &\longrightarrow \int_\Omega \langle A(x,du),d\varphi\rangle + \langle B(x,u),\varphi\rangle dx \\ &= (\mathfrak{A}u,\varphi),\end{aligned} \tag{3.8}$$

that is to say, $\mathfrak{A}$ is strong-weakly continuous on V. □

Lemma 3.3. *$\mathfrak{A}$ is coercive on $\mathfrak{K}_0$, that is,*

$$\lim_{\|u\|_{\mathfrak{K}}\to\infty} \frac{(\mathfrak{A}u,u)}{\|u\|_{\mathfrak{K}}} = +\infty, \quad \forall u \in \mathfrak{K}_0. \tag{3.9}$$

Proof. By (H3) and (H4),

$$\begin{aligned}(\mathfrak{A}u,u) &= \int_\Omega \langle A(x,du),du\rangle + \langle B(x,u),u\rangle dx \\ &\geq \int_\Omega \left(a|du|^{p(x)} - |h(x)| + \overline{a}|u|^{p(x)} - \left|\overline{h}(x)\right|\right)dx \\ &\geq \int_\Omega a|du|^{p(x)}dx - C\left(h,\overline{h}\right).\end{aligned} \tag{3.10}$$

By $d\vartheta_\Omega = 0$ and Lemma 2.6, we have

$$\|u\|_{L^{p(x)}(\Omega,\Lambda^{l-1})} = \|Td\vartheta\|_{L^{p(x)}(\Omega,\Lambda^{l-1})} \leq 2^n C(n,p)\mu(\Omega)(\operatorname{diam}\Omega)\|du\|_{L^{p(x)}(\Omega,\Lambda^{l})}, \tag{3.11}$$

for all $u = \vartheta - \vartheta_\Omega \in \mathfrak{K}_0$. Then $\|du\|_{L^{p(x)}(\Omega,\Lambda^l)} \to \infty$, as $\|u\|_{\mathfrak{K}} \to \infty$. Taking

$$\delta = \frac{1}{2}\|du\|_{L^{p(x)}(\Omega,\Lambda^l)} > 1, \tag{3.12}$$

we have

$$\frac{\int_\Omega |du|^{p(x)}dx}{\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}} = \int_\Omega \left(\frac{|du|}{\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}-\delta}\right)^{p(x)} \frac{\left(\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}-\delta\right)^{p(x)}}{\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}}dx$$
$$\geq \frac{\left(\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}-\delta\right)^{p_*}}{\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}} = \left(\frac{1}{2}\right)^{p_*}\|du\|^{p_*-1}_{L^{p(x)}(\Omega,\Lambda^l)}. \tag{3.13}$$

Therefore,

$$\frac{\int_\Omega |du|^{p(x)}dx}{\|du\|_{L^{p(x)}(\Omega,\Lambda^l)}} \longrightarrow \infty \quad \text{as } \|du\|_{L^{p(x)}(\Omega,\Lambda^l)} \longrightarrow \infty. \tag{3.14}$$

Then it is immediate to obtain that

$$\frac{(\mathfrak{A}u,u)}{\|u\|_{\mathfrak{K}}} \longrightarrow \infty \quad \text{as } \|u\|_{\mathfrak{K}} \longrightarrow \infty. \tag{3.15}$$

That is to say, $\mathfrak{A}$ is coercive on $\mathfrak{K}_0$. □

Lemma 3.4 (see[16]). *Suppose $g = A(x)$ is a mapping from $\mathbb{R}^m$ into itself such that*

$$\lim_{|x|\to\infty}\frac{A(x)\cdot x}{|x|} = \infty. \tag{3.16}$$

Then the range of A is the whole of $\mathbb{R}^m$.

Lemma 3.5. *There exists a sequence $\{u_k\} \subset \mathfrak{K}_0$ and $u_0 \in \mathfrak{K}_0$, such that*

$$(\mathfrak{A}u_k, u_k - u_0) \longrightarrow 0 \quad \textit{as } k \longrightarrow \infty. \tag{3.17}$$

Proof. By Lemmas 2.7 and 2.8, we can choose a Schauder basis $\{\omega_s\}$ of $\mathfrak{K}_0$ such that the union of subspace finitely generated from ω_s is dense in $\mathfrak{K}_0$. Let $\mathfrak{K}_0^k$ be the subspace of $\mathfrak{K}_0$ generated by $\omega_1, \omega_2, \ldots, \omega_k$. Since $\mathfrak{K}_0^k$ is topologically isomorphic to $\mathbb{R}^k$. By By Lemmas 3.3, and 3.4, there exists $u_k \in \mathfrak{K}_0^k$ such that

$$(\mathfrak{A}u_k, \omega) = 0 \quad \forall\omega \in \mathfrak{K}_0^k. \tag{3.18}$$

By Lemma 3.3 again, we know that $\|u_k\|_{\mathfrak{K}} \leq C$, where C is independent of k. Since $\mathfrak{K}_0$ is reflexive, by Remark 2.14 and (H1), we can extract a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } \mathfrak{K}_0, \qquad \mathfrak{A}u_k \rightharpoonup \xi \quad \text{weakly}^* \text{ in } \mathfrak{K}_0^*, \quad (\xi,\omega) = 0, \tag{3.19}$$

where ω is in a dense subset of $\mathfrak{K}_0$. For fixed ξ, by the continuity of $(\xi, \cdot)$, we get $(\xi, \omega) = 0$ for all $\omega \in \mathfrak{K}_0$. For $(\mathfrak{A}u_k, u_k - u_0)$, we have

$$(\mathfrak{A}u_k, u_k - u_0) = (\mathfrak{A}u_k, u_k) - (\mathfrak{A}u_k, u_0) = -(\mathfrak{A}u_k, u_0) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.20}$$

This completes the proof of Lemma 3.5. □

Set $v_k = u_k - u_0 = \sum v_{kI} dx_I$. Then

$$v_k \rightharpoonup 0 \quad \text{weakly in } \mathfrak{K}_0 \quad \text{as } k \longrightarrow \infty. \tag{3.21}$$

Consider $(\mathfrak{A}u_k, u_k - u_0)$ once more, then

$$(\mathfrak{A}u_k, u_k - u_0) = \int_\Omega \langle A(x, du_0 + dv_k), dv_k \rangle + \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0, \tag{3.22}$$

as $k \to \infty$. By Remark 2.13, we get

$$v_k \longrightarrow 0 \quad \text{strongly} \quad \text{in } L^{p(x)}\left(\Omega, \Lambda^{l-1}\right). \tag{3.23}$$

In view of (3.23) and (H2), it is immediate that

$$\int_\Omega \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.24}$$

that is to say,

$$\int_\Omega \langle A(x, du_0 + dv_k), dv_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.25}$$

Now if we can prove that there exists a subsequence of $\{v_k\}$ which is strongly convergent in $\mathfrak{K}_0$, then from the strong-weakly continuity of $\mathfrak{A}$, we get $\mathfrak{A}u_k \rightharpoonup \mathfrak{A}u_0 = \xi$ weakly in $\mathfrak{K}_0$ as $k \to \infty$ and u_0 will be a weak solution of (1.1). We need the following lemmas.

Definition 3.6. Let Ω be an open subset of $\mathbb{R}^n$ provided with the Lebesgue measure. The mapping $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is said Carathéodory function if for almost all $x \in \Omega$, $f(x, \cdot)$ is continuous on $\mathbb{R}^N$, for all $\xi \in \mathbb{R}^N$ is measurable on Ω.

Lemma 3.7 (see[17]). *A mapping $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function if and only if for all compact sets $K \subset \Omega$ and all $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset K$ such that* $\text{meas}(K - K_\varepsilon) < \varepsilon$ *for with the restriction of f to $K_\varepsilon \times \mathbb{R}^N$ is continuous.*

Lemma 3.8 (see[15]). *Let $\{f_k\}$ be a sequence of bounded function in $L^1(\mathbb{R}^n)$. For each $\varepsilon > 0$ there exists $(A_\varepsilon, \delta, N)$ (where A_ε is measurable and $\operatorname{meas}(A_\varepsilon) < \varepsilon, \delta > 0, N$ is an infinite subset of natural numbers set $\mathbb{N}$) such that for each $k \in N$,*

$$\int_B |f_k(x)| dx < \varepsilon, \tag{3.26}$$

where B and A_ε are disjoint and $\operatorname{meas}(B) < \delta$.

Definition 3.9. For $u \in C_0^1(\mathbb{R}^n)$, define

$$(M^* u)(x) = (Mu)(x) + \sum_{\alpha=1}^{n} \left(M \frac{\partial u}{\partial x_\alpha} \right)(x). \tag{3.27}$$

Lemma 3.10 (see[18]). *If $u \in C_0^\infty(\mathbb{R}^n)$, then $M^* u \in C^0(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,*

$$|u(x)| + \sum_{\alpha=1}^{n} \left| \frac{\partial u}{\partial x_\alpha}(x) \right| \leq (M^* u)(x). \tag{3.28}$$

Furthermore, if $p > 1$, then

$$\|M^* u\|_{L^p(\mathbb{R}^n)} \leq C(n,p) \|u\|_{W^{1,p}(\mathbb{R}^n)}, \tag{3.29}$$

and if $p = 1$, then

$$\operatorname{meas}(\{x \in \mathbb{R}^n : (M^* u)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}, \tag{3.30}$$

for all $\lambda > 0$.

Lemma 3.11 (see[19]). *Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. Set*

$$H^\lambda = \{x \in \mathbb{R}^n : (M^* u)(x) < \lambda\}. \tag{3.31}$$

Then for all $x, y \in H^\lambda$, we have

$$|u(y) - u(x)| \leq C(n) \lambda |y - x|. \tag{3.32}$$

Lemma 3.12 (see[16]). *Let X be a metric space, E be a subspace of X, and k be a positive number. Then any k-Lipchitz mapping from E into $\mathbb{R}$ can be extended to a k-Lipchitz mapping from X into $\mathbb{R}$.*

Proof of Theorem 3.1. We need only to show that there exists subsequence of $\{v_k\}$ which is strongly convergent in $\mathfrak{K}_0$.

For each measurable set $S \subset \Omega$, define

$$F(v,S) = \int_S \langle A(x, du_0 + dv), dv \rangle dx, \tag{3.33}$$

where $v \in \mathfrak{K}_0$. Similar to the proof of Lemma 3.2, $F(\cdot, S)$ is strongly continuous on $\mathfrak{K}_0$. Since $C_0^\infty(\Omega, \Lambda^{l-1})$ is dense in $\mathfrak{K}_0$, there exists $h_k \subset C_0^\infty(\Omega, \Lambda^{l-1})$ such that

$$\|h_k - v_k\|_{\mathfrak{K}} < \frac{1}{k}, \qquad |F(h_k, \Omega) - F(v_k, \Omega)| < \frac{1}{k}. \tag{3.34}$$

So we can suppose that $\{v_k\} \subset C_0^\infty(\Omega, \Lambda^{l-1})$ is bounded in $\mathfrak{K}_0$.

Next define

$$v_k(x) = 0 \quad \text{when } x \in \mathbb{R}^n \setminus \Omega. \tag{3.35}$$

In this way, we extend the domain of v_k to $\mathbb{R}^n$ and supp $v_k \subset \Omega$.

Let $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function satisfying $\beta(0) = 0$ and for each measurable set $D \subset \Omega$,

$$\int_D \left(|G(x)|^{p'(x)} + |h(x)| + (C_1 + 1)|du_0|^{p(x)} \right) dx \le \beta(\text{meas}(D)), \tag{3.36}$$

where C_1 is the constant in (H2).

Let $\{\varepsilon_j\}$ be a positive decreasing sequence with $\varepsilon_j \to 0$ as $j \to \infty$. For ε_1, by Lemma 3.8, we get a subsequence $\{k_1\}$ of $\{k\}$, a set $A_{\varepsilon_1} \subset \Omega$ satisfying meas $(A_{\varepsilon_1}) < \varepsilon_1$, and a real number $\delta_1 > 0$ such that

$$\int_B (M^* v_{k_1 I})^{p(x)} dx < \varepsilon_1, \tag{3.37}$$

for each k_1, I and $B \subset \Omega \setminus A_{\varepsilon_1}$ satisfying $\text{meas}(B) < \delta_1$. By Lemma 3.10, we can choose $\lambda > 1$ so large that for all I and k_1,

$$\text{meas}(\{x \in \mathbb{R}^n : (M^* v_{k_1 I})(x) \ge \lambda\}) \le \min\{\varepsilon_1, \delta_1\}. \tag{3.38}$$

For each I and k_1, define

$$H_{k_1 I}^\lambda = \{x \in \mathbb{R}^n : (M^* v_{k_1 I})(x) < \lambda\}, \quad H_{k_1}^\lambda = \bigcap_I H_{k_1 I}^\lambda. \tag{3.39}$$

In view of Lemma 3.11, we have

$$\frac{|v_{k_1 I}(y) - v_{k_1 I}(x)|}{|y - x|} \le C(n)\lambda \quad \forall x, y \in H_{k_1}^\lambda \text{ and } I. \tag{3.40}$$

Form Lemma 3.12, there exists a Lipschitz function g_{k_1I} which extends v_{k_1I} outside $H_{k_1}^{\lambda}$ and Lipschitz constant of g_{k_1I} is no more than $C(n)\lambda$. As $H_{k_1}^{\lambda}$ is an open set, we have $g_{k_1I} = v_{k_1I}$ and $\nabla g_{k_1I}(x) = \nabla v_{k_1I}(x)$ for all $x \in H_{k_1}^{\lambda}$, and $\|\nabla g_{k_1I}\|_{L^\infty(\mathbb{R}^n)} \le C(n)\lambda$. We can further suppose that

$$\left\|g_{k_1I}\right\|_{L^\infty(\mathbb{R}^n)} \le \|v_{k_1I}\|_{L^\infty(H_{k_1}^{\lambda})} \le \lambda, \qquad \left\|g_{k_1I}\right\|_{W^{1,\infty}(\Omega)} \le C(n)\lambda. \tag{3.41}$$

By the uniformly boundedness of $\{\|g_{k_1I}\|_{W^{1,\infty}(\Omega)}\}$, there exists a subsequence of $\{g_{k_1I}\}$ (still denoted by $\{g_{k_1I}\}$) such that

$$g_{k_1I} \rightharpoonup \omega_I \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega) \quad \text{as } k_1 \longrightarrow \infty \ \forall I. \tag{3.42}$$

Set $\omega = \sum_I \omega_I dx_I$ and $g_{k_1} = \sum_I g_{k_1I} dx_I$. We have

$$F(v_{k_1},\Omega) = F(g_{k_1},\Omega \setminus A_{\varepsilon_1}) - F\left(g_{k_1},(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda}\right) + F\left(v_{k_1},A_{\varepsilon_1} \cup \left(\Omega \setminus H_{k_1}^{\lambda}\right)\right). \tag{3.43}$$

Next we estimate $F(v_{k_1},\Omega)$ in four steps.

(1) *The estimate of* $F(g_{k_1},(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda})$ *and* $F(v_{k_1},A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^{\lambda}))$. Since

$$\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda}\right) \le \sum_I \text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1I}^{\lambda}\right) \le C_n^{l-1} \min\{\varepsilon_1,\delta_1\}, \tag{3.44}$$

where $C_n^{l-1} = n(n-1)\cdots(n-l+2)/(l-1)(l-2)\cdots 1$, from (H2), (H3), and the choose of A_{ε_1}, we have

$$\begin{aligned}
&\left|F\left(g_{k_1},(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda}\right)\right| \\
&\quad\le \int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1}^{\lambda}} \left(C_1\left|du_0 + dg_{k_1}\right|^{p(x)-1}\left|dg_{k_1}\right| + |G(x)|\left|dg_{k_1}\right|\right)dx \\
&\quad\le \int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1}^{\lambda}} \left(C_1 2^{p^*-1}\left(|du_0|^{p(x)} + \left|dg_{k_1}\right|^{p(x)}\right) + C_1\left|dg_{k_1}\right|^{p(x)} + |G(x)|^{p'(x)} + \left|dg_{k_1}\right|^{p(x)}\right)dx \\
&\quad\le 2^{p^*-1}\beta\left(\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda}\right)\right) + 2^{p^*}(C_1+1)\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1}^{\lambda}} \left|dg_{k_1}\right|^{p(x)}dx \\
&\quad\le 2^{p^*-1}\beta\left(\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^{\lambda}\right)\right) + 2^{p^*}(C_1+1)\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1}^{\lambda}} \left(\sum_I\left|\nabla g_{k_1I}\right|\right)^{p(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right)+2^{p^*}C(C_1,n,l)\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1}^{\lambda}}\lambda^{p(x)}dx\\
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right)+2^{p^*}C(C_1,n,l)\sum_I\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H_{k_1I}^{\lambda}}(M^*v_{k_1I})^{p(x)}dx\\
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right)+2^{p^*}C(C_1,n,l)\varepsilon_1\leq O(\varepsilon_1),
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
&F\left(v_{k_1},A_{\varepsilon_1}\cup\left(\Omega\setminus H_{k_1}^{\lambda}\right)\right)\\
&\quad=\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}\langle A(x,du_0+dv_{k_1}),du_0+dv_{k_1}\rangle-\langle A(x,du_0+dv_{k_1}),du_0\rangle dx\\
&\quad\geq\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}\left(a|du_0+dv_{k_1}|^{p(x)}-h(x)\right)-\left(C_1|du_0+dv_{k_1}|^{p(x)-1}|du_0|+|G(x)||du_0|\right)dx\\
&\quad\geq\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}\left(\left(a2^{-(p^*-1)}-C_1\mu 2^{p^*-1}\right)|dv_{k_1}|^{p(x)}-|h(x)|-|G(x)|^{p'(x)}\right)\\
&\qquad-\left(-a2^{-(p^*-1)}+C_1\mu 2^{p^*-1}+C_1C(\mu)+1\right)|du_0|^{p(x)}dx\\
&\quad\geq\left(a2^{-(p^*-1)}-C_1\mu 2^{p^*-1}\right)\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}|dv_{k_1}|^{p(x)}dx-C(a,p,C_1,\mu)\beta\left(\operatorname{meas}\left(A_{\varepsilon_1}\cup\left(\Omega\setminus H_{k_1}^{\lambda}\right)\right)\right)\\
&\quad\geq a2^{-p^*}\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}|dv_{k_1}|^{p(x)}dx-O(\varepsilon_1),
\end{aligned}
\tag{3.46}
$$

where $\mu>0$ is small enough.

From (3.43)–(3.46), we get

$$
F(v_{k_1},\Omega)\geq F\left(g_{k_1},\Omega\setminus A_{\varepsilon_1}\right)+a2^{-p^*}\int_{A_{\varepsilon_1}\cup(\Omega\setminus H_{k_1}^{\lambda})}|dv_{k_1}|^{p(x)}dx-O(\varepsilon_1).
\tag{3.47}
$$

(2) *The estimate of* $F(g_{k_1},\Omega\setminus A_{\varepsilon_1})$. Set $f_{k_1I}=g_{k_1I}-\omega_I$, where ω_I is defined by (3.42). Then

$$
\begin{gathered}
f_{k_1I}\rightharpoonup 0 \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega)\quad\text{as } k_1\longrightarrow\infty\quad\forall I,\\
\left\|f_{k_1I}\right\|_{L^\infty(\Omega)}\leq 2\lambda,\qquad\left\|df_{k_1I}\right\|_{L^\infty(\Omega,\Lambda^l)}\leq 2C(n)\lambda.
\end{gathered}
\tag{3.48}
$$

Let $G = \bigcup_I G_I$ with $G_I = \{x \in \Omega : \omega_I(x) \neq 0\}$. According to Acerbi and Fusco [19], we have $\text{meas}(G) \leq (\mathrm{C}_n^{l-1} + 1)\varepsilon_1$ where $\mathrm{C}_n^{l-1} = n(n-1)\cdots(n-l+2)/(l-1)(l-2)\cdots 1$, and set $f_{k_1} = \sum_I f_{k_1 I} dx_I$, then

$$\begin{aligned} F\left(g_{k_1}, \Omega \setminus A_{\varepsilon_1}\right) = F&\left(f_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus G\right) \\ &+ F\Big(v_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^{\lambda} \cap G\Big) \\ &+ F\Big(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap \Big(G \setminus H_{k_1}^{\lambda}\Big)\Big). \end{aligned} \tag{3.49}$$

Define

$$\begin{aligned} \Omega_1^{\varepsilon_1, k_1} &= A_{\varepsilon_1} \cup \Big(\Omega \setminus H_{k_1}^{\lambda}\Big), \qquad \Omega_2^{\varepsilon_1} = (\Omega \setminus A_{\varepsilon_1}) \setminus G, \\ \Omega_3^{\varepsilon_1, k_1} &= (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^{\lambda} \cap G, \qquad \Omega_4^{\varepsilon_1, k_1} = (\Omega \setminus A_{\varepsilon_1}) \cap \Big(G \setminus H_{k_1}^{\lambda}\Big). \end{aligned} \tag{3.50}$$

Similar to the proof of (3.46), we get

$$F\Big(v_{k_1}, \Omega_3^{\varepsilon_1, k_1}\Big) \geq a 2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \tag{3.51}$$

Since on $\Omega_4^{\varepsilon_1, k_1}$ we have

$$\int_{\Omega_4^{\varepsilon_1, k_1}} |dg_{k_1}|^{p(x)} dx \leq C(n, p)\Big(\mathrm{C}_n^{l-1} + 1\Big)\varepsilon_1, \tag{3.52}$$

then similar to the proof of (3.45), we get

$$\Big|F\Big(g_{k_1}, \Omega_4^{\varepsilon_1, k_1}\Big)\Big| \leq O(\varepsilon_1). \tag{3.53}$$

By (3.49)–(3.53), we have

$$F\left(g_{k_1}, \Omega \setminus A_{\varepsilon_1}\right) \geq F\left(f_{k_1}, \Omega_2^{\varepsilon_1}\right) + a 2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \tag{3.54}$$

Thus, we have

$$F(v_{k_1}, \Omega) \geq F\left(f_{k_1}, \Omega_2^{\varepsilon_1}\right) + a 2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1), \tag{3.55}$$

where $\Omega_5^{\varepsilon_1, k_1} = \Omega_1^{\varepsilon_1, k_1} \cup \Omega_3^{\varepsilon_1, k_1}$.

Choose an open set $\Omega' \subset \Omega$ which contains $\Omega_2^{\varepsilon_1}$ such that

$$\left|F\left(f_{k_1},\Omega'\right)-F\left(f_{k_1},\Omega_2^{\varepsilon_1}\right)\right|<\varepsilon_1. \tag{3.56}$$

From (3.55), we get

$$F(v_{k_1},\Omega)\geq F\left(f_{k_1},\Omega'\right)+a2^{-p^*}\int_{\Omega_5^{\varepsilon_1,k_1}}|dv_{k_1}|^{p(x)}dx-O(\varepsilon_1). \tag{3.57}$$

Approximate Ω' by hypercubes with edges parallel to coordinate axes, that is, construct

$$\begin{gathered}H_j\subset\Omega',\\ \operatorname{meas}\left(\Omega'\setminus H_j\right)\longrightarrow 0\quad\text{as } j\longrightarrow\infty,\\ H_j=\bigcup_{s=1}^{h_j}D_{j,s},\\ \operatorname{meas}\left(D_{j,s}\right)=1/2^{nj},\quad 1\leq s\leq h_j.\end{gathered} \tag{3.58}$$

Let $j>0$ be large enough such that for all $k_1>0$, we have

$$\begin{gathered}\left|F\left(f_{k_1},\Omega'\right)-F\left(f_{k_1},H_j\right)\right|<\varepsilon_1,\qquad\int_{\Omega'\setminus H_j}\left|df_{k_1}\right|^{p(x)}dx<\varepsilon_1,\\ \operatorname{meas}\left(\Omega'\setminus H_j\right)<\min\{\varepsilon_1,\delta_1\}.\end{gathered} \tag{3.59}$$

Thus,

$$F(v_{k_1},\Omega)\geq F\left(f_{k_1},H_j\right)+a2^{-p^*}\int_{\Omega_5^{\varepsilon_1,k_1}}|dv_{k_1}|^{p(x)}dx-O(\varepsilon_1)-2\varepsilon_1. \tag{3.60}$$

(3) *The estimate of* $F(f_{k_1},H_j)$. Let $\alpha>0$ be large enough such that for $E=\{x\in\Omega' : \eta(x)\leq\alpha\}$. Then

$$\operatorname{meas}\left(\Omega'\setminus E\right)<\frac{\varepsilon_1}{N},\qquad\int_{\Omega'\setminus E}\eta(x)dx<\varepsilon_1, \tag{3.61}$$

where $\|df_{k_1}\|_{L^\infty(\Omega,\Lambda^l)}\leq 2C_n^{l-1}C(n)\lambda=N$ and $\eta(x)=|G(x)|^{p'(x)}+2^{p^*-1}(C_1+1)|du_0|^{p(x)}$.

For $x\in\Omega,\xi\in\Lambda^l(\Omega)$, define

$$\psi(x,\xi)=\langle A(x,du_0(x)+\xi),\xi\rangle. \tag{3.62}$$

By Lemma 3.7 and (H1), there exists a compact subset $K \subset H_j$ such that $\psi(x,\xi)$ is continuous on $K \times \Lambda^l(\Omega)$ and $\operatorname{meas}(H_j \setminus K) < \varepsilon_1/(\alpha + N)$. Hence, $\psi(x,\xi)$ is uniformly continuous on bounded subsets of $K \times \Lambda^l(\Omega)$.

Divide each $D_{j,s}$ into 2^{nm} hypercubes $Q^m_{t,j,s}$ with edge length $2^{-jm}, 1 \le t \le 2^{nm}$. For all j, s, m, t, take $x^m_{t,j,s} \in Q^m_{t,j,s} \cap K \cap E$ (if this set is empty, take $x^m_{t,j,s} \in Q^m_{t,j,s}$) such that

$$\eta\left(x^m_{t,j,s}\right) \operatorname{meas}\left(Q^m_{t,j,s}\right) \le \int_{Q^m_{t,j,s}} \eta(x)dx. \tag{3.63}$$

Then

$$\begin{aligned}
&F\left(f_{k_1}, H_j\right) \\
&\quad = F\left(f_{k_1}, H_j \cap K \cap E\right) + F\left(f_{k_1}, H_j \setminus E\right) + F\left(f_{k_1}, \left(H_j \cap E\right) \setminus K\right) \\
&\quad \ge F\left(f_{k_1}, H_j \cap K \cap E\right) - \int_{H_j \setminus E} \eta(x)dx - \int_{(H_j \cap E)\setminus K} \eta(x)dx \\
&\qquad - 2^{p^*}(C_1+1)\left(\int_{H_j\setminus E} \left|df_{k_1}\right|^{p(x)}dx + \int_{(H_j\cap E)\setminus K} \left|df_{k_1}\right|^{p(x)}dx\right) \\
&\quad = F\left(f_{k_1}, H_j \cap K \cap E\right) - O(\varepsilon_1) \\
&\quad = b^{m,j}_{k_1} + c^{m,j}_{k_1} + d^{m,j}_{k_1} - O(\varepsilon_1),
\end{aligned} \tag{3.64}$$

where

$$\begin{aligned}
b^{m,j}_{k_1} &= \sum_{t,s} \int_{Q^m_{t,j,s}\cap K \cap E} \left(\psi\left(x, df_{k_1}(x)\right) - \psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)\right)dx, \\
c^{m,j}_{k_1} &= \sum_{t,s} \int_{Q^m_{t,j,s}} \psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)dx, \\
d^{m,j}_{k_1} &= -\sum_{t,s} \int_{Q^m_{t,j,s}\setminus(K\cap E)} \psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)dx.
\end{aligned} \tag{3.65}$$

By (3.25), we have

$$\lim_{k_1 \to \infty} F\left(v_{k_1}, \Omega\right) = 0. \tag{3.66}$$

Note that if $Q^m_{t,j,s} \cap K \cap E$ is an empty set, then

$$\int_{Q^m_{t,j,s}\cap K\cap E} \left[\psi\left(x, df_{k_1}(x)\right) - \psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)\right]dx = 0. \tag{3.67}$$

Now we only consider $Q^m_{t,j,s}$ which satisfies $Q^m_{t,j,s} \cap K \cap E \neq \phi$. Since $du_0(x)$ is uniformly continuous on H_j, then by the uniform continuity of ψ on bounded subsets of $K \times \Lambda^l(\Omega)$, we obtain that for $x \in Q^m_{t,j,s}$, there exists a constant $L > 0$ such that

$$\begin{aligned} &\left|\psi(x, df_{k_1}(x)) - \psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)\right| \\ &\quad = \left|\left\langle A(x, du_0(x) + df_{k_1}(x)) - A\left(x^m_{t,j,s}, du_0\left(x^m_{t,j,s}\right) + df_{k_1}(x)\right), df_{k_1}(x)\right\rangle\right| \\ &\quad < \frac{1}{\operatorname{meas}(H_j)}\varepsilon_1 \end{aligned} \tag{3.68}$$

holds for all $m > L$ and each k_1. Therefore, $|b^{m,j}_{k_1}| < \varepsilon_1$ for all k_1.

$$\begin{aligned} \left|d^{m,j}_{k_1}\right| &\le \sum_{t,s}\int_{Q^m_{t,j,s}\setminus(K\cap E)}\left|\psi\left(x^m_{t,j,s}, df_{k_1}(x)\right)\right|dx \\ &= \sum_{t,s}\int_{Q^m_{t,j,s}\setminus(K\cap E)}\left\langle A\left(x^m_{t,j,s}, du_0\left(x^m_{t,j,s}\right) + df_{k_1}(x)\right), df_{k_1}(x)\right\rangle dx \\ &\le \sum_{t,s}\int_{Q^m_{t,j,s}\setminus(K\cap E)} C_1\left|du_0\left(x^m_{t,j,s}\right) + df_{k_1}(x)\right|^{p(x)-1}|df_k(x)| + \left|G\left(x^m_{t,j,s}\right)\right||df_{k_1}(x)|dx \\ &\le \sum_{t,s}\int_{Q^m_{t,j,s}\setminus(K\cap E)}\left(\eta\left(x^m_{t,j,s}\right) + 2^{p^*}(C_1+1)N\right)dx \\ &\le \int_{(H_j\cap E)\setminus K}\left(\eta\left(x^m_{t,j,s}\right) + 2^{p^*}(C_1+1)N\right)dx + C(C_1,p)\sum_{t,s}\int_{Q^m_{t,j,s}\setminus E}\left(\eta\left(x^m_{t,j,s}\right) + N\right)dx \\ &\le C(\alpha, N, C_1, p)\operatorname{meas}((H_j\cap E)\setminus K) + C(C_1,p)\int_{H_j\setminus E}[\eta(x) + N]dx \\ &\le C(\alpha, N, C_1, p)\varepsilon_1 \le O(\varepsilon_1). \end{aligned} \tag{3.69}$$

Now we suppose that m is large enough that $|b^{m,j}_{k_1}| < \varepsilon_1$ for each $k_1 > 0$ and there exists $\overline{k}_1 > 0$ such that $F(v_{k_1}\Omega) < \varepsilon_1$ for $k_1 > \overline{k}_1$. Therefore, from (3.25), (3.60), and (3.64), we have

$$\begin{aligned} \varepsilon_1 &\ge F(v_{k_1}, \Omega) \\ &\ge c^{m,j}_{k_1} + a2^{-p^*}\int_{\Omega_5^{\varepsilon_1,k_1}}|dv_{k_1}|^{p(x)}dx - O(\varepsilon_1) - 3\varepsilon_1 - C(C_1,p)\varepsilon_1 \\ &= c^{m,j}_{k_1} + a2^{-p^*}\int_{\Omega_5^{\varepsilon_1,k_1}}|dv_{k_1}|^{p(x)}dx - O(\varepsilon_1). \end{aligned} \tag{3.70}$$

(4) *The estimate of* $c_{k_1}^{m,j}$. By $f_{k_1 I} \rightharpoonup 0$ weakly* in $W^{1,\infty}(\Omega)$ as $k_1 \to \infty$, we obtain $\|f_{k_1 I}\|_{L^\infty(\Omega)} \to 0$ as$k_1 \to \infty$ for each I. Then

$$R_{t,s,j}^{k_1,m} = \left\| |f_{k_1}| \right\|_{L^\infty(Q_{t,s,j}^m)} \longrightarrow 0 \quad \text{as } k_1 \longrightarrow \infty \text{ for fixed } m. \tag{3.71}$$

Define a hypercube $E_{t,s,j}^{k_1,m}$ contained in $Q_{t,s,j}^m$ with edge length $1/2^{jm} - 2R_{t,s,j}^{k_1,m}$ such that $\operatorname{dist}(\partial Q_{t,s,j}^m, E_{t,s,j}^{k_1,m}) = R_{t,s,j}^{k_1,m}$.

Next define

$$\begin{aligned} \varphi_{k_1}(x) &= 0, \quad x \in \partial Q_{t,s,j}^m, \\ \varphi_{k_1}(x) &= f_{k_1}(x), \quad x \in E_{t,s,j}^m. \end{aligned} \tag{3.72}$$

Since $\varphi_{k_1 I}$ is a Lipschitz mapping on set $E_{t,s,j}^m \cup \partial Q_{t,s,j}^m$ and its Lipschitz constant is no more than $2C(n)\lambda$, by Lemma 3.12, $\varphi_{k_1 I}$ can be extended to the whole $Q_{t,s,j}^m$, where it is also a Lipschitz mapping with the same Lipchistz constant. We still denote the extension by $\varphi_{k_1 I}$ and suppose that it is defined on the whole H_j. Then by [20]

$$\nabla \varphi_{k_1 I} - \nabla f_{k_1 I} \longrightarrow 0 \quad \text{a.e.} \quad \text{on} \ \ H_j. \tag{3.73}$$

Thus, there exists a $\overline{\overline{k_1}} > \overline{k_1}$ such that for all $k_1 > \overline{\overline{k_1}}$, we have

$$\begin{gathered} \int_{H_j} \left| d\varphi_{k_1} - df_{k_1} \right|^{p(x)} dx \le \frac{\varepsilon_1}{2}, \\ \sum_{t,s} \left| \int_{Q_{t,j,s}^m} \psi\left(x_{t,j,s}^m, df_{k_1}(x)\right) - \psi\left(x_{t,j,s}^m, d\varphi_{k_1}(x)\right) dx \right| \le \frac{\varepsilon_1}{2}. \end{gathered} \tag{3.74}$$

In view of (H5), we obtain that

$$\begin{aligned} c_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m} \psi\left(x_{t,j,s}^m, df_{k_1}(x)\right) dx \\ &\ge \sum_{t,s} \int_{Q_{t,j,s}^m} \psi\left(x_{t,j,s}^m, d\varphi_{k_1}(x)\right) dx - \frac{\varepsilon_1}{2} \\ &= \sum_{t,s} \int_{Q_{t,j,s}^m} \left\langle A\left(x_{t,j,s}^m, du_0\left(x_{t,j,s}^m\right) + d\varphi_{k_1}(x)\right), d\varphi_{k_1}(x) \right\rangle dx - \frac{\varepsilon_1}{2} \\ &\ge \gamma \sum_{t,s} \int_{Q_{t,j,s}^m} \left| d\varphi_{k_1} \right|^{p(x)} dx - \frac{\varepsilon_1}{2} \\ &\ge \frac{\gamma}{2^{p^*-1}} \int_{H_j} \left| df_{k_1} \right|^{p(x)} dx - \frac{(\gamma+1)\varepsilon_1}{2}. \end{aligned} \tag{3.75}$$

Thus in (3.70) for $k_1 > \overline{\overline{k_1}}$, we obtain the estimate of $F(v_{k_1}, \Omega)$ from the four steps above

$$\begin{aligned} \varepsilon_1 &\geq F(v_{k_1}, \Omega) \\ &\geq a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \frac{\gamma}{2^{p^*-1}} \int_{H_j} \left|df_{k_1}\right|^{p(x)} dx - \frac{(\gamma+1)\varepsilon_1}{2} - O(\varepsilon_1). \end{aligned} \tag{3.76}$$

Let $K(\varepsilon_1) = (\gamma+1)\varepsilon_1 / (2 + o(\varepsilon_1)) / \min\{a2^{-p^*}, \gamma/2^{p^*-1}\}$. Then

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \int_{H_j} \left|df_{k_1}\right|^{p(x)} dx \leq K(\varepsilon_1), \quad \text{for } k_1 > \overline{\overline{k_1}}. \tag{3.77}$$

Form (3.59) and (3.77), we deduce that

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1), \qquad \int_{\Omega'} \left|df_{k_1}\right|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.78}$$

According to the definition of $\Omega_2^{\varepsilon_1}$, we have

$$\int_{\Omega_{\varepsilon_1}^2} \left|dg_{k_1}\right|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.79}$$

Since $dg_{k_1}(x) = dv_{k_1}(x)$ for each $x \in H_{k_1}^{\lambda}$, we get

$$\int_{\Omega_{\varepsilon_1}^2 \cap H_{k_1}^{\lambda}} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.80}$$

By the definitions of $\Omega_2^{\varepsilon_1}$ and $\Omega_5^{\varepsilon_1, k_1}$, it is immediate that

$$\left(\Omega_2^{\varepsilon_1} \cap H_{k_1}^{\lambda}\right) \cup \Omega_5^{\varepsilon_1, k_1} = \Omega, \tag{3.81}$$

which implies that

$$\int_{\Omega} |dv_{k_1}|^{p(x)} dx \leq 2K(\varepsilon_1) + \varepsilon_1 \leq O(\varepsilon_1). \tag{3.82}$$

For $\varepsilon_2 > 0$ and the sequence $\{v_{k_1}\}$, repeating the above arguments we can extract a subsequence $\{v_{k_2}\}$ of $\{v_{k_1}\}$ such that

$$\int_{\Omega} |dv_{k_2}|^{p(x)} dx \leq O(\varepsilon_2), \tag{3.83}$$

whenever $k_2 > \overline{\overline{k_2}}$ for some $\overline{\overline{k_2}}$. If $\{v_{k_n}\}$ has been obtained, repeating the above process, we can extract a subsequence $\{k_{n+1}\}$ of $\{k_n\}$ such that

$$\int_\Omega |dv_{k_{n+1}}|^{p(x)}dx \le O(\varepsilon_{n+1}), \tag{3.84}$$

whenever $k_{n+1} > \overline{\overline{k}}_{n+1}$ for some $\overline{\overline{k}}_{n+1}$. Finally, by a diagonal argument we get a subsequence $\{v_{k_i}\}_{i=1}^\infty$ which satisfies

$$\int_\Omega |dv_{k_i}|^{p(x)}dx \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{3.85}$$

Therefore,

$$\|dv_{k_i}\|_{L^{p(x)}(\Omega,\Lambda^l)} \longrightarrow 0 \quad \text{as } i \longrightarrow \infty, \tag{3.86}$$

and by (3.23), $\{v_{k_i}\}_{i=1}^\infty$ strongly converges to zero in $\mathfrak{K}_0$ as $i \to \infty$. This completes the proof of Theorem 3.1. □

4. Applications

In this section, we explore applications of our results developed in this paper.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex Lipschitz domain. Suppose that maps $A : \Omega \times \Lambda^l(\Omega) \to \Lambda^l(\Omega)$ and $B : \Omega \times \Lambda^{l-1}(\Omega) \to \Lambda^{l-1}(\Omega)$, where $l = 1, 2, \dots, n$.

Example 4.1. If $p(x)$ satisfies (1.3), let $l = 1$, $A(x,\xi) = \xi|\xi|^{p(x)-2}$ and $B(x,\varsigma) = \varsigma|\varsigma|^{p(x)-2} - f(x)$, where $f(x) \in L^{p'(x)}(\Omega)$. Then A, B satisfy the required conditions, and (1.1) reduce to the following $p(x)$-Laplacian equations:

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = f(x), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.1}$$

Now by Theorem 3.1, we deduce that the $p(x)$-Laplacian equations (4.1) have at least one weak solution in $\mathfrak{K}^{1,p(x)}(\Omega)$ with $u = 0$ on $\partial\Omega$.

Example 4.2. If $l = 1$, $A(x,\xi) = \sum_{i,j} A_{ij}(x)\xi_j dx_i$, $B(x,\varsigma) = B(x)\varsigma - f(x)$, where $f(x) \in L^2(\Omega)$, and $A_{ij}(x)$, $B(x)$ satisfy the following conditions:

$$A_{ij}(x) = A_{ji}(x), \qquad \wedge|\xi|^2 \ge \sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \quad \lambda \le B(x) \le \wedge, \tag{4.2}$$

for some constants $\lambda, \wedge > 0$. Then A, B satisfy the required conditions, and (1.1) reduce to the following Divergence form equations:

$$\sum_{i,j=1}^{n} \nabla_j \left(A_{ij}(x) \nabla_i u(x)\right) + B(x)u(x) = f(x), \quad x \in \Omega, \tag{4.3}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{4.4}$$

where $\nabla_i = (\partial/\partial x_i)$. Now by Theorem 3.1, we deduce that the divergence form (4.3) have at least one weak solution $u(x)$ in $\mathfrak{K}^{1,2}(\Omega)$ with $u = 0$ on $\partial\Omega$. The comparison principles, the maximum principles, and the existence of weak solutions for divergence form equation (4.3) can be found in [21].

Example 4.3. If $p(x)$ satisfies (1.3), let $A(x, \xi) = \xi|\xi|^{p(x)-2}$ and $B(x, \varsigma) = \varsigma|\varsigma|^{p(x)-2} - f(x)$, where $f(x) \in L^{p'(x)}(\Omega, \wedge^{l-1})$. Then A, B satisfy the required conditions, and (1.1) reduce to the following $p(x)$-harmonic equations for differential forms:

$$d^* \left(du|du|^{p(x)-2}\right) + u|u|^{p(x)-2} = f(x), \quad x \in \Omega, \tag{4.5}$$

$$u(x) = 0, \quad x \in \partial\Omega. \tag{4.6}$$

Now by Theorem 3.1, we deduce that (4.5) have at least one weak solution $u(x)$ in $\mathfrak{K}_0^{1,p(x)}(\Omega, \wedge^{l-1})$. If $p(x)$ is a constant q and $1 < q < \infty$, the equation (4.5) is called nonhomogeneous q-harmonic equation. In [2], Iwaniec and Lutoborski studied the L^q theory of weak solution for homogeneous q-harmonic equations.

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