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Research Article

Residual Iterative Method for Solving Absolute Value Equations

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We suggest and analyze a residual iterative method for solving absolute value equations $Ax-|x|=b$ where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown, using the projection technique. We also discuss the convergence of the proposed method. Several examples are given to illustrate the implementation and efficiency of the method. Comparison with other methods is also given. Results proved in this paper may stimulate further research in this fascinating field.

1. Introduction

The residual methods were proposed for solving large sparse systems of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in R^{n\times n}$ is a positive definite matrix and $x, b \in R^n$. Paige and Saunders [1] minimized the residual norm over the Krylov subspace and proposed an algorithm for solving indefinite systems. Saad and Schultz [2] used Arnoldi process and suggested generalized minimal residual method which minimize norm of the residual at each step. The residual methods have been studied extensively [3–5].

We show that the Petrov-Galerkin process can be extended for solving absolute value equations of the form:

$$Ax - |x| = b, \tag{1.2}$$

where $A \in R^{n\times n}$, $b \in R^n$. Here $|x|$ is the vector in R^n with absolute values of the components of x and $x \in R^n$, is unknown. The absolute value equations (1.1) were investigated extensively in [6]. It was Managasarian [7, 8], who proved that the absolute value equations (1.2) are equivalent to the linear complementarity problems. This equivalent formulation was used by Managasarian [7, 8] to solve the absolute value equations. We would like to remark that the complementarity problems are also equivalent to the variational inequalities. Thus, we conclude that the absolute value equations are equivalent to the variational inequalities. There are several methods for solving the variational inequalities; see Noor [9–11], Noor et al. [12, 13] and the references therein. To the best our knowledge, this alternative equivalent formulation has not exploited up to now. This is another direction for future direction. We hope that this interlink among these fields may lead to discover novel and innovative techniques for solving the absolute value equations and related optimization problems. Noor et al. [14, 15] have suggested some iterative methods for solving absolute value equation (1.2) using minimization technique with symmetric positive definite matrix. For more details, see [3, 4, 6–12, 14–19].

In this paper, we suggest and analyse residual iterative method for solving absolute value equations (1.2) using projection technique. Our method is easy to implement. We discuss the convergence of the residual method for nonsymmetric positive definite matrices.

We denote by K and L the search subspace and the constraints subspace, respectively, and let m be their dimension and $x_0 \in R^n$ an initial guess. A projection method onto the subspace K and orthogonal to L is a process to find an approximate solution $x \in R^n$ to (1.2) by imposing the Petrov-Galerkin conditions that x belong to affine space $x_0 + K$ such that the new residual vector orthogonal to L, that is,

$$\text{find } x \in x_0 + K \quad \text{such that } b - (A - D(x))\, x \perp L, \tag{1.3}$$

where $D(x)$ is diagonal matrix corresponding to $\operatorname{sign}(x)$. For different choices of the subspace L, we have different iterative methods. Here we use the constraint space $L = (A - D(x))K$. The residual method approximates the solution of (1.2) by the vector $x \in x_0 + K$ that minimizes the norm of residual.

The inner product is denoted by $\langle \cdot, \cdot \rangle$ in the n-dimensional Euclidean space R^n. For $x \in R^n$, $\operatorname{sign}(x)$ will denote a vector with components equal to $1, 0, -1$ depending on whether the corresponding component of x is positive, zero, or negative. The diagonal matrix $D(x)$ corresponding to $\operatorname{sign}(x)$ is defined as

$$D(x) = \partial|x| = \operatorname{diag}(\operatorname{sign}(x)), \tag{1.4}$$

where $\partial|x|$ represent the generalized Jacobean of $|x|$ based on a subgradient [20, 21].

We denote the following by

$$\begin{aligned}
a &= \langle Cv_1, Cv_1 \rangle, \\
c &= \langle Cv_1, Cv_2 \rangle, \\
d &= \langle Cv_2, Cv_2 \rangle, \\
p_1 &= \langle b - Ax_k + |x_k|, Cv_1 \rangle = \langle b - Cx_k, Cv_1 \rangle, \\
p_2 &= \langle b - Ax_k + |x_k|, Cv_2 \rangle = \langle b - Cx_k, Cv_2 \rangle,
\end{aligned} \tag{1.5}$$

where $0 \neq v_1$, $v_2 \in R^n$, and $C = A - D(x_k)$. We consider A such that C is a positive definite matrix. We remark that $D(x_k)x_k = |x_k|$.

2. Residual Iterative Method

Consider the iterative scheme of the type:

$$x_{k+1} = x_k + \alpha v_1 + \beta v_2, \qquad 0 \neq v_1, v_2 \in R^n,\ k = 0, 1, 2, \ldots. \tag{2.1}$$

These vectors can be chosen by different ways. To derive residual method for solving absolute value equations in the first step, we choose the subspace

$$K_1 = \operatorname{span}\{v_1\}, \qquad L_1 = \operatorname{span}\{Cv_1\}, \qquad x_0 = x_k. \tag{2.2}$$

For $D(\tilde{x}_{k+1}) = D(x_k)$, we write the residual in the following form:

$$\begin{aligned} b - A\tilde{x}_{k+1} + |\tilde{x}_{k+1}| &= b - (A - D(\tilde{x}_{k+1}))\tilde{x}_{k+1} \\ &= b - (A - D(x_k))\tilde{x}_{k+1} \\ &= b - C\tilde{x}_{k+1}. \end{aligned} \tag{2.3}$$

From (1.3) and (2.3), we calculate

$$\tilde{x}_{k+1} \in x_k + K_1 \quad \text{such that } b - C\ \tilde{x}_{k+1} \perp L_1; \tag{2.4}$$

that is, we find the approximate solution by the iterative scheme

$$\tilde{x}_{k+1} = x_k + \alpha v_1. \tag{2.5}$$

Now, we rewrite (2.4) in the inner product as

$$\langle b - C\tilde{x}_{k+1}, Cv_1 \rangle = 0; \tag{2.6}$$

from the above discussion, we have

$$\begin{aligned} \langle b - Cx_k - \alpha Cv_1, Cv_1 \rangle &= \langle b - Cx_k, Cv_1 \rangle - \alpha \langle Cv_1, Cv_1 \rangle \\ &= p_1 - a\alpha = 0, \end{aligned} \tag{2.7}$$

from which we have

$$\alpha = \frac{p_1}{a}. \tag{2.8}$$

The next step is to choose the subspace

$$K_2 = \operatorname{span}\{v_2\}, \qquad L_2 = \operatorname{span}\{Cv_2\}, \qquad x_0 = \tilde{x}_{k+1}, \tag{2.9}$$

and to find the approximate solution x_{k+1} such that

$$x_{k+1} \in \tilde{x}_{k+1} + K_2 \quad \text{such that } b - Cx_{k+1} \perp L_2, \tag{2.10}$$

where

$$\begin{gathered} x_{k+1} = \tilde{x}_{k+1} + \beta v_2, \\ b - Ax_{k+1} + |x_{k+1}| = b - Cx_{k+1}, \qquad D(x_{k+1}) = D(x_k). \end{gathered} \tag{2.11}$$

Rewriting (2.10) in terms of the inner product, we have

$$\langle b - Cx_{k+1}, Cv_2 \rangle = 0. \tag{2.12}$$

Thus, we have

$$\begin{aligned} \langle b - Cx_{k+1}, Cv_2 \rangle &= \langle b - Cx_k - \alpha Cv_1 - \beta Cv_2, Cv_2 \rangle \\ &= \langle b - Cx_k, Cv_2 \rangle - \alpha \langle Cv_1, Cv_2 \rangle - \beta \langle Cv_2, Cv_2 \rangle \\ &= p_2 - c\alpha - d\beta = 0. \end{aligned} \tag{2.13}$$

From (2.8) and (2.13), we obtain

$$\beta = \frac{ap_2 - cp_1}{ad}. \tag{2.14}$$

We remark that one can choose $v_1 = r_k$ and v_2 in different ways. However, we consider the case $v_2 = s_k$ (s_k is given in Algorithm 2.1).

Based upon the above discussion, we suggest and analyze the following iterative method for solving the absolute value equations (1.2) and this is the main motivation of this paper.

Algorithm 2.1. Choose an initial guess $x_0 \in R^n$,

For $k = 0, 1, 2, \ldots$ until convergence do

$r_k = b - Ax_k + |x_k|$

$g_k = (A - D(x_k))^T (Ax_k - |x_k| - b)$

$H_k = ((A - D(x_k))^{-1} (A - D(x_k)))^T$

$s_k = -H_k g_k$

If $\|r_k\| = 0$, then stop; else

$\alpha_k = p_1/a, \beta_k = (ap_2 - cp_1)/ad$
Set $x_{k+1} = x_k + \alpha_k r_k + \beta_k s_k$
If $\|x_{k+1} - x_k\| < 10^{-6}$
then stop
End if
End for k.

If $\beta = 0$, then Algorithm 2.1 reduces to minimal residual method; see [2, 5, 21, 22]. For the convergence analysis of Algorithm 2.1, we need the following result.

Theorem 2.2. *Let $\{x_k\}$ and $\{r_k\}$ be generated by Algorithm 2.1; if $D(x_{k+1}) = D(x_k)$, then*

$$\|r_k\|^2 - \|r_{k+1}\|^2 = \frac{p_1^2}{a} + \frac{(ap_2 - cp_1)^2}{a^2 d}, \tag{2.15}$$

where $r_{k+1} = b - Ax_{k+1} + |x_{k+1}|$ and $D(x_{k+1}) = \operatorname{diag}(\operatorname{sign}(x_{k+1}))$.

Proof. Using (2.1), we obtain

$$\begin{aligned} r_{k+1} &= b - Ax_{k+1} + |x_{k+1}| \\ &= b - (A - D(x_{k+1}))x_{k+1} \\ &= b - (A - D(x_k))x_{k+1} \\ &= b - (A - D(x_k))x_k - \alpha(A - D(x_k))v_1 - \beta(A - D(x_k))v_2 \\ &= b - Ax_k + |x_k| - \alpha C v_1 - \beta C v_2 \\ &= r_k - \alpha C v_1 - \beta C v_2. \end{aligned} \tag{2.16}$$

Now consider

$$\begin{aligned} \|r_{k+1}\|^2 &= \langle r_{k+1}, r_{k+1}\rangle \\ &= \langle r_k - \alpha C v_1 - \beta C v_2, r_k - \alpha C v_1 - \beta C v_2\rangle \\ &= \langle r_k, r_k\rangle - 2\alpha\langle r_k, Cv_1\rangle - 2\alpha\beta\langle Cv_1, Cv_2\rangle - 2\beta\langle r_k, Cv_2\rangle + \alpha^2\langle Cv_1, Cv_1\rangle + \beta^2\langle Cv_2, Cv_2\rangle \\ &= \|r_k\|^2 - 2\alpha p_1 + 2c\alpha\beta - 2\beta p_2 + a\alpha^2 + \beta^2 d. \end{aligned} \tag{2.17}$$

From (2.8), (2.14), and (2.17), we have

$$\|r_k\|^2 - \|r_{k+1}\|^2 = \frac{p_1^2}{a} + \frac{(ap_2 - cp_1)^2}{a^2 d}, \tag{2.18}$$

the required result (2.15). □

Since $p_1^2/a + (ap_2 - cp_1)^2/a^2d \geq 0$, so from (2.18) we have

$$\|r_k\|^2 - \|r_{k+1}\|^2 = \frac{p_1^2}{a} + \frac{(ap_2 - cp_1)^2}{a^2 d} \geq 0. \tag{2.19}$$

From (2.19) we have $\|r_{k+1}\|^2 \leq \|r_k\|^2$. For any arbitrary vectors $0 \neq v_1, v_2 \in R^n$, α, β are defined by (2.8), and (2.14) minimizes norm of the residual.

We now consider the convergence criteria of Algorithm 2.1, and it is the motivation of our next result.

Theorem 2.3. *If C is a positive definite matrix, then the approximate solution obtained from Algorithm 2.1 converges to the exact solution of the absolute value equations* (1.2).

Proof. From (2.15), we have

$$\|r_k\|^2 - \|r_{k+1}\|^2 \geq \frac{p_1^2}{a} = \frac{\langle r_k, Cr_k\rangle^2}{\langle Cr_k, Cr_k\rangle} \geq \frac{\lambda_{\min}^2 \|r_k\|^4}{\lambda_{\max}^2 \|r_k\|^2} = \frac{\lambda_{\min}^2}{\lambda_{\max}^2}\|r_k\|^2. \tag{2.20}$$

This means that the sequence $\|r_k\|^2$ is decreasing and bounded. Thus the above sequence is convergent which implies that the left-hand side tends to zero. Hence $\|r_k\|^2$ tends to zero, and the proof is complete. □

3. Numerical Results

To illustrate the implementation and efficiency of the proposed method, we consider the following examples. All the experiments are performed with Intel(R) Core(TM) 2 × 2.1 GHz, 1 GB RAM, and the codes are written in Mat lab 7.

Example 3.1. Consider the ordinary differential equation:

$$\frac{d^2x}{dt^2} - |x| = \left(1 - t^2\right), \quad 0 \leq t \leq 1, \quad x(0) = -1 \quad x(1) = 0. \tag{3.1}$$

We discredited the above equation using finite difference method to obtain the system of absolute value equations of the type:

$$Ax - |x| = b, \tag{3.2}$$

where the system matrix A of size $n = 10$ is given by

$$a_{i,j} = \begin{cases} -242, & \text{for } j = i, \\ 121, & \text{for } \begin{cases} j = i+1, & i = 1, 2, \dots, n-1, \\ j = i-1, & i = 2, 3, \dots, n, \end{cases} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

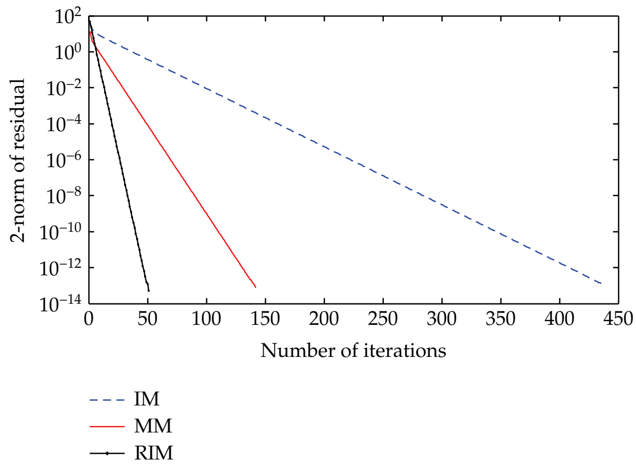

Figure 1

Table 1

Problems with sad $(A) > 1$	GNM	RIM
Problem size	1000	
Number of problem solved	100	
Total number of iterations	297	268
Accuracy	10^{-6}	10^{-6}
Total time in seconds	870.30	977.45

The exact solution is

$$x = \begin{cases} .1915802528 \sin t - 4 \cos t + 3 - t^2, & x < 0, \\ -1.462117157 e^{-t} - 0.5378828428 e^{t} + 1 + t^2, & x > 0. \end{cases} \tag{3.4}$$

In Figure 1, we compare residual method with Noor et al. [14, 15]. The residual iterative method, minimization method [14], and the iterative method [10] solve (3.1) in 51, 142, and 431 iterations, respectively. For the next two examples, we interchange v_1, v_2 with each other as Algorithm 2.1 converges for nonzero vectors $v_1, v_2 \in R^n$.

Example 3.2 (see [17]). We first chose a random A from a uniform distribution on $[-10, 10]$, then we chose a random x from a uniform distribution on $[-1, 1]$. Finally we computed $b = Ax - |x|$. We ensured that the singular values of each A exceeded 1 by actually computing the minimum singular value and rescaling A by dividing it by the minimum singular value multiplied by a random number in the interval $[0, 1]$. The computational results are given in Table 1.

In Table 1, GNM and RIM denote generalized Newton method [17] and residual iterative method. From Table 1 we conclude that residual method for solving absolute value equations (1.2) is more effective.

Table 2

Order	Residual iterative method		Yong method [23]	
	No. of iterations	TOC	No. of iterations	TOC
4	2	0.006	2	2.230
8	2	0.022	2	3.340
16	2	0.025	3	3.790
32	2	0.053	2	4.120
64	2	0.075	3	6.690
128	2	0.142	3	12.450
256	2	0.201	3	34.670
512	3	1.436	5	76.570
1024	2	6.604	5	157.12

Example 3.3 (see [23]). Consider random matrix A and b in Mat lab code as

$$\begin{gathered} n = \text{input}(\text{“dimension of matrix } A = \text{”}); \\ \text{rand } (\text{“state”}, 0); \\ R = \text{rand } (n, n); \\ b = \text{rand } (n, 1); \\ A = R'^{*}\text{Run}^{*}\text{eye}(n), \end{gathered} \tag{3.5}$$

with random initial guess. The comparison between the residual iterative method and the Yong method [23] is presented in Table 2.

In Table 2 TOC denotes time taken by CPU. Note that for large problem sizes the residual iterative method converges faster than the Yong method [23].

4. Conclusions

In this paper, we have used the projection technique to suggest an iterative method for solving the absolute value equations. The convergence analysis of the proposed method is also discussed. Some examples are given to illustrate the efficiency and implementation of the new iterative method. The extension of the proposed iterative method for solving the general absolute value equation of the form $Ax + B|x| = b$ for suitable matrices is an open problem. We have remarked that the variational inequalities are also equivalent to the absolute value equations. This equivalent formulation can be used to suggest and analyze some iterative methods for solving the absolute value equations. It is an interesting and challenging problem to consider the variational inequalities for solving the absolute value equations.

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