

Research Article

Analysis of an Ecoepidemiological Model with Prey Refuges

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An ecoepidemiological system with prey refuges and disease in prey is proposed. Bilinear incidence and Holling III functional response are used to model the contact process and the predation process, respectively. We will study the stability behavior of the basic system from a local to a global perspective. Permanence of the considered system is also investigated.

1. Introduction

Ecoepidemiology is the branch of biomathematics that understands the dynamics of disease spread on the predator-prey system. Modeling researches on such ecoepidemiological issues have received much attention recently [1–13]. Anderson and May [1] investigated a prey-predator model with prey infection and observed destabilization due to the spread of infectious diseases within animal and plant communities. Chattopadhyay analyzed predator-prey system with disease in the prey [5] and applied the ecoepidemiological study to the Pelicans at risk in the Salton Sea [6]. Bairagi et al. [13] made a comparative study on the role of prey infection in the stability aspects of a predator-prey system with several functional responses. An ecoepidemiological model with prey harvesting and predator switching was investigated by Bhattacharyya and Mukhopadhyay [14]. Kooi et al. [15] studied stabilization and complex dynamics in a predator-prey system with disease in predator. Most of the above-mentioned studies focused on the role of disease in regulating the dynamical consequences of the interacting populations concerned, such as disease-induced stabilization and destabilization of population states [13].

In fact, the dynamical consequences of the predator-prey model can be determined by much ecological effect, such as the Allee effect and prey refuge. Theoretical research

and field observations on population dynamics of prey refuges lead to the conclusion that prey refuges have two influences (stabilizing and destabilizing effect) on predator-prey models and prey extinction can be prevented by the addition of prey refuges [13–34]. Here, stabilization (destabilization) of stability refers to cases where an equilibrium point changes from an attractor (a repeller) to a repeller (an attractor) due to increase in the value of a control parameter [17]. Ruxton [16] proposed a continuous-time predator-prey model under the assumption that the rate of prey moving to refuges is proportional to predator density and the results showed that the hiding behavior of prey has a stabilizing effect. The stabilizing effect was also observed in a simple predator-prey system by González-Olivares and Ramos-Jiliberto [17]. Ma et al. [23] formulated a predator-prey model with a class of functional response incorporating the effect of prey refuges and observed the stabilizing and destabilizing effect due to the increases in the prey refuges.

In the present research, we formulate a mathematical model of prey-predator interaction with prey refuges and disease in prey. We mainly study the positivity and boundedness, the stability behavior of the disease-free equilibrium point, and the permanence of the basic model.

2. Model Formulation

The basic model comprises two population subclasses—(i) prey population with density $N(t)$ and (ii) predators with density $Y(t)$. To formulate our model, we make the following assumptions.

- (1) The prey population increases logistically with intrinsic growth rate r and environmental carrying capacity K .
- (2) The prey population is divided into two subclasses—the susceptible prey ($S(t)$) and the infected prey ($I(t)$) due to infectious disease. We also assumed that $N(t) = S(t) + I(t)$ at any instant of time t .
- (3) The susceptible prey is capable of reproducing only and the infected prey is removed by death at a rate d_1 .
- (4) The disease is spread only among the prey population and the disease is not genetically inherited. The infected prey does not become immune.
- (5) Susceptible prey becomes infected with the simple mass action law βSI , where β measures the force of infection.
- (6) The predators vanish due to natural death at a constant rate d_2 . They consume susceptible and infected prey following the Holling III functional response with predation coefficients c_1 and c_2 , respectively. The consumed prey is converted into predator with efficiency e .
- (7) It is assumed that there is a quantity γX of (the susceptible and the infected) prey population incorporating refuges. γ ($0 < \gamma < 1$) denotes that a constant proportion of (the susceptible and the infected) prey use refuges. All the above-mentioned parameters are assumed to be positive.

With the previously mentioned assumptions, the generalized predator-prey system with prey refuges and disease in prey can be represented by the following equations:

$$\begin{aligned}\frac{dS}{dt} &= rS \left(1 - \frac{S+I}{K}\right) - \beta SI - \frac{c_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2}, \\ \frac{dI}{dt} &= \beta SI - \frac{c_2(1-\gamma)^2 I^2 Y}{m + (1-\gamma)^2 I^2} - d_1 I, \\ \frac{dY}{dt} &= \frac{ec_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2} + \frac{ec_2(1-\gamma)^2 I^2 Y}{m + (1-\gamma)^2 I^2} - d_2 Y.\end{aligned}\tag{2.1}$$

Defining $G = \{(S, I, Y) \in R_+^3 \mid S + I \leq K, S \geq 0, I \geq 0, Y \geq 0\}$, it is easy to show that the set G is the positively invariable set of system (2.1).

3. The Positivity and Boundedness

Theorem 3.1. *All solutions of system (2.1) initiating R_+^3 are positive and ultimately bounded.*

Proof. Let $(S(t), I(t), Y(t))$ be one of the solutions of system (2.1).

Integrating (2.1) with initial conditions (S_0, I_0, Y_0) , we have

$$\begin{aligned}S(t) &= S_0 \exp \left[\int_0^t \left(r \left(1 - \frac{S(\xi) + I(\xi)}{K} \right) - \beta I(\xi) - \frac{c_1(1-\gamma)^2 S(\xi) Y(\xi)}{m + (1-\gamma)^2 S^2(\xi)} \right) d\xi \right] > 0, \\ I(t) &= I_0 \exp \left[\int_0^t \left(\beta S(\xi) - \frac{c_2(1-\gamma)^2 I(\xi) Y(\xi)}{m + (1-\gamma)^2 I^2(\xi)} - d_1 \right) d\xi \right] > 0, \\ Y(t) &= Y_0 \exp \left[\int_0^t \left(\frac{ec_1(1-\gamma)^2 S^2(\xi)}{m + (1-\gamma)^2 S^2(\xi)} + \frac{ec_2(1-\gamma)^2 I^2(\xi)}{m + (1-\gamma)^2 I^2(\xi)} - d_2 \right) d\xi \right] > 0.\end{aligned}\tag{3.1}$$

Hence all solutions starting in R_+^3 remain in R_+^3 for all $t \geq 0$. □

Next, we will prove the boundedness of the solutions.

Because $dS/dt \leq rS(1 - S/K)$, then we have

$$\lim_{t \rightarrow +\infty} \sup S(t) \leq K.\tag{3.2}$$

Let $W = eS + eI + Y$, then we obtain that

$$\begin{aligned}\frac{dW}{dt} &= reS\left(1 - \frac{S}{K}\right) - \frac{re}{K}SI - ed_1I - d_2Y \\ &\leq reS - ed_1I - d_2Y \\ &= (r+1)eS - (eS + d_1eI + d_2Y) \\ &\leq e(r+1)K - \delta W,\end{aligned}\tag{3.3}$$

where $\delta = \min\{1, d_1, d_2\}$.

Hence, we have

$$\frac{dW}{dt} + \delta W \leq e(r+1)K.\tag{3.4}$$

That is

$$\begin{aligned}0 \leq W(S, I, Y) &\leq \frac{e(r+1)K}{\delta} + \frac{W(S(0), I(0), Y(0))}{e^{\delta t}}, \\ 0 \leq W &\leq \frac{e(r+1)K}{\delta} \quad (t \rightarrow +\infty).\end{aligned}\tag{3.5}$$

Thus, all curves of system (2.1) will enter the following region:

$$B = \left\{ (S, I, Y) : 0 \leq W \leq \frac{e(r+1)K}{\delta} + \varepsilon, \forall \varepsilon > 0 \right\}.\tag{3.6}$$

4. The Equilibrium Point

All equilibrium points of system (2.1) can be obtained by solving the following equations:

$$\begin{aligned}rS\left(1 - \frac{S+I}{K}\right) - \beta SI - \frac{c_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2} &= 0, \\ \beta SI - \frac{c_2(1-\gamma)^2 I^2 Y}{m + (1-\gamma)^2 I^2} - d_1 I &= 0, \\ \frac{ec_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2} + \frac{ec_2(1-\gamma)^2 I^2 Y}{m + (1-\gamma)^2 I^2} - d_2 Y &= 0,\end{aligned}\tag{4.1}$$

These points are as follows:

- (1) the trivial equilibrium point $E_0(0, 0, 0)$,
- (2) the equilibrium point $E_K(K, 0, 0)$,

(3) the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$,

(4) the disease-free equilibrium point $E^*(S^*, 0, Y^*)$,

where

$$\begin{aligned}\bar{S} &= \frac{d_1}{\beta} \doteq S_M, & \bar{I} &= \frac{r(\beta K - d_1)}{\beta(r + \beta K)} \doteq I_M, \\ S^* &= \frac{1}{1 - \gamma} \sqrt{\frac{d_2 m}{ec_1 - d_2}}, & Y^* &= \frac{erS^*}{d_2} \left(1 - \frac{S^*}{K}\right).\end{aligned}\tag{4.2}$$

Let $R_1 = (ec_1(1 - \gamma)^2 K^2 / (m + (1 - \gamma)^2 K^2)) / d_2$. It is clear to show that the disease-free equilibrium point $E^*(S^*, 0, Y^*)$ has its ecological meaning when $R_1 > 1$.

Let $R_0 = \beta K / d_1$, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is nonnegative when $R_0 > 1$.

5. The Stability Property

In this section, we will study the local and global stability of the equilibrium points of system (2.1).

Theorem 5.1. Let $\bar{R}_1 = ((ec_1(1 - \gamma)^2 S_M^2 / (m + (1 - \gamma)^2 S_M^2)) + (ec_2(1 - \gamma)^2 I_M^2 / (m + (1 - \gamma)^2 I_M^2))) / d_2$, $\bar{R}_2 = \max(R_1, \bar{R}_1)$, then one has the following.

- (1) The trivial equilibrium point $E_0(0, 0, 0)$ is always unstable.
- (2) If $R_0 < 1$, $R_1 < 1$, then the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is globally asymptotically stable. If $R_0 > 1$ or $R_1 > 1$, the equilibrium point $E_K(K, 0, 0)$ is unstable.
- (3) If $R_0 > 1$ and $\bar{R}_1 < 1$, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is locally asymptotically stable. If $\bar{R}_2 < 1$, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is locally asymptotically stable. If $\bar{R}_1 > 1$, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is unstable.

Proof. The Jacobian matrix of system (2.1) at the trivial equilibrium point $E_0(0, 0, 0)$ is

$$\begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.\tag{5.1}$$

□

Clearly, the trivial equilibrium point $E_0(0, 0, 0)$ is unstable.

The Jacobian matrix of system (2.1) at the equilibrium point $E_K(K, 0, 0)$ is

$$\begin{pmatrix} -r & -r - \beta K & -\frac{c_1(1-\gamma)^2 K^2}{m + (1-\gamma)^2 K^2} \\ 0 & \beta K - d_1 & 0 \\ 0 & 0 & \frac{ec_1(1-\gamma)^2 K^2}{m + (1-\gamma)^2 K^2} - d_2 \end{pmatrix}. \quad (5.2)$$

According to the theorem about the local stability, the local stability of the equilibrium point $E_K(K, 0, 0)$ is determined only by the sign of $(ec_1(1-\gamma)^2 K^2 / (m + (1-\gamma)^2 K^2)) - d_2$ and $\beta K - d_1$.

Therefore, if $R_0 < 1$ and $R_1 < 1$, the equilibrium point $E_K(K, 0, 0)$ is locally asymptotically stable.

Next, we will prove the global stability of the equilibrium point $E_K(K, 0, 0)$.

Defining the Lyapunov function $V = eI + Y$, then we obtain that

$$\begin{aligned} \frac{dV}{dt} &= e(\beta S - d_1)I + \left(\frac{ec_1(1-\gamma)^2 S^2}{m + (1-\gamma)^2 S^2} - d_2 \right)Y \\ &\leq e(\beta K - d_1)I + \left(\frac{ec_1(1-\gamma)^2 K^2}{m + (1-\gamma)^2 K^2} - d_2 \right)Y. \end{aligned} \quad (5.3)$$

Hence, if $R_0 < 1$ and $R_1 < 1$, $dV/dt \leq 0$.

Having $E = \{(S, I, Y) \in G \mid dV/dt = 0\} = \{I = 0, Y = 0\}$, the maximum invariable set of system (2.1) is $M = E = \{I = 0, Y = 0\}$.

According to the LaSalle invariable set theorem, $\lim_{t \rightarrow +\infty} I(t) = 0$, $\lim_{t \rightarrow +\infty} Y(t) = 0$.

Thus, the limit equation of system (2.1) is

$$\frac{dS}{dt} = rS \left(1 - \frac{S}{K} \right). \quad (5.4)$$

Clearly, the equilibrium point $S = K$ is globally asymptotically stable.

According to the limit system theorem, if $R_0 < 1$ and $R_1 < 1$, the equilibrium point $E_K(K, 0, 0)$ is globally asymptotically stable.

Setting $R_0 > 1$, the Jacobian matrix of system (2.1) at the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is

$$\begin{pmatrix} -\frac{r\bar{S}}{K} & -\frac{r\bar{S}}{K} - \beta\bar{S} & -\frac{c_1(1-\gamma)^2\bar{S}^2}{m + (1-\gamma)^2\bar{S}^2} \\ \beta\bar{I} & 0 & -\frac{c_2(1-\gamma)^2\bar{I}^2}{m + (1-\gamma)^2\bar{I}^2} \\ 0 & 0 & \frac{ec_1(1-\gamma)^2\bar{S}^2}{m + (1-\gamma)^2\bar{S}^2} + \frac{ec_2(1-\gamma)^2\bar{I}^2}{m + (1-\gamma)^2\bar{I}^2} - d_2 \end{pmatrix}. \quad (5.5)$$

The characteristic equation of system (2.1) at the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is

$$(\lambda - A)(\lambda^2 - B\lambda + C) = 0, \quad (5.6)$$

where

$$\begin{aligned} A &= \frac{c_1(1-\gamma)^2 S_M^2}{m + (1-\gamma)^2 S_M^2} + \frac{ec_2(1-\gamma)^2 I_M^2}{m + (1-\gamma)^2 I_M^2} - d_2, \\ B &= -\frac{2S_M}{K} < 0, \quad C = \beta I_M \left(\frac{S_M}{K} + \beta S_M \right) > 0. \end{aligned} \quad (5.7)$$

According to the Routh-Hurwitz rule, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is locally asymptotically stable when $\bar{R}_1 < 1$.

Next, we will prove the global stability of the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$.

Defining the Lyapunov function $V = Y$, then we have

$$\frac{dV}{dt} = \left(\frac{ec_1(1-\gamma)^2 S^2}{m + (1-\gamma)^2 S^2} + \frac{ec_2(1-\gamma)^2 I^2}{m + (1-\gamma)^2 I^2} - d_2 \right) Y. \quad (5.8)$$

Let $\Phi(S, I) = (c_1(1-\gamma)^2 S^2 / (m + (1-\gamma)^2 S^2)) + (ec_2(1-\gamma)^2 I^2 / (m + (1-\gamma)^2 I^2))$ in which $(S, I) \in D = \{(S, I) \mid S + I \leq K, S \geq 0, I \geq 0\}$.

Thus

$$\Phi'(t) = \frac{2c_1em(1-\gamma)^2 S}{(m + (1-\gamma)^2 S^2)^2} \frac{dS}{dt} + \frac{2c_2em(1-\gamma)^2 I}{(m + (1-\gamma)^2 I^2)^2} \frac{dI}{dt}. \quad (5.9)$$

Clearly, $(2c_1em(1-\gamma)^2 S / (m + (1-\gamma)^2 S^2)^2) > 0$ and $(2c_2em(1-\gamma)^2 I / (m + (1-\gamma)^2 I^2)^2) > 0$ on the set $D' = D - \{S = 0 \mid I = 0\}$.

Thus, $\Phi'(t) = 0$ on the set D' , if and only if $dS/dt = 0$ and $dI/dt = 0$, then $S = S_M$, $I = I_M$.

If $S = I = 0$, then $\Phi(S, I) = 0$. If $S = S_M$, then $I = I_M$. If $S < S_M$, then $dI/dt < 0$; that is, $I < I_M$.

Therefore, the maximum value of the function $\Phi(t)$ is obtained at the point $(K, 0)$ or (S_M, I_M) .

Hence, if $\bar{R}_2 < 1$, $dV/dt \leq 0$; that is, $E = \{(S, I, Y) \in G \mid dV/dt = 0\} = \{Y = 0\}$.

According to the LaSalle invariable set theorem, $\lim_{t \rightarrow +\infty} Y(t) = 0$. Thus, the limit equation of system (2.1) is

$$\begin{aligned} \frac{dS}{dt} &= rS \left(1 - \frac{S}{K}\right) - \beta SI, \\ \frac{dI}{dt} &= \beta SI - d_1 I. \end{aligned} \quad (5.10)$$

According to the results of the appendix section, if $R_0 > 1$, the equilibrium point (S_M, I_M) is globally asymptotically stable.

According to the limit system theorem, if $R_0 > 1$ and $\bar{R}_2 < 1$, the predator-extinction equilibrium point $\bar{E}(\bar{S}, \bar{I}, 0)$ is globally asymptotically stable.

Theorem 5.2. Let $R_1^* = (\beta/d_1(1-\gamma))\sqrt{(d_2 m/(ec_1 - d_2))}$, $R_2^* = ((2d_2 - ec_1)(1-\gamma)K)/2d_2\sqrt{(ec_1 - d_2)/d_2 m}$. If $R_1 > 1$, then the disease-free equilibrium point $E^*(S^*, 0, I^*)$ is nonnegative,

- (1) if $R_1^* < 1$ and $R_2^* < 1$, then the disease-free equilibrium point $E^*(S^*, 0, I^*)$ is locally asymptotically stable;
- (2) if $R_0 < 1$ and $R_2^* < 1$, then the disease-free equilibrium point $E^*(S^*, 0, I^*)$ is globally asymptotically stable;
- (3) if $R_1^* > 1$ or $R_2^* > 1$, then the disease-free equilibrium point $E^*(S^*, 0, I^*)$ is unstable.

Proof. Assuming $R_1 > 1$, the Jacobian matrix of system (2.1) at the disease-free equilibrium point $E^*(S^*, 0, Y^*)$ is

$$\begin{pmatrix} a_{11} & -\beta S^* - \frac{rS^*}{K} - \frac{c_1(1-\gamma)^2(S^*)^2}{m + (1-\gamma)^2(S^*)^2} \\ 0 & a_{22} & 0 \\ \frac{2ec_1 m(1-\gamma)^2 S^* Y^*}{(m + (1-\gamma)^2(S^*)^2)^2} & 0 & 0 \end{pmatrix}, \quad (5.11)$$

where

$$a_{11} = r \left(1 - \frac{2S^*}{K} \right) - \frac{2c_1 m (1-\gamma)^2 S^* Y^*}{(m + (1-\gamma)^2 (S^*)^2)^2}, \quad a_{22} = \beta S^* - d_1. \quad (5.12)$$

□

The characteristic equation of system (2.1) at the disease-free equilibrium point $E^*(S^*, 0, Y^*)$ is

$$(\lambda - a_{22}) \left(\lambda^2 - a_{11}\lambda + \frac{2c_1^2 e m (1-\gamma)^4 (S^*)^3 Y^*}{(m + (1-\gamma)^2 (S^*)^2)^3} \right) = 0. \quad (5.13)$$

Clearly, if $R_1^* < 1$, $a_{22} = \beta S^* - d_1 < 0$.

Again, we have

$$\begin{aligned} a_{11} &= r \left(1 - \frac{2S^*}{K} \right) - \frac{2c_1 m (1-\gamma)^2 S^* Y^*}{(m + (1-\gamma)^2 (S^*)^2)^2} \\ &= r \left(1 - \frac{2S^*}{K} \right) - 2r \left(1 - \frac{d_2}{ec_1} \right) \left(1 - \frac{S^*}{K} \right) \\ &= r \left[1 - \frac{2S^*}{K} - 2 + \frac{2S^*}{K} + \frac{2d_2}{ec_1} - \frac{2d_2}{ec_1} \frac{S^*}{K} \right] \\ &= \frac{r}{ec_1} \left[(2d_2 - ec_1) - \frac{2d_2}{(1-\gamma)K} \sqrt{\frac{d_2 m}{ec_1 - d_2}} \right]. \end{aligned} \quad (5.14)$$

Hence, if $R_2^* < 1$, then $a_{11} < 0$.

According to the Routh-Hurwitz rule, if $R_1^* < 1$ and $R_2^* < 1$, then the disease-free equilibrium point $E^*(S^*, 0, Y^*)$ is locally asymptotically stable.

Next, we will study the global stability of the disease-free equilibrium point $E^*(S^*, 0, Y^*)$.

Defining the Lyapunov function $V = I$, then we obtain

$$\frac{dV}{dt} = \left(\beta S - \frac{c_2 (1-\gamma)^2 Y}{m + (1-\gamma)^2 I} - d_1 \right) I \leq (\beta K - d_1) I. \quad (5.15)$$

Thus, if $R_0 < 1$, $dV/dt \leq 0$ and $dV/dt = 0$ if and only if $I = 0$; that is, $E = \{(S, I, Y) \in G \mid dV/dt = 0\} = \{I = 0\}$.

According to the LaSalle invariable set theorem, $I(t) \rightarrow 0$ when $t \rightarrow +\infty$. The limit system of system (2.1) is

$$\begin{aligned}\frac{dS}{dt} &= rS\left(1 - \frac{S}{K}\right) - \frac{c_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2}, \\ \frac{dY}{dt} &= \frac{ec_1(1-\gamma)^2 S^2 Y}{m + (1-\gamma)^2 S^2} - d_2 Y.\end{aligned}\tag{5.16}$$

Clearly, the equilibrium points of the system (5.16) are

$$E'_0(0,0), \quad E'_K(K,0), \quad E'_*(S^*, Y^*)\tag{5.17}$$

in which S^* and Y^* are similar as the equilibria expression of system (2.1).

It is easy to show that the equilibrium point E'_* has its ecological meaning when $R_1 > 1$.

According to the Routh-Hurwitz rule, the equilibrium point E'_0 is unstable. If $R_1 > 1$, then $E'_K(K,0)$ is unstable.

Again, the Jacobian matrix of system (5.16) at the equilibrium point E'_* is

$$\begin{pmatrix} a_{11} & -\frac{c_1(1-\gamma)^2 (S^*)^2}{m + (1-\gamma)^2 (S^*)^2} \\ \frac{2ec_1m(1-\gamma)^2 S^* Y^*}{(m + (1-\gamma)^2 (S^*)^2)^2} & 0 \end{pmatrix},\tag{5.18}$$

where

$$a_{11} = r\left(1 - \frac{2S^*}{K}\right) - \frac{2c_1m(1-\gamma)^2 S^* Y^*}{(m + (1-\gamma)^2 (S^*)^2)^2}.\tag{5.19}$$

The characteristic equation of system (5.16) at the equilibrium point E'_* is

$$\lambda^2 - a_{11}\lambda + \frac{2c_1^2 em(1-\gamma)^4 (S^*)^3 Y^*}{(m + (1-\gamma)^2 (S^*)^2)^3} = 0.\tag{5.20}$$

According to the above study, if $R_2^* < 1$, then $a_{11} < 0$.

Hence, if $R_2^* < 1$, then the equilibrium point E'_* is locally asymptotically stable in the region D by the Routh-Hurwitz rule.

It is easy to note that the globally asymptotically stability of the equilibrium point E'_* implies that there is no close orbit in the region D for the considered system.

Let

$$\begin{aligned} S &= \sqrt{\frac{d_2 m}{ec_1 - d_2}} \tilde{S}, & Y &= \frac{1}{c_1} \sqrt{d_2 m(ec_1 - d_2)} \tilde{Y}, \\ dt &= \frac{d_2(1-\gamma)^2 \tilde{S}^2 + ec_1 - d_2}{d_2(ec_1 - d_2)} d\tilde{t}, \end{aligned} \quad (5.21)$$

and rewrite \tilde{S} , \tilde{Y} , and \tilde{t} into S , Y , and t , then system (5.16) becomes as follows:

$$\begin{aligned} \frac{dS}{dt} &= S(A_0 + A_1 S + A_2 S^2 + A_3 S^3) - (1-\gamma)^2 S^2 Y \equiv P(S, Y), \\ \frac{dY}{dt} &= (1-\gamma)^2 S^2 Y - Y \equiv Q(S, Y), \end{aligned} \quad (5.22)$$

where $A_0 = r/d_2 > 0$, $A_1 = -r/(d_2 K) \sqrt{d_2 m/(ec_1 - d_2)} < 0$, $A_2 = (1-\gamma)^2 r/(ec_1 - d_2) > 0$, $A_3 = -(1-\gamma)^2 r/(ec_1 - d_2) \sqrt{d_2 m/(ec_1 - d_2)} < 0$.

Thus, the positive equilibrium point $E'_*(S^*, Y^*)$ of system (5.16) becomes the positive equilibrium point $E_P(S_P, Y_P)$ of system (5.22), where

$$\begin{aligned} S_P &= \frac{1}{1-\gamma}, \\ Y_P &= \frac{1}{1-\gamma} \frac{ec_1}{d_2(ec_1 - d_2)} \left(r - \frac{r}{(1-\gamma)K} \sqrt{\frac{d_2 m}{ec_1 - d_2}} \right) \\ &= \frac{rec_1}{d_2(ec_1 - d_2)(1-\gamma)} \left(1 - \frac{1}{(1-\gamma)K} \sqrt{\frac{d_2 m}{ec_1 - d_2}} \right). \end{aligned} \quad (5.23)$$

Considering the Dulac function $B(S, Y) = S^{-2}Y^{n-1}$, then we have

$$\begin{aligned} \frac{\partial(BP)}{\partial S} + \frac{\partial(BQ)}{\partial Y} &= S^{-2}Y^{n-1} \left[2A_3 S^3 + (A_2 + n(1-\gamma)^2) S^2 - (A_0 + n) \right] \\ &\equiv S^{-2}Y^{n-1} \Phi(S, n). \end{aligned} \quad (5.24)$$

In order to prove the global stability, we will prove only that there exists a real number n such that $\Phi(S, n) \leq 0$.

Clearly, if $S = 0$ or $S = -(A_2 + n(1-\gamma)^2)/3A_3$, then $2S[3A_3 S + A_2 + n(1-\gamma)^2] = 0$ since $\partial\Phi(S, n)/\partial S = 2S[3A_3 S + A_2 + n(1-\gamma)^2]$.

Let $-(A_2 + n(1-\gamma)^2)/3A_3 > 0$ ($A_2 > 0$, $A_3 < 0$), that is, $A_2 + n(1-\gamma)^2 > 0$, then we obtain that if $0 < S < -(A_2 + n(1-\gamma)^2)/3A_3$, then $\partial\Phi(S, n)/\partial S > 0$. Otherwise, $\partial\Phi(S, n)/\partial S < 0$.

Therefore, the function $\Phi(S, n)$ has the maximum value at the point $S = -(A_2 + n(1-\gamma)^2)/3A_3$ and $\Phi(0, n) = -(A_0 + n) < 0$.

Hence, there exists only one real number n , such that $A_2 + n(1 - \gamma)^2 > 0$ and

$$\begin{aligned} \max_{0 < S < +\infty} \Phi(S, n) &= \Phi(S, n)|_{S=-(A_2+n(1-\gamma)^2)/3A_3} \\ &= \frac{[A_2 + n(1 - \gamma)^2]^3}{27A_3^2} - \frac{A_2 + n(1 - \gamma)^2}{(1 - \gamma)^2} + \frac{A_2}{(1 - \gamma)^2} - A_0 = 0. \end{aligned} \quad (5.25)$$

Thus, we will prove only that there exists $A_2 + n(1 - \gamma)^2 > 0$, such that

$$[A_2 + n(1 - \gamma)^2]^3 + M[A_2 + n(1 - \gamma)^2] + N = 0, \quad (5.26)$$

where $M = -27A_3^2/(1 - \gamma)^2 < 0$, $N = 27A_3^2[A_2/(1 - \gamma)^2 - A_0] > 0$.

The discriminant of the cubic equation is

$$\begin{aligned} \Delta &= \frac{N^2}{4} + \frac{M^2}{27} = \frac{27A_3^4}{4} \left[\left(\frac{A_2}{(1 - \gamma)^2} - A_0 \right)^2 - \frac{4A_3^2}{(1 - \gamma)^6} \right] \\ &= [A_2S_p^2 - A_0 + 2A_3S_p^3][A_2S_p^2 - A_0 - 2A_3S_p^3]. \end{aligned} \quad (5.27)$$

It is easy to show that if $d_2 < ec_1 < 2d_2$, then $A_2S_p^2 - A_0 = r(2d_2 - ec_1)/d_2(ec_1 - d_2) > 0$.

Again, if $R_1 > 1$ and $R_2^* < 1$, then $A_2S_p^2 - A_0 - 2A_3S_p^3 > 0$ and $2A_3S_p^3 + A_2S_p^2 - A_0 \leq 0$. If $2A_3S_p^3 + A_2S_p^2 - A_0 = 0$, then $\Delta = 0$.

According to Shengjin's distinguishing means, the cubic equation has one negative real root and two positively real roots.

If $2A_3S_p^3 + A_2S_p^2 - A_0 < 0$, the cubic equation has three roots which are not equal.

According to the Descartes rule of signs, the cubic equation has two positively real roots and one negatively real root at most.

Therefore, the cubic equation has at least one positively real root. That is to say, there exists a number n , such that $A_2 + n(1 - \gamma)^2 > 0$.

Furthermore, we obtain that

$$\frac{\partial(BP)}{\partial S} + \frac{\partial(BQ)}{\partial Y} \leq 0. \quad (5.28)$$

According to the Bendixson-Dulac theorem, there does not exist the limit cycle for the limit system.

Hence, the equilibrium point E'_* is globally asymptotically stable.

Therefore, if $R_0 < 1$ and $R_2^* < 1$, then the equilibrium point E'_* is globally asymptotically stable according to the limit system theorem.

6. Permanence

Theorem 6.1. *If $R_0 > 1$, $R_1 > 1$, $\bar{R}_1 > 1$, and $R_1^* > 1$, then system (2.1) is permanent.*

Proof. Considering the average Lyapunov function $V(S, I, Y) = S^{\alpha_1} I^{\alpha_2} Y^{\alpha_3}$, where α_i ($i = 1, 2, 3$) is positive, then in the region R_+^3 , we have

$$\begin{aligned} \frac{\dot{V}}{V} = & \alpha_1 \left[r \left(1 - \frac{S+I}{K} \right) - \beta I - \frac{c_1(1-\gamma)^2 SY}{m + (1-\gamma)^2 S^2} \right] \\ & + \alpha_2 \left[\beta S - \frac{c_2(1-\gamma)^2 IY}{m + (1-\gamma)^2 I^2} - d_1 \right] \\ & + \alpha_3 \left[\frac{ec_1(1-\gamma)^2 S^2}{m + (1-\gamma)^2 S^2} + \frac{ec_2(1-\gamma)^2 I^2}{m + (1-\gamma)^2 I^2} - d_2 \right] \equiv \Psi(S, I, Y). \end{aligned} \quad (6.1)$$

□

In order to prove the permanence of system (2.1), we only indicate the following results: the function $\Psi(S, I, Y) > 0$ for all boundary equilibrium points.

Let $\alpha_1 > (d_1\alpha_2 + d_2\alpha_3)/r$, then

$$\begin{aligned} \Psi(E_0) &= \alpha_1 r - \alpha_2 d_1 - \alpha_3 d_2 > 0, \\ \Psi(E_K) &= \alpha_2(\beta K - d_1) + \alpha_3 \left(\frac{ec_1(1-\gamma)^2 K^2}{m + (1-\gamma)^2 K^2} - d_2 \right), \\ \Psi(\bar{E}) &= \alpha_3 \left(\frac{ec_1(1-\gamma)^2 S_M^2}{m + (1-\gamma)^2 S_M^2} + \frac{ec_2(1-\gamma)^2 I_M^2}{m + (1-\gamma)^2 I_M^2} - d_2 \right), \\ \Psi(E^*) &= \alpha_2(\beta S^* - d_1). \end{aligned} \quad (6.2)$$

Hence, if $R_0 > 1$ and $R_1 > 1$, then $\Psi(E_K) > 0$. If $\bar{R}_2 > 1$, then $\Psi(\bar{E}) > 0$. If $R_1^* > 1$, then $\Psi(E^*) > 0$.

Therefore, system (2.1) is permanent when $R_0 > 1$, $R_1 > 1$, $\bar{R}_1 > 1$, and $R_1^* > 1$ by the average Lyapunov function theorem [34].

By simple computation, $R_1 > 1 (< 1)$ is equivalent to $(1-\gamma)K\sqrt{(ec_1 - d_2)/d_2 m} > 1 (< 1)$.

Hence, if $R_1 > 1$, then $R_1^* > 1$ implies $R_0 > 1$.

Appendix

Considering the following *SI* model

$$\begin{aligned} \frac{dS}{dt} &= rS \left(1 - \frac{S+I}{K} \right) - \beta SI, \\ \frac{dI}{dt} &= \beta SI - d_1 I, \end{aligned} \quad (A.1)$$

where $S(t)$ and $I(t)$ are density of susceptible and infected prey population at time t . The parameters r , K , β , and d_1 are all positive.

The positive equilibrium point of this system is $E_M(S_M, I_M)$ in which

$$S_M = \frac{d_1}{\beta}, \quad I_M = \frac{r(\beta K - d_1)}{\beta(r + \beta K)}. \quad (\text{A.2})$$

Let $R_0 = \beta K / d_1$.

Clearly, if $R_0 = \beta K / d_1 > 1$, then the equilibrium point $E_M(S_M, I_M)$ is positive.

Theorem A.1. *Let $R_0 = \beta K / d_1$. If $R_0 > 1$, then the positive equilibrium point $E_M(S_M, I_M)$ is globally asymptotically stable.*

Proof. The Jacobian matrix of system (A.1) at the positive equilibrium point $E_M(S_M, I_M)$ is

$$\begin{pmatrix} -\frac{rS_M}{K} & -\frac{rS_M}{K} - \beta S_M \\ 0 & \beta S_M - d_1 \end{pmatrix}. \quad (\text{A.3})$$

□

Clearly, if $R_0 > 1$, then the positive equilibrium point $E_M(S_M, I_M)$ is locally asymptotically stable.

Next, defining the Dulac function $B(S, I) = 1/SI$, then we have

$$\frac{\partial(BP)}{\partial S} + \frac{\partial(BQ)}{\partial I} = -\frac{r}{KI} < 0, \quad (\text{A.4})$$

where $P(S, I) = rS(1 - ((S + I)/K)) - \beta SI$, $Q(S, I) = \beta SI - d_1 I$.

Therefore, there does not exist closed curve in the region D according to the Dulac theorem.

This implies that the positive equilibrium point $E_M(S_M, I_M)$ is globally asymptotically stable.

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