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Research Article

The Hyers-Ulam-Rassias Stability of $(m,n)_{(\sigma,\tau)}$-Derivations on Normed Algebras

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We study the Hyers-Ulam-Rassias stability of $(m,n)_{(\sigma,\tau)}$-derivations on normed algebras.

1. Introduction

A classical question in the theory of functional equations is as follows. *Under what conditions is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$*? This problem was formulated by Ulam in 1940 (see [1, 2]). He investigated the stability of group homomorphisms. *Let* $(\mathcal{G}_1,\circ)$ *be a group, and let* $(\mathcal{G}_2,*,\delta)$ *be a metric group with a metric* $\delta(\cdot,\cdot)$. *Suppose that* $f:\mathcal{G}_1\to\mathcal{G}_2$ *is a map and* $\epsilon>0$ *a fixed scalar. Does there exists* $\lambda>0$*such that if* f *satisfies the inequality*

$$\delta(f(x\circ y),f(x)*f(y))\le\lambda \tag{1.1}$$

for all $x,y\in\mathcal{G}_1$, *then there exists a group homomorphism* $F:\mathcal{G}_1\to\mathcal{G}_2$ *with the property*

$$\delta(f(x),F(x))\le\epsilon \tag{1.2}$$

for all $x\in\mathcal{G}_1$?

One year later, Ulam's problem was affirmatively solved by Hyers [3] for the Cauchy functional equation $f(x+y)=f(x)+f(y)$.: *Let* $\mathcal{X}_1$ *be a normed space,* $\mathcal{X}_2$ *a Banach space, and* $\epsilon>0$ *a fixed scalar. Suppose that* $f:\mathcal{X}_1\to\mathcal{X}_2$ *is a map with the property*

$$\|f(x+y)-f(x)-f(y)\|<\epsilon \tag{1.3}$$

for all $x, y \in \mathcal{X}_1$*. Then there exists a unique additive mapping* $F : \mathcal{X}_1 \to \mathcal{X}_2$ *such that*

$$\|f(x) - F(x)\| < \epsilon \tag{1.4}$$

for all $x \in \mathcal{X}_1$. This gave rise to the stability theory of functional equations.

The famous Hyers stability result has been generalized in the stability of additive mappings involving a sum of powers of norms by Aoki [4] which allowed the Cauchy difference to be unbounded. In 1978, Rassias [5] proved the stability of linear mappings in the following way. *Let* $\mathcal{X}_1$ *be a real normed space and* $\mathcal{X}_2$ *a real Banach space. If there exist scalars* $\epsilon \geq 0$ *and* $0 \leq p < 1$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.5}$$

for all $x, y \in \mathcal{X}_1$*, then there exists a unique additive mapping* $F : \mathcal{X}_1 \to \mathcal{X}_2$ *with the property*

$$\|f(x) - F(y)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.6}$$

for all $x \in \mathcal{X}_1$*. Moreover, if the map* $r \mapsto f(rx)$ *is continuous on* $\mathbb{R}$ *for each* $x \in \mathcal{X}_1$*, then* F *is linear.* This result has provided a lot of influence in the development of what we now call the Hyers-Ulam-Rassias stability of functional equations.

Later, Găvruţa [6] generalized the Rassias' theorem as follows: *Let* $(\mathcal{G}, +)$ *be an Abelian group and* $\mathcal{X}$ *a Banach space. Suppose that the so-called admissible control function* $\varphi : \mathcal{G} \times \mathcal{G} \to [0, \infty)$ *satisfies*

$$\sum_{k=0}^{\infty} \frac{\varphi(2^k x, 2^k y)}{2^{k+1}} < \infty \tag{1.7}$$

for all $x, y \in \mathcal{G}$. If $f : \mathcal{G} \to \mathcal{X}$ *is a mapping with the property*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y) \tag{1.8}$$

for all $x, y \in \mathcal{G}$*, then there exists a unique additive mapping* $F : \mathcal{G} \to \mathcal{X}$ *such that*

$$\|f(x) - F(x)\| \leq \sum_{k=0}^{\infty} \frac{\varphi(2^k x, 2^k y)}{2^{k+1}} \tag{1.9}$$

for all $x \in \mathcal{G}$.

In the last few decades, various approaches to the problem have been introduced by several authors. Moreover, it is surprising that in some cases the *approximate mapping* is actually a *true mapping*. In such cases we call the equation $\mathcal{E}$ superstable. For the history and various aspects of this theory we refer the reader to monographs [7–9].

As we are aware, the stability of derivations was first investigated by Jun and Park [10]. During the past few years, approximate derivations were studied by a number of mathematicians (see [11–18] and references therein).

Moslehian [19] studied the stability of (σ,τ)-derivations and generalized some results obtained in [18]. He also established the generalized Hyers-Ulam-Rassias stability of (σ,τ)-derivations on normed algebras into Banach bimodules. This motivated us to investigate approximate $(m,n)_{(\sigma,\tau)}$-derivations on normed algebras. The aim of this paper is to study the stability of $(m,n)_{(\sigma,\tau)}$-derivations and to generalize some results given in [19].

2. Preliminaries

Throughout, $\mathcal{A}$ will be a normed algebra and $\mathcal{M}$ a Banach $\mathcal{A}$-bimodule. Let σ and τ be two linear operators on $\mathcal{A}$. An additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called an (σ,τ)-derivation if

$$d(xy)=d(x)\sigma(y)+\tau(x)d(y) \tag{2.1}$$

holds for all $x,y\in\mathcal{A}$. Ordinary derivations from $\mathcal{A}$ to $\mathcal{M}$ and maps defined by $x\mapsto a\sigma(x)-\tau(x)a$, where $a\in\mathcal{A}$ is a fixed element and σ,τ are endomorphisms on $\mathcal{A}$, are natural examples of (σ,τ)-derivations on $\mathcal{A}$. Moreover, if ψ is an endomorphism on $\mathcal{A}$, then ψ is a $((1/2)\psi,(1/2)\psi)$-derivation on $\mathcal{A}$. We refer the reader to [20], where further information about (σ,τ)-derivations can be found.

In [19] Moslehian studied stability of (σ,τ)-derivations. The natural question here is, whether the analogue results hold true for $(m,n)_{(\sigma,\tau)}$-derivations. Theorem 3.1 answers this question in the affirmative.

Let m and n be nonnegative integers with $m+n\neq 0$. An additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called a $(m,n)_{(\sigma,\tau)}$-derivation if

$$(m+n)d(xy)=2md(x)\sigma(y)+2n\tau(x)d(y) \tag{2.2}$$

holds for all $x,y\in\mathcal{A}$. Clearly, $(m,n)_{(\sigma,\tau)}$-derivations are one of the natural generalizations of (σ,τ)-derivations (the case $m=n$). If $\sigma,\tau=id$, where id denotes the identity map on $\mathcal{A}$, and an additive mapping $d:\mathcal{A}\to\mathcal{M}$ satisfies (2.2), then d is called a (m,n)-derivation. In the last few decades a lot of work has been done on the field of (m,n)-derivations on rings and algebras (see, e.g, [21–25]). This motivated us to study the Hyers-Ulam-Rassias stability of functional inequalities associated with $(m,n)_{(\sigma,\tau)}$-derivations.

In the following, we will assume that m and n are nonnegative integers with $m+n\neq 0$. We will use the same symbol $\|\cdot\|$ in order to represent the norms on a normed algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule $\mathcal{M}$. For a given (admissible control) function $\varphi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ we will use the following abbreviation:

$$\phi(x,y):=\sum_{k=0}^{\infty}\frac{\varphi(2^kx,2^ky)}{2^{k+1}},\quad x,y\in\mathcal{A}. \tag{2.3}$$

Let us start with one well-known lemma.

Lemma 2.1 (see [6]). *Suppose that a function $\varphi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ satisfies $\phi(x,y)<\infty$, $x,y\in\mathcal{A}$. If $f:\mathcal{A}\to\mathcal{M}$ is a mapping with*

$$\|f(x+y)-f(x)-f(y)\|\leq\varphi(x,y) \tag{2.4}$$

for all $x, y \in \mathcal{A}$, then there exists a unique additive mapping $F : \mathcal{A} \to \mathcal{M}$ such that

$$\|f(x) - F(x)\| \leq \phi(x, x) \tag{2.5}$$

for all $x \in \mathcal{A}$.

We say that an additive mapping $f : \mathcal{A} \to \mathcal{M}$ is $\mathbb{C}$-linear if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{A}$ and all scalars $\lambda \in \mathbb{C}$. In the following, Λ will denote the set of all complex units, that is,

$$\Lambda = \{\lambda \in C : |\lambda| = 1\}. \tag{2.6}$$

For a given additive mapping $f : \mathcal{A} \to \mathcal{M}$, Park [26] obtained the next result.

Lemma 2.2. *If $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{A}$ and all $\lambda \in \Lambda$, then f is $\mathbb{C}$-linear.*

3. The Results

Our first result is a generalization of [19, Theorem 2.1] (the case $m = n$). We use the direct method to construct a unique $\mathbb{C}$-linear mapping from an approximate one and prove that this mapping is an appropriate $(m, n)_{(\sigma,\tau)}$-derivation on $\mathcal{A}$. This method was first devised by Hyers [3]. The idea is taken from [19].

Theorem 3.1. *Let $d : \mathcal{A} \to \mathcal{M}$ and $f, g : \mathcal{A} \to \mathcal{A}$ be mappings with $d(0) = f(0) = g(0) = 0$. Suppose that there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that $\phi(x, y) < \infty$ for all $x, y \in \mathcal{A}$ and*

$$\|d(\lambda x + \lambda y) - \lambda d(x) - \lambda d(y)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \varphi(x, y), \tag{3.2}$$

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \varphi(x, y), \tag{3.3}$$

$$\|(m + n)d(xy) - 2md(x)f(y) - 2ng(x)d(y)\| \leq \varphi(x, y) \tag{3.4}$$

for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$. Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(x) - \sigma(x)\| \leq \phi(x, x), \qquad \|g(x) - \tau(x)\| \leq \phi(x, x) \tag{3.5}$$

for all $x \in \mathcal{A}$, and a unique $\mathbb{C}$-linear $(m, n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that

$$\|d(x) - D(x)\| \leq \phi(x, x) \tag{3.6}$$

for all $x \in \mathcal{A}$.

Proof. Taking $\lambda = 1$ in (3.1) and using Lemma 2.1, it follows that there exists a unique additive mapping $D : \mathcal{A} \to \mathcal{M}$ such that $\|d(x) - D(x)\| \leq \phi(x, x)$ holds for all $x \in \mathcal{A}$. More precisely, using the induction, it is easy to see that

$$\left\| \frac{d(2^l x)}{2^l} - d(x) \right\| \leq \sum_{k=0}^{l-1} \frac{\varphi(2^k x, 2^k x)}{2^{k+1}}, \tag{3.7}$$

$$\left\| \frac{d(2^p x)}{2^p} - \frac{d(2^q x)}{2^q} \right\| \leq \sum_{k=q}^{p-1} \frac{\varphi(2^k x, 2^k x)}{2^{k+1}} \tag{3.8}$$

for all $x \in \mathcal{A}$, all positive integers l, and all $0 \leq q < p$. According to the assumptions on $\phi(x, y)$, it follows that the sequence $\{d(2^k x)/2^k\}_{k=0}^{\infty}$ is Cauchy. Thus, by the completeness of $\mathcal{M}$, this sequence is convergent and we can define a map $D : \mathcal{A} \to \mathcal{M}$ as

$$D(x) := \lim_{k \to \infty} \frac{d(2^k x)}{2^k}, \quad x \in \mathcal{A}. \tag{3.9}$$

Using (3.1), we get

$$\begin{aligned} &\|D(\lambda x + \lambda y) - \lambda D(x) - \lambda D(y)\| \\ &\quad = \lim_{k \to \infty} 2^{-k} \left\| d\left(\lambda 2^k x + \lambda 2^k y\right) - \lambda d\left(2^k x\right) - \lambda d\left(2^k y\right) \right\| \\ &\quad \leq \lim_{k \to \infty} 2^{-k} \varphi\left(2^k x, 2^k y\right) = 0. \end{aligned} \tag{3.10}$$

This yields that

$$D(\lambda x + \lambda y) = \lambda D(x) + \lambda D(y) \tag{3.11}$$

for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$. Using Lemma 2.2, it follows that the map D is $\mathbb{C}$-linear. Moreover, according to inequality (3.7), we have

$$\|d(x) - D(x)\| = \lim_{k \to \infty} \left\| d(x) - \frac{d(2^k x)}{2^k} \right\| \leq \phi(x, x) \tag{3.12}$$

for all $x \in \mathcal{A}$.

Next, we have to show the uniqueness of D. So, suppose that there exists another $\mathbb{C}$-linear mapping $\widetilde{D} : \mathcal{A} \to \mathcal{M}$ such that $\|d(x) - \widetilde{D}(x)\| \leq \phi(x,x)$ for all $x \in \mathcal{A}$. Then

$$\begin{aligned}\left\|D(x) - \widetilde{D}(x)\right\| &= \lim_{k\to\infty} 2^{-k}\left\|d\left(2^k x\right) - \widetilde{D}\left(2^k x\right)\right\| \\ &\leq \lim_{k\to\infty} 2^{-k}\phi\left(2^k x, 2^k x\right) \\ &= \lim_{k\to\infty} 2^{-k}\sum_{j=0}^{\infty}\frac{\varphi(2^{j+k}x, 2^{j+k}x)}{2^{j+1}} \\ &= \lim_{k\to\infty}\sum_{j=k}^{\infty}\frac{\varphi(2^j x, 2^j x)}{2^{j+1}} = 0.\end{aligned} \tag{3.13}$$

Therefore, $D(x) = \widetilde{D}(x)$ for all $x \in \mathcal{A}$, as desired.

Similarly we can show that there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}\sigma(x) &:= \lim_{k\to\infty}\frac{f(2^k x)}{2^k}, \quad x \in \mathcal{A}, \\ \tau(x) &:= \lim_{k\to\infty}\frac{g(2^k x)}{2^k}, \quad x \in \mathcal{A}.\end{aligned} \tag{3.14}$$

Furthermore,

$$\|f(x) - \sigma(x)\| \leq \phi(x,x), \qquad \|g(x) - \tau(x)\| \leq \phi(x,x) \tag{3.15}$$

for all $x \in \mathcal{A}$.

It remains to prove that D is an $(m,n)_{(\sigma,\tau)}$-derivation. Writing $2^k x$ in the place of x and $2^k y$ in the place of y in (3.4), we obtain

$$\left\|(m+n)d\left(4^k xy\right) - 2md\left(2^k x\right)f\left(2^k y\right) - 2ng\left(2^k x\right)d\left(2^k y\right)\right\| \leq \varphi\left(2^k x, 2^k y\right). \tag{3.16}$$

This yields that

$$\begin{aligned}&\|((m+n)D(xy) - 2mD(x)\sigma(y) - 2n\tau(x)D(y)\| \\ &\quad = \lim_{k\to\infty} 4^{-k}\left\|(m+n)d\left(4^k xy\right) - 2md\left(2^k x\right)f\left(2^k y\right) - 2ng\left(2^k x\right)d\left(2^k y\right)\right\| \\ &\quad \leq \lim_{k\to\infty} 4^{-k}\varphi\left(2^k x, 2^k y\right) = 0\end{aligned} \tag{3.17}$$

for all $x, y \in \mathcal{A}$. Thus, mappings D and σ, τ satisfy (2.2). The proof is completed. □

Remark 3.2. If there exists $x_0 \in \mathcal{A}$ such that d and the map $x \mapsto \phi(x,x)$ are continuous at point x_0, then D is continuous on $\mathcal{A}$. Namely, if D was not continuous, then there would exist an integer C and a sequence $\{x_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x_k = 0$ and $\|D(x_k)\| > 1/C$, $k \geq 0$. Let $t > C(2\phi(x_0,x_0)+1)$. Then

$$\lim_{k\to\infty} d(tx_k + x_0) = d(x_0) \tag{3.18}$$

since d is continuous at point x_0. Thus, there exists an integer k_0 such that for every $k > k_0$ we have

$$\|d(tx_k + x_0) - d(x_0)\| < 1. \tag{3.19}$$

Therefore,

$$\begin{aligned} 2\phi(x_0,x_0) + 1 &< \frac{t}{C} < \|D(tx_k)\| = \|D(tx_k + x_0) - D(x_0)\| \\ &\leq \|D(tx_k + x_0) - d(tx_k + x_0)\| + \|d(tx_k + x_0) - d(x_0)\| + \|d(x_0) - D(x_0)\| \\ &< \phi(tx_k + x_0, tx_k + x_0) + 1 + \phi(x_0,x_0) \end{aligned} \tag{3.20}$$

for every $k > k_0$. Letting $k \to \infty$ and using the continuity of the map $x \mapsto \phi(x,x)$ at point x_0, we get a contradiction.

Let $\epsilon \geq 0$ and $0 \leq p < 1$. Applying Theorem 3.1 for the case

$$\varphi(x,y) := \epsilon\left(\|x\|^p + \|y\|^p\right), \quad x,y \in \mathcal{A}. \tag{3.21}$$

Corollary 3.3. *Let $d : \mathcal{A} \to \mathcal{M}$ and $f,g : \mathcal{A} \to \mathcal{A}$ be mappings with $d(0) = f(0) = g(0) = 0$. Suppose that* (3.1), (3.2), (3.3), *and* (3.4) *hold true for all $x,y \in \mathcal{A}$ and $\lambda \in \Lambda$, where a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ is defined as above. Then there exist unique $\mathbb{C}$-linear mappings $\sigma,\tau : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\|f(x) - \sigma(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p, \quad \|g(x) - \tau(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{3.22}$$

for all $x \in \mathcal{A}$ and a unique $\mathbb{C}$-linear $(m,n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that

$$\|d(x) - D(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{3.23}$$

Proof. Note that $\phi(x,y) < \infty$ for all $x,y \in \mathcal{A}$ and

$$\phi(x,y) = \frac{\epsilon}{2-2^p}\left(\|x\|^p + \|y\|^p\right), \quad x,y \in \mathcal{A}. \tag{3.24}$$

□

Remark 3.4. Recall that we can actually take any map $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ in the form

$$\varphi(x, y) := \nu + \epsilon(\|x\|^p + \|y\|^p), \quad x, y \in \mathcal{A}, \tag{3.25}$$

where $\nu \geq 0$. In this case we have

$$\phi(x, y) = \nu + \frac{\epsilon(\|x\|^p + \|y\|^p)}{(2 - 2^p)}, \quad x, y \in \mathcal{A}. \tag{3.26}$$

Before stating our next result, let us write one well-known lemma about the continuity of measurable functions (see, e.g., [27]).

Lemma 3.5. *If a measurable function $\psi : \mathbb{R} \to \mathbb{R}$ satisfies $\psi(r_1+r_2) = \psi(r_1)+\psi(r_2)$ for all $r_1, r_2 \in \mathbb{R}$, then ψ is continuous.*

Now we are in the position to state a result for normed algebras $\mathcal{A}$ which are spanned by a subset $\mathcal{S}$ of $\mathcal{A}$. For example, $\mathcal{A}$ can be a C^*-algebra spanned by the unitary group of $\mathcal{A}$ or the positive part of $\mathcal{A}$

Theorem 3.6. *Let $\mathcal{A}$ be a normed algebra which is spanned by a subset $\mathcal{S}$ of $\mathcal{A}$ and $d : \mathcal{A} \to \mathcal{M}$, $f, g : \mathcal{A} \to \mathcal{A}$ mappings with $d(0) = f(0) = g(0) = 0$. Suppose that there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that $\phi(x, y) < \infty$ for all $x, y \in \mathcal{A}$ and* (3.1), (3.2), (3.3) *holds true for all $x, y \in \mathcal{A}$ and $\lambda = 1, i$. Moreover, suppose that* (3.4) *holds true for all $x, y \in \mathcal{S}$. If for all $x \in \mathcal{A}$ the functions $r \mapsto d(rx)$, $r \mapsto f(rx)$, and $r \mapsto g(rx)$ are continuous on $\mathbb{R}$, then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\|f(x) - \sigma(x)\| \leq \phi(x, x), \qquad \|g(x) - \tau(x)\| \leq \phi(x, x) \tag{3.27}$$

for all $x \in \mathcal{A}$ and a unique $\mathbb{C}$-linear $(m, n)_{(\sigma,\tau)}$-derivation $D : \mathcal{A} \to \mathcal{M}$ such that

$$\|d(x) - D(x)\| \leq \phi(x, x) \tag{3.28}$$

for all $x \in \mathcal{A}$.

We will give just a sketch of the proof since most of the steps are the same as in the proof of Theorem 3.1.

Proof. As in the proof of Theorem 3.1, we can show that there exists a unique additive mapping $D : \mathcal{A} \to \mathcal{M}$ defined by $D(x) := \lim_{k\to\infty}(d(2^k x)/2^k)$, $x \in \mathcal{A}$. Moreover, $\|d(x) - D(x)\| \leq \phi(x, x)$ for all $x \in \mathcal{A}$.

Writing $y = 0$, $\lambda = i$ in (3.1), we get

$$\|d(ix) - id(x)\| \leq \varphi(x, 0). \tag{3.29}$$

Therefore,

$$\|D(ix) - iD(x)\| = \lim_{k\to\infty} 2^{-k} \left\| d\left(2^k ix\right) - id\left(2^k x\right) \right\| \leq \lim_{k\to\infty} 2^{-k} \varphi\left(2^k x, 0\right) = 0. \tag{3.30}$$

This yields that

$$D(ix) = iD(x) \tag{3.31}$$

for all $x \in \mathcal{A}$. In the next step we will show that D is $\mathbb{R}$-linear, that is,

$$D(rx) = rD(x) \tag{3.32}$$

for all $x \in \mathcal{A}$ and all $r \in \mathbb{R}$.

Since D is additive, we have $D(qx) = qx$ for every $x \in \mathcal{A}$ and all rational numbers q. Let us fix elements $x_0 \in \mathcal{A}$ and $\rho \in \mathcal{M}^*$, where $\mathcal{M}^*$ denotes the dual space of $\mathcal{M}$. Then we can define a function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(r) = \rho(D(rx_0)), \quad r \in \mathbb{R}. \tag{3.33}$$

Firstly, we would like to prove that ψ is continuous. Recall that

$$\psi(r_1 + r_2) = \rho(D((r_1 + r_2)x_0)) = \rho(D(r_1x_0)) + \rho(D(r_2x_0)) = \psi(r_1) + \psi(r_2) \tag{3.34}$$

for all $r_1, r_2 \in \mathbb{R}$. Furthermore,

$$\psi(r) = \lim_{k\to\infty} \rho\left(\frac{d(2^k r x_0)}{2^k}\right) \tag{3.35}$$

for all $r \in \mathbb{R}$. Set

$$\psi_k(r) = \rho\left(\frac{d(2^k r x_0)}{2^k}\right), \quad k \geq 0. \tag{3.36}$$

Obviously, $\{\psi_k\}_{k=0}^{\infty}$ is a sequence of continuous functions and ψ is its pointwise limit. This yields that ψ is a Borel function and, by Lemma 3.5 it is continuous. Therefore, we have $\psi(r) = r\psi(1)$ for all $r \in \mathbb{R}$. This implies $D(rx_0) = rD(x_0)$. Since x_0 was an arbitrary element from $\mathcal{A}$, we proved that D is $\mathbb{R}$-linear.

Now, let $\lambda \in \mathbb{C}$. Then $\lambda = r_1 + ir_2$ for some real numbers r_1, r_2. Using (3.31), we have

$$D(\lambda x) = D((r_1 + ir_2)x) = D(r_1x) + D(ir_2x) = r_1D(x) + ir_2D(x) = \lambda D(x) \tag{3.37}$$

for all $x \in \mathcal{A}$. This means that D is $\mathbb{C}$-linear.

Similarly we can show that there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(x) - \sigma(x)\| \leq \phi(x,x), \qquad \|g(x) - \tau(x)\| \leq \phi(x,x) \tag{3.38}$$

for all $x \in \mathcal{A}$. Moreover, (2.2) holds true for all $x, y \in \mathcal{S}$. Since $\mathcal{A}$ is linearly generated by $\mathcal{S}$, we conclude that D is an $(m,n)_{(\sigma,\tau)}$-derivation on $\mathcal{A}$. The proof is completed. □

Remark 3.7. As above, we can apply Theorem 3.6 for the case

$$\varphi(x,y) := \nu + \epsilon\left(\|x\|^p + \|y\|^p\right), \quad x, y \in \mathcal{A}, \tag{3.39}$$

where $\nu, \epsilon \geq 0$ and $0 \leq p < 1$.

Remark 3.8. If $\epsilon \geq 0$ and $0 \leq p < 1/2$, then we can use in Theorem 3.1 as well as in Theorem 3.6 a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ given by

$$\varphi(x,y) := \epsilon\|x\|^p\|y\|^p, \quad x, y \in \mathcal{A}. \tag{3.40}$$

In this case

$$\phi(x,y) = \frac{\epsilon}{2-4^p}\|x\|^p\|y\|^p, \quad x, y \in \mathcal{A}. \tag{3.41}$$

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