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*Research Article*

# Global Stability of Multigroup Dengue Disease Transmission Model

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We investigate a class of multigroup dengue epidemic model. We show that the global dynamics are determined by the basic reproductive number $R_0$. We present that when $R_0 \leq 1$, there is a unique disease-free equilibrium which is globally asymptotically stable; when $R_0 > 1$, there exists a unique endemic equilibrium and it is globally asymptotically stable proved by a graph-theoretic approach to the method of global Lyapunov function.

## 1. Introduction

To understand and control the spread of infectious disease in population, mathematical epidemic models have been paid more attention. One essential assumption in most classical epidemic models is that the individuals are homogeneously mixed. However, many infectious diseases, such as measles, mumps, and gonorrhea, occur in heterogeneous host population, so multigroup epidemic models seem more reasonable. One of the earliest multigroup models is analysed by Lajmanovich and Yorke [1] for gonorrhea in a nonhomogeneous population. However, because of the large scale and complexity of multigroup models, progresses in the mathematical analysis of their global dynamics have been slow, particularly, the question of uniqueness and global stability of the endemic equilibrium. Recently, a graph-theoretic approach to the method of global Lyapunov functions in [2, 3] was proposed to resolve the open problem on the uniqueness and global stability of the endemic equilibrium. Subsequently, a series of good results were produced about multigroup epidemic models in [4–8].

In this paper, we study a multigroup dengue disease transmission model by the method in [2, 3]. In the model, the population is divided into $n$ groups. Each group is divided into five disjoint classes: susceptible individuals, infective individuals, removed individuals, susceptible mosquitoes, and infective mosquitoes whose numbers of individuals at time $t$ are denoted by $S_{H_i}(t)$, $I_{H_i}(t)$, $R_{H_i}(t)$, $S_{V_i}(t)$, $I_{V_i}(t)$, respectively. The model to be studied takes the following form:

$$\begin{aligned} S'_{H_i} &= A_{H_i} - \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - \mu_{H_i} S_{H_i}, \\ I'_{H_i} &= \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - (\mu_{H_i} + \gamma_{H_i}) I_{H_i}, \\ R'_{H_i} &= \gamma_{H_i} I_{H_i} - \mu_{H_i} R_{H_i}, \\ S'_{V_i} &= A_{V_i} - \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} S_{V_i}, \\ I'_{V_i} &= \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} I_{V_i}, \end{aligned} \tag{1.1}$$

where $i = 1, 2, \ldots, n$. Here $A_{H_i}$ and $A_{V_i}$ represent the recruitment rate of the humans and the mosquitoes in the $i$th group, $\beta_{H_{ij}}$ represents the contact rate between susceptible humans $S_{H_i}$ and infectious mosquitoes $I_{V_j}$, $\beta_{V_{ij}}$ is the contact rate between infected people $I_{H_j}$ and susceptible mosquitoes $S_{V_i}$, $\mu_{H_i}$ and $\mu_{V_i}$ represent the death rate of the humans and the mosquitoes in the $i$th group, and $\gamma_{H_i}$ represents the recovery rate of the humans in the $i$th group. All parameter values are assumed to be nonnegative and $A_{H_i}, A_{V_i}, \mu_{H_i}, \mu_{V_i} > 0$.

Dengue fever (DF) is an acute mosquito-transmitted disease, with a recorded prevalence in 101 countries [9–11]. An estimated 50–100 million people per year are infected, with approximately 25,000 deaths annually [12]. Thus, the study of DF is perceived as signification and receives much attention. When $n = 1$, the model (1.1) had been studied extensively. For example, the global stability of the equilibria was proved with the results of the theory of competitive systems and stability of periodic orbits in [13]; in [14], the global stability of the equilibria was proved with Lyapunov functions under some conditions.

The organization of this paper is as follows. In Section 2, we quote some results from graph theory which will be used in the proof of our main results. In Section 3, we present a global analysis of the system (1.1). At Section 4, we give a further discussion.

## 2. Preliminaries

In this section, we will give some previous results which will be useful for our main results.

*Definition 2.1* (see [15]). *Let $U = (u_{ij})_{n\times n}$. We say that $U \geq 0$ ($U$ is nonnegative), if all its entries $u_{ij}$ are real and nonnegative.*

If $U = (u_{ij})_{n\times n}$ and $W = (w_{ij})_{n\times n}$ are both nonnegative, we write $U \geq W$ if $u_{ij} \geq w_{ij}$ for all $i$ and $j$, and $U > W$ if $u_{ij} \geq w_{ij}$ and $U \neq W$.

*Definition 2.2* (see [15]). *A matrix $U = (u_{ij})_{n\times n}$ is said to be reducible if either*

(i) $n = 1$ and $U = 0$; or

(ii) $n \geq 2$, there is a permutation matrix $P$,

$$PUP^T = \begin{pmatrix} U_1 & 0 \\ U_2 & U_3 \end{pmatrix}, \tag{2.1}$$

where $U_1$ and $U_3$ are square matrices. Otherwise, $U$ is irreducible.

Let $\Gamma(U)$ denote the directed graph of $(u_{ij})_{n\times n}$. We have the following proposition.

**Proposition 2.3** (see [16]). *For matrix $U$, one has*

(i) *If $U$ is nonnegative, then the spectral radius $\rho(U)$ of $U$ is an eigenvalue, and $U$ has a nonnegative eigenvector corresponding to $\rho(U)$.*

(ii) *If $U$ is nonnegative and irreducible, then $\rho(U)$ is a simple eigenvalue, and $U$ has a positive eigenvector $x$ corresponding to $\rho(U)$.*

(iii) *If $0 < W < U$, then $\rho(W) \leq \rho(U)$. Moreover, if $0 < W < U$ and $W + U$ is irreducible, then $\rho(W) < \rho(U)$.*

(iv) *If $U$ is nonnegative and irreducible, and $W$ is diagonal and positive (namely, all of its entries are positive), then $UW$ is irreducible.*

(v) *Matrix $U$ is irreducible if and only if $\Gamma(U)$ is strongly connected.*

## 3. Mathematical Analysis

From the first and the fourth equation in (1.1), we know

$$\limsup_{t\longrightarrow\infty} S_{H_i} \leq \frac{A_{H_i}}{\mu_{H_i}}, \qquad \limsup_{t\longrightarrow\infty} S_{V_i} \leq \frac{A_{V_i}}{\mu_{V_i}}. \tag{3.1}$$

For each $i$, adding the five equations in (1.1), we obtain

$$\begin{aligned}(S_{H_i} + I_{H_i} + R_{H_i} + S_{V_i} + I_{V_i})' &= A_{H_i} + A_{V_i} - \mu_{H_i}(S_{H_i} + I_{H_i} + R_{H_i}) - \mu_{V_i}(S_{V_i} + I_{V_i}) \\ &\leq A_{H_i} + A_{V_i} - \mu_i^*(S_{H_i} + I_{H_i} + R_{H_i} + S_{V_i} + I_{V_i}),\end{aligned} \tag{3.2}$$

where $\mu_i^* = \min\{\mu_{H_i}, \mu_{V_i}\}$. Thus,

$$\limsup_{t\longrightarrow\infty}(S_{H_i} + I_{H_i} + R_{H_i} + S_{V_i} + I_{V_i}) \leq \frac{A_{H_i} + A_{V_i}}{\mu_i^*}. \tag{3.3}$$

Before going into any detail, we simplify the system. For each $i$-group, since the variable $R_{H_i}$ dose not appear in the first two and the last two equations of (1.1), it suffices to consider the following reduced system:

$$
\begin{aligned}
S'_{H_i} &= A_{H_i} - \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - \mu_{H_i} S_{H_i}, \\
I'_{H_i} &= \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - (\mu_{H_i} + \gamma_{H_i}) I_{H_i}, \\
S'_{V_i} &= A_{V_i} - \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} S_{V_i}, \\
I'_{V_i} &= \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} I_{V_i},
\end{aligned}
\tag{3.4}
$$

where $i = 1, 2, \ldots, n$, in the feasible region

$$
\begin{aligned}
D = \Bigg\{ (S, I) \ \in R_{+}^{4n} \mid S_{H_i} \le \frac{A_{H_i}}{\mu_{H_i}},\ S_{V_i} \le \frac{A_{V_i}}{\mu_{V_i}},\ & S_{H_i} + I_{H_i} + S_{V_i} + I_{V_i} \\
\le \frac{A_{H_i} + A_{V_i}}{\mu_i^*},\ i = 1, 2, \ldots, n \Bigg\},&
\end{aligned}
\tag{3.5}
$$

where $S = (S_H, S_V)$, $I = (I_H, I_V)$, $S_H = (S_{H_1}, \ldots, S_{H_n})$, $S_V = (S_{V_1}, \ldots, S_{V_n})$, $I_H = (I_{H_1}, \ldots, I_{H_n})$, and $I_V = (I_{V_1}, \ldots, I_{V_n})$. It can be verified that $D$ is positively invariant with respect to system (3.4). Behaviors of $R_{H_i}$ can then be determined from the third equation in (1.1). Our results in this paper will be stated for system (3.4) in $D$ and can be translated straightforwardly to system (1.1). Let $\mathring{D}$ denote the interior of $D$.

An equilibrium $(S, I)$ of (3.4) satisfies

$$
\begin{aligned}
A_{H_i} - \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - \mu_{H_i} S_{H_i} &= 0, \\
\sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - (\mu_{H_i} + \gamma_{H_i}) I_{H_i} &= 0, \\
A_{V_i} - \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} S_{V_i} &= 0, \\
\sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} I_{V_i} &= 0,
\end{aligned}
\tag{3.6}
$$

where $i = 1, 2, \ldots, n$. It is easy to see that the disease-free equilibrium denoted by $E^0 = (S^0, I^0)$ exists for all positive parameter values, where $S^0_{H_i} = A_{H_i}/\mu_{H_i}$, $S^0_{V_i} = A_{V_i}/\mu_{V_i}$, and $I^0_{H_i} = I^0_{V_i} = 0$, $i = 1, 2, \ldots, n$.

Denote

$$M(S) = \begin{pmatrix} 0 & M_H(S) \\ M_V(S) & 0 \end{pmatrix}, \tag{3.7}$$

where $M_H(S) = (\beta_{H_{ij}} S_{H_i} / \mu_{H_i} + \gamma_{H_i})_{n\times n}$, $M_V(S) = (\beta_{V_{ij}} S_{V_i} / \mu_{V_i})_{n\times n}$. We also denote $M_H(S_H^0) = M_{H_0}$ and $M_V(S_V^0) = M_{V_0}$

$$M_0 = \begin{pmatrix} 0 & M_{H_0} \\ M_{V_0} & 0 \end{pmatrix}. \tag{3.8}$$

We know that for all $S \in D$, $S \leq S^0$, so for all $S \in D$, $M(S) \leq M_0$. We define the basic reproduction number $R_0$ as the spectral radius of $M_0$; that is $R_0 = \rho(M_0)$. We set

$$B_H = \begin{pmatrix} \beta_{H_{11}} & \beta_{H_{21}} & \cdots & \beta_{H_{n1}} \\ \beta_{H_{12}} & \beta_{H_{22}} & \cdots & \beta_{H_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{H_{1n}} & \beta_{H_{2n}} & \cdots & \beta_{H_{nn}} \end{pmatrix}, \tag{3.9}$$

$$B_V = \begin{pmatrix} \beta_{V_{11}} & \beta_{V_{21}} & \cdots & \beta_{V_{n1}} \\ \beta_{V_{12}} & \beta_{V_{22}} & \cdots & \beta_{V_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{V_{1n}} & \beta_{V_{2n}} & \cdots & \beta_{V_{nn}} \end{pmatrix}, \qquad B_M = \begin{pmatrix} 0 & B_H \\ B_V & 0 \end{pmatrix}. \tag{3.10}$$

**Theorem 3.1.** *Assume that $B_H$, $B_V$, and $B_M$ are irreducible.*

(1) *If $R_0 \leq 1$, then the disease-free equilibrium $E^0$ of system* (3.4) *is globally asymptotically stable in $D$.*

(2) *If $R_0 > 1$, then $E^0$ is unstable and system* (3.4) *is uniformly persistent in $\mathring{D}$.*

*Proof.* Since $B_M$ is irreducible and nonnegative, we know that $M(S)$ and $M_0$ are irreducible and nonnegative. Therefore, by Proposition 2.3(ii), there exists a left eigenvector $\omega = (\omega_H, \omega_V) > 0$ of $M_0$ corresponding to $\rho(M_0)$, where $\omega_H = (\omega_{H_1}, \omega_{H_2}, \ldots, \omega_{H_n})$, $\omega_V = (\omega_{V_1}, \omega_{V_2}, \ldots, \omega_{V_n})$; that is, $\omega\rho(M_0) = \omega M_0$. Define

$$L = \sum_{i=1}^{n} \left( \frac{\omega_{H_i}}{\mu_{H_i} + \gamma_{H_i}} I_{H_i} + \frac{\omega_{V_i}}{\mu_{V_i}} I_{V_i} \right). \tag{3.11}$$

Denote the transpose of $I$ as $I^T$. Differentiating $L$ along the solution of system (3.4), we obtain

$$\begin{aligned}
L' &= \sum_{i=1}^{n}\left\{ \frac{\omega_{H_i}}{\mu_{H_i}+\gamma_{H_i}}\left[\sum_{j=1}^{n}\beta_{H_{ij}}S_{H_i}I_{V_j} - \left(\mu_{H_i}+\gamma_{H_i}\right)I_{H_i}\right] \right. \\
&\qquad\qquad \left. + \frac{\omega_{V_i}}{\mu_{V_i}}\left(\sum_{j=1}^{n}\beta_{V_{ij}}S_{V_i}I_{H_j} - \mu_{V_i}I_{V_i}\right)\right\} \\
&= \sum_{i=1}^{n}\left[\omega_{H_i}\left(\sum_{j=1}^{n}\frac{\beta_{H_{ij}}S_{H_i}I_{V_j}}{\mu_{H_i}+\gamma_{H_i}} - I_{H_i}\right) + \omega_{V_i}\left(\sum_{j=1}^{n}\frac{\beta_{V_{ij}}S_{V_i}I_{H_j}}{\mu_{V_i}} - I_{V_i}\right)\right] \\
&= \omega\left(M(S)I^T - I^T\right) \\
&\le \omega\left(M_0 I^T - I^T\right) \\
&= \left(\rho(M_0)-1\right)\omega I^T \\
&\le 0.
\end{aligned} \tag{3.12}$$

Therefore, we obtain

(i) if $R_0 < 1$, $L' = 0 \Leftrightarrow I = 0$;

(ii) if $R_0 = 1$, $L' = 0 \Leftrightarrow S = S^0$ or $I = 0$.

Thus, we know that the singleton $\{E^0\}$ is the only compact invariant subset of $\{L' = 0\}$. By LaSalle's Invariance Principle [17], $E^0$ is globally asymptotically stable in $D$, if $R_0 \le 1$.

If $R_0 > 1$ and $I > 0$, it is easy to see that

$$\omega\left(M_0 I^T - I^T\right) = \left(\rho(M_0)-1\right)\omega I^T > 0. \tag{3.13}$$

Then, according to continuity, there exists a neighborhood $\mathfrak{B}(E^0)$ of $E^0$, $\mathfrak{B}(E^0) \subseteq D$, such that for all $(S, I) \in \mathfrak{B}(E^0)$

$$L' = \omega\left(M(S)I^T - I^T\right) > 0. \tag{3.14}$$

This implies that $E^0$ is unstable. Using a uniform persistence result from [18] and a similar argument as in the proof of Proposition 3.3 of [19], we know that, when $R_0 > 1$, the instability of $E^0$ implies the uniform persistence of (3.4). The proof is complete. $\square$

Uniform persistence of (3.4), together with uniform boundedness of solutions in $\overset{\circ}{D}$, implies the existence of an equilibrium of system (3.4) in $\overset{\circ}{D}$ [20, 21].

**Corollary 3.2.** *Assume $B_H$, $B_V$, and $B_M$ are irreducible. If $R_0 > 1$, then* (3.4) *has at least one endemic equilibrium.*

Denote the endemic equilibrium by $E^* = (S^*, I^*)$, where $S^*_{H_i}, S^*_{V_i}, I^*_{H_i}, I^*_{V_i} > 0$, $i = 1, 2, \ldots, n$. One has the following result on the endemic equilibrium $E^*$.

**Theorem 3.3.** *Assume that $B_H$, $B_V$, and $B_M$ are irreducible. If $R_0 > 1$, then the endemic equilibrium $E^*$ of system* (3.4) *is globally asymptotically stable in $\mathring{D}$.*

*Proof.* The uniqueness of endemic equilibrium is obvious in $\mathring{D}$, if we prove that the endemic equilibrium $E^*$ is globally stable when $R_0 > 1$. We denote $\overline{\beta}_{H_{ij}} = \beta_{H_{ij}} S^*_{H_i} I^*_{V_j}$, $\overline{\beta}_{V_{ij}} = \beta_{V_{ij}} S^*_{V_i} I^*_{H_j}$

$$\overline{B} = \begin{pmatrix} \sum\limits_{k,j\neq 1}^{n} \overline{\beta}_{V_{1j}} \overline{\beta}_{H_{1k}} & -\sum\limits_{k\neq 1}^{n} \overline{\beta}_{V_{21}} \overline{\beta}_{H_{k1}} & \cdots & -\sum\limits_{k\neq 1}^{n} \overline{\beta}_{V_{n1}} \overline{\beta}_{H_{k1}} \\ -\sum\limits_{k\neq 2}^{n} \overline{\beta}_{V_{12}} \overline{\beta}_{H_{k2}} & \sum\limits_{k,j\neq 2}^{n} \overline{\beta}_{V_{2j}} \overline{\beta}_{H_{2k}} & \cdots & -\sum\limits_{k\neq 2}^{n} \overline{\beta}_{V_{n2}} \overline{\beta}_{H_{k2}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum\limits_{k\neq n}^{n} \overline{\beta}_{V_{1n}} \overline{\beta}_{H_{kn}} & -\sum\limits_{k\neq n}^{n} \overline{\beta}_{V_{2n}} \overline{\beta}_{H_{k2}} & \cdots & \sum\limits_{k,j\neq n}^{n} \overline{\beta}_{V_{nj}} \overline{\beta}_{H_{nk}} \end{pmatrix}. \tag{3.15}$$

It is easy to see that

$$\begin{aligned} \overline{B} = &\begin{pmatrix} \sum\limits_{k,j\neq 1}^{n} \overline{\beta}_{V_{1j}} \overline{\beta}_{H_{1k}} & 0 & \cdots & 0 \\ 0 & \sum\limits_{k,j\neq 2}^{n} \overline{\beta}_{V_{2j}} \overline{\beta}_{H_{2k}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum\limits_{k,j\neq n}^{n} \overline{\beta}_{V_{nj}} \overline{\beta}_{H_{nk}} \end{pmatrix} \\ &- \begin{pmatrix} \sum\limits_{k\neq 1}^{n} \overline{\beta}_{H_{k1}} & 0 & \cdots & 0 \\ 0 & \sum\limits_{k\neq 2}^{n} \overline{\beta}_{H_{k2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum\limits_{k\neq n}^{n} \overline{\beta}_{H_{kn}} \end{pmatrix} \begin{pmatrix} 0 & \overline{\beta}_{V_{21}} & \cdots & \overline{\beta}_{V_{n1}} \\ \overline{\beta}_{V_{12}} & 0 & \cdots & \overline{\beta}_{V_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\beta}_{V_{1n}} & \overline{\beta}_{V_{2n}} & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{3.16}$$

Since $B_H$ is irreducible and nonnegative, we get $\sum_{k\neq j}^{n} \overline{\beta}_{H_{kj}} \neq 0$, $j = 1, 2, \ldots, n$. Together with $B_V$ being irreducible and nonnegative, by Proposition 2.3(iv), we know that $\overline{B}$ is irreducible. Let $C_{ij}$ denote the cofactor of the $(i, j)$ entry of $\overline{B}$. According to Lemma 2.1 in [2], we have that the equation $\overline{B}v = 0$ has a positive solution $v = (v_1, v_2, \ldots, v_n)$, where $v_i = C_{ii} > 0$ for $i = 1, 2, \ldots, n$. Define a Lyapunov function as follows:

$$\begin{aligned} V = \sum_{i=1}^{n} v_i \Bigg[ &\sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \left( S_{H_i} - S_{H_i}^{*} \ln S_{H_i} + I_{H_i} - I_{H_i}^{*} \ln I_{H_i} \right) \\ &+ \sum_{k=1}^{n} \overline{\beta}_{H_{ik}} \left( S_{V_i} - S_{V_i}^{*} \ln S_{V_i} + I_{V_i} - I_{V_i}^{*} \ln I_{V_i} \right) \Bigg]. \end{aligned} \tag{3.17}$$

Together with (3.6), we get the derivative of $V$ along the solution of system (3.4)

$$\begin{aligned} V' = \sum_{i=1}^{n} v_i \Bigg\{ &\sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \Bigg[ A_{H_i} - \mu_{H_i} S_{H_i} - \frac{S_{H_i}^{*}}{S_{H_i}} \left( A_{H_i} - \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - \mu_{H_i} S_{H_i} \right) \\ &\qquad - (\mu_{H_i} + \gamma_{H_i}) I_{H_i} - \frac{I_{H_i}^{*}}{I_{H_i}} \left( \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} - (\mu_{H_i} + \gamma_{H_i})\, I_{H_i} \right) \Bigg] \\ &+ \sum_{k=1}^{n} \overline{\beta}_{H_{ik}} \Bigg[ A_{V_i} - \mu_{V_i} S_{V_i} - \frac{S_{V_i}^{*}}{S_{V_i}} \left( A_{V_i} - \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} S_{V_i} \right) \\ &\qquad - \mu_{V_i} I_{V_i} - \frac{I_{V_i}^{*}}{I_{V_i}} \left( \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} - \mu_{V_i} I_{V_i} \right) \Bigg] \Bigg\}, \\ = \sum_{i,k=1}^{n} v_i &\left[ \overline{\beta}_{V_{ik}} \mu_{H_i} S_{H_i}^{*} \left( 2 - \frac{S_{H_i}}{S_{H_i}^{*}} - \frac{S_{H_i}^{*}}{S_{H_i}} \right) + \overline{\beta}_{H_{ik}} \mu_{V_i} S_{V_i}^{*} \left( 2 - \frac{S_{V_i}}{S_{V_i}^{*}} - \frac{S_{V_i}^{*}}{S_{V_i}} \right) \right] \\ + \sum_{i,k=1}^{n} v_i \overline{\beta}_{V_{ik}} &\left( 2 \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} - \frac{S_{H_i}^{*}}{S_{H_i}} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} + \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \frac{I_{V_j}}{I_{V_j}^{*}} - \frac{I_{H_i}}{I_{H_i}^{*}} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} - \frac{I_{H_i}^{*}}{I_{H_i}} \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} \right) \\ + \sum_{i,k=1}^{n} v_i \overline{\beta}_{H_{ik}} &\left( 2 \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} - \frac{S_{V_i}^{*}}{S_{V_i}} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} + \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} \frac{I_{H_j}}{I_{H_j}^{*}} - \frac{I_{V_i}}{I_{V_i}^{*}} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} - \frac{I_{V_i}^{*}}{I_{V_i}} \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} \right). \end{aligned} \tag{3.18}$$

According to $(x_1/x_2) + (x_2/x_1) \geq 2$ for each $x_1, x_2 > 0$, with equality holding if and only if $x_1 = x_2$, we have

$$\begin{aligned} \mu_{H_i} S_{H_i}^{*} \left( 2 - \frac{S_{H_i}^{*}}{S_{H_i}} - \frac{S_{H_i}}{S_{H_i}^{*}} \right) &\leq 0, \\ \mu_{V_i} S_{V_i}^{*} \left( 2 - \frac{S_{V_i}^{*}}{S_{V_i}} - \frac{S_{V_i}}{S_{V_i}^{*}} \right) &\leq 0, \end{aligned} \tag{3.19}$$

where $i = 1, 2, \ldots, n$ and equalities hold, respectively, if and only if

$$S_{H_i} = S_{H_i}^*, \quad S_{V_i} = S_{V_i}^*, \quad i = 1, 2, \ldots, n. \tag{3.20}$$

Hence,

$$\begin{aligned}
V' \leq{}& \sum_{i,k=1}^{n} v_i \overline{\beta}_{V_{ik}} \left( 2\sum_{j=1}^{n} \overline{\beta}_{H_{ij}} - \frac{S_{H_i}^*}{S_{H_i}} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} + \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \frac{I_{V_j}}{I_{V_j}^*} - \frac{I_{H_i}}{I_{H_i}^*} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} - \frac{I_{H_i}^*}{I_{H_i}} \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i} I_{V_j} \right) \\
&+ \sum_{i,k=1}^{n} v_i \overline{\beta}_{H_{ik}} \left( 2\sum_{j=1}^{n} \overline{\beta}_{V_{ij}} - \frac{S_{V_i}^*}{S_{V_i}} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} + \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} \frac{I_{H_j}}{I_{H_j}^*} - \frac{I_{V_i}}{I_{V_i}^*} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} - \frac{I_{V_i}^*}{I_{V_i}} \sum_{j=1}^{n} \beta_{V_{ij}} S_{V_i} I_{H_j} \right) \\
={}& \sum_{i=1}^{n} v_i \left[ \sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \left( \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \frac{I_{V_j}}{I_{V_j}^*} - \frac{I_{H_i}}{I_{H_i}^*} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \right) + \sum_{k=1}^{n} \overline{\beta}_{H_{ik}} \left( \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} \frac{I_{H_j}}{I_{H_j}^*} - \frac{I_{V_i}}{I_{V_i}^*} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} \right) \right] \\
&+ \sum_{i,j,k=1}^{n} v_i \overline{\beta}_{V_{ik}} \overline{\beta}_{H_{ij}} \left( 4 - \frac{S_{H_i}^*}{S_{H_i}} - \frac{S_{V_i}^*}{S_{V_i}} - \frac{I_{H_i}^* S_{H_i} I_{V_j}}{I_{H_i} S_{H_i}^* I_{V_j}^*} - \frac{I_{V_i}^* S_{V_i} I_{H_j}}{I_{V_i} S_{V_i}^* I_{H_j}^*} \right) \\
=:{}& K_1 + K_2.
\end{aligned} \tag{3.21}$$

We first show $K_1 \equiv 0$ for all $(S, I) \in \mathring{D}$. It follows from $\overline{B} v = 0$ that

$$\sum_{k,j=1}^{n} \overline{\beta}_{V_{ij}} \overline{\beta}_{H_{ik}} v_i = \sum_{k,j=1}^{n} \overline{\beta}_{V_{ji}} \overline{\beta}_{H_{ki}} v_j, \tag{3.22}$$

$i = 1, 2, \ldots, n$. This implies that

$$\begin{aligned}
\sum_{i=1}^{n} v_i \sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \frac{I_{V_j}}{I_{V_j}^*} &= \sum_{j=1}^{n} \frac{I_{V_j}}{I_{V_j}^*} \sum_{k=1}^{n} \sum_{i=1}^{n} v_i \overline{\beta}_{V_{ik}} \overline{\beta}_{H_{ij}} \\
&= \sum_{j=1}^{n} \frac{I_{V_j}}{I_{V_j}^*} v_j \sum_{k,i=1}^{n} \overline{\beta}_{V_{jk}} \overline{\beta}_{H_{ji}}.
\end{aligned} \tag{3.23}$$

Thus,

$$\sum_{i=1}^{n} v_i \sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} \frac{I_{V_j}}{I_{V_j}^*} - \sum_{i=1}^{n} v_i \sum_{k=1}^{n} \overline{\beta}_{H_{ik}} \frac{I_{V_i}}{I_{V_i}^*} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} = 0. \tag{3.24}$$

Similarly, we produce

$$\sum_{i=1}^{n} v_i \sum_{k=1}^{n} \overline{\beta}_{H_{ik}} \sum_{j=1}^{n} \overline{\beta}_{V_{ij}} \frac{I_{H_j}}{I_{H_j}^*} - \sum_{i=1}^{n} v_i \sum_{k=1}^{n} \overline{\beta}_{V_{ik}} \frac{I_{H_i}}{I_{H_i}^*} \sum_{j=1}^{n} \overline{\beta}_{H_{ij}} = 0. \tag{3.25}$$

Therefore, $K_1 \equiv 0$ for all $(S, I) \in \mathring{D}$.

$\Gamma(\overline{B})$ has vertices $\{1, 2, \ldots, n\}$ with a directed arc $(i, j)$ from $i$ to $j$ if and only if $\sum_{k \neq j}^{n} \overline{\beta}_{V_{ij}} \overline{\beta}_{H_{kj}} \neq 0$. Since $\overline{B}$ is irreducible, by a similar argument in [2], we obtain $K_2 \leq 0$ for all $(S, I) \in \overset{\circ}{D}$. Furthermore, we produce that

$$V' \leq 0. \tag{3.26}$$

If (3.20) holds, we have

$$K_2 = 0 \Longleftrightarrow I_{H_i} = \eta I_{H_i}^*, \quad I_{V_i} = \eta I_{V_i}^*, \quad i = 1, 2, \ldots, n, \tag{3.27}$$

where $\eta$ is arbitrary positive numbers.

According to (3.20) and (3.27), we know that $V' = 0 \Leftrightarrow S_{H_i} = S_{H_i}^*$, $S_{V_i} = S_{V_i}^*$, $I_{H_i} = \eta I_{H_i}^*$, $I_{V_i} = \eta I_{V_i}^*$, $i = 1, 2, \ldots, n$. Substituting (3.20) and (3.27) into system (3.4), we obtain

$$\begin{aligned} 0 &= A_{H_i} - \eta \sum_{j=1}^{n} \beta_{H_{ij}} S_{H_i}^* I_{V_j}^* - \mu_{H_i} S_{H_i}^*, \\ 0 &= A_{V_i} - \eta \sum_{j=1}^{n} \beta_{V_i} S_{V_i}^* I_{H_j}^* - \mu_{V_i} S_{V_i}^*. \end{aligned} \tag{3.28}$$

Since the right-hand side of (3.28) is strictly decreasing in $\eta$, by (3.6), we get that (3.28) holds if and only if $\eta = 1$, namely, at $E^*$. By LaSalle's Invariance Principle, $E^*$ is globally asymptotically stable in $\overset{\circ}{D}$. The proof is complete. □

From the process of proof of Theorem 3.3 and the definition of matrix $\overline{B}$, it is easy to get a corollary as follows.

**Corollary 3.4.** *Assume that $B_V$ and $B_M$ are irreducible and $\sum_{k \neq j}^{n} \beta_{H_{kj}} \neq 0$ (or $B_H$, $B_M$ are irreducible and $\sum_{k \neq j}^{n} \beta_{V_{kj}} \neq 0$), $j = 1, 2, \ldots, n$. If $R_0 > 1$, then the endemic equilibrium $E^*$ of system* (3.4) *is globally asymptotically stable in $\overset{\circ}{D}$.*

## 4. Discussion

Taking the basic reproduction number $R_0$ as a sharp threshold parameter, we establish the global dynamics of system (3.4). Our result implies that, if $R_0 \leq 1$, then the dengue disease always dies out in all groups; if $R_0 > 1$, then the dengue disease always persists at the unique endemic equilibrium level in all groups, independent of the initial condition.

Biologically, our assumptions in Theorem 3.3 and Corollary 3.4 mean that mosquitoes in $I_{V_j}$ can infect ones in individuals $S_{H_i}$ directly or indirectly; individuals in $I_{H_j}$ can infect ones in mosquitoes $S_{V_i}$ directly or indirectly, and individuals in $I_{H_j}$ can infect ones in $S_{H_i}$ by mosquitoes indirectly, respectively.

## References


[1] A. Lajmanovich and J. A. Yorke, "A deterministic model for gonorrhea in a nonhomogeneous population," *Mathematical Biosciences*, vol. 28, no. 3/4, pp. 221–236, 1976.

[2] H. B. Guo, M. Y. Li, and Z. Shuai, "Global stability of the endemic equilibrium of multigroup SIR epidemic models," *Canadian Applied Mathematics Quarterly*, vol. 14, no. 3, pp. 259–284, 2006.

[3] H. B. Guo, M. Y. Li, and Z. Shuai, "A graph-theoretic approach to the method of global Lyapunov functions," *Proceedings of the American Mathematical Society*, vol. 136, no. 8, pp. 2793–2802, 2008.

[4] M. Y. Li, Z. Shuai, and C. Wang, "Global stability of multi-group epidemic models with distributed delays," *Journal of Mathematical Analysis and Applications*, vol. 361, no. 1, pp. 38–47, 2010.

[5] Z. Yuan and L. Wang, "Global stability of epidemiological models with group mixing and nonlinear incidence rates," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 2, pp. 995–1004, 2010.

[6] R. Sun and J. Shi, "Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 280–286, 2011.

[7] H. Chen and J. Sun, "Global stability of delay multigroup epidemic models with group mixing and nonlinear incidence rates," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4391–4400, 2011.

[8] R. Sun, "Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2286–2291, 2010.

[9] B. Kay, "Dengue vector surveillance and control," *Current Opinion in Infectious Diseases*, vol. 12, no. 5, pp. 425–432, 1999.

[10] J. G. Rigau-Pérez, G. Clark, D. Gubler, P. Reiter, E. Sanders, and A. Vorndam, "Dengue and dengue heamorrhagic fever," *The Lancet*, vol. 352, no. 3, pp. 971–977, 1998.

[11] U. Bulugahapitiya, S. Siyambalapitiya, S. L. Seneviratne, and D. J. S. Fernando, "Dengue fever in travellers: a challenge for European physicians," *European Journal of Internal Medicine*, vol. 18, no. 3, pp. 185–192, 2007.

[12] I. Stephenson, J. Roper, M. Fraser, K. Nicholson, and M. Wiselka, "Dengue fever in febrile returning travellers to a UK regional infectious diseases unit," *Travel Medicine and Infectious Disease*, vol. 1, no. 2, pp. 89–93, 2003.

[13] L. Esteva and C. Vargas, "Analysis of a dengue disease transmission model," *Mathematical Biosciences*, vol. 150, no. 2, pp. 131–151, 1998.

[14] L. Cai, S. Guo, X. Li, and M. Ghosh, "Global dynamics of a dengue epidemic mathematical model," *Chaos, Solitons and Fractals*, vol. 42, no. 4, pp. 2297–2304, 2009.

[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.

[16] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1979.

[17] J. P. LaSalle, *The Stability of Dynamical Systems*, Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1976.

[18] H. I. Freedman, S. G. Ruan, and M. X. Tang, "Uniform persistence and flows near a closed positively invariant set," *Journal of Dynamics and Differential Equations*, vol. 6, no. 4, pp. 583–600, 1994.

[19] M. Y. Li, J. R. Graef, L. Wang, and J. Karsai, "Global dynamics of a SEIR model with varying total population size," *Mathematical Biosciences*, vol. 160, no. 2, pp. 191–213, 1999.

[20] H. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, UK, 1995.

[21] N. P. Bhatia and G. P. Szegő, *Dynamical Systems: Stability Theory and Applications*, vol. 35 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, Germany, 1967.