Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 313725, 14 pages
doi:10.1155/2012/313725

Research Article

Nonoscillatory Solutions of Second-Order Differential Equations without Monotonicity Assumptions

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Received 25 March 2012; Accepted 7 June 2012

Academic Editor: Chong Lin

The continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions for a class of second-order nonlinear differential equations $[p(t)h(x(t))f(x'(t))]' = q(t)g(x(t))$ are discussed without monotonicity assumption for function g. It is proved that all solutions can be extended to infinity, are eventually monotonic, and can be classified into disjoint classes that are fully characterized in terms of several integral conditions. Moreover, necessary and sufficient conditions for the existence of solutions in each class and for the boundedness of all solutions are established.

1. Introduction

This paper studies the continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions for a class of second-order nonlinear differential equations

$$[p(t)h(x(t))f(x'(t))]' = q(t)g(x(t)), \quad t \geq a. \tag{1.1}$$

Some special cases of (1.1) such as half-linear equation

$$[p(t)\Phi_p(x'(t))]' = q(t)\Phi_p(x(t)), \tag{1.2}$$

where $\Phi_p(r) = |r|^{p-2}r$, $p > 1$, is the so-called p-Laplacian operator, Emden-Fowler equation

$$[p(t)\Phi_p(x'(t))]' = q(t)\Phi_\beta(x(t)), \tag{1.3}$$

and differential equation

$$[p(t)h(x(t))x'(t)]' = q(t)g(x(t)) \tag{1.4}$$

have been extensively discussed in the literature; see, for example, [1–15] and references cited therein. Equation (1.1) with general nonlinear function $f(r)$ is investigated in [16–18]. It is worth to point out that $g(r)$ is assumed to be monotonic in most cited papers, but [2, 6] explain that this assumption does not hold in some applications. The aim of this paper is to investigate the continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions of (1.1) without monotonic assumption for g. Some techniques and ideas have been used by the authors in [17].

By solution of (1.1), we mean a differentiable function x such that $p(t)h(x(t))f(x'(t))$ is differentiable and satisfies (1.1) on the maximum existence interval $[a, \alpha_x)$, $\alpha_x \leq \infty$. A solution x of (1.1) is said to be eventually monotonic if there exists a $t_x \geq a$ such that x is monotonic on $[t_x, \alpha_x)$. In this paper, we consider only solutions that are not eventually identically equal to zero.

Throughout the paper, we always assume that

(H) $p(t), q(t) : [a, \infty) \to \mathbb{R}$ are continuous and $p(t) > 0$ and $q(t) > 0$;
$h(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $h(r) > 0$;
$g(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $rg(r) > 0$ for $r \neq 0$;
$f(r) : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and $rf(r) > 0$ for $r \neq 0$.

(H1) There exists a constant $M_1 > 0$ such that

$$\left|f^{-1}(uv)\right| \leq M_1 \left|f^{-1}(u)\right| \left|f^{-1}(v)\right|, \quad \forall u, v \in \mathbb{R}. \tag{1.5}$$

Remark 1.1. (H1) holds for p-Laplacian operator; indeed,

$$f^{-1}(uv) = f^{-1}(u) f^{-1}(v). \tag{1.6}$$

However, there are nonlinear functions f that satisfy (H1) but not (1.6); see [17].

The paper is organized as follows: Section 1 briefly addresses the background and the motivation of the paper. Continuability, classification, and boundedness of solutions are discussed in Section 2. Sections 3 and 4 deal with the existence of class A and class B solutions, respectively. Finally, several remarks are provided in Section 5 to compare our results with existing ones.

2. Continuability, Classification, and Boundedness of Solutions

In this section we discuss continuability, classification, and boundedness of solutions of (1.1). First of all, we cite a result from [17] that will be used later on.

Lemma 2.1. *If x is a solution of* (1.1) *with maximal existence interval $[a, \alpha_x)$, $\alpha_x \leq \infty$, then x is eventually monotonic. Moreover, if x is bounded on all finite subinterval of $[a, \alpha_x)$, then $\alpha_x = \infty$.*

Remark 2.2. From Lemma 2.1 all solutions of (1.1) except eventually trivial solutions can be classified into two classes

$$
\begin{aligned}
A &= \{x \text{ is defined on } [a, \alpha_x) : x(t)x'(t) > 0 \text{ in a left neighborhood of } \alpha_x\}, \\
B &= \{x \text{ is defined on } [a, \infty) : x(t)x'(t) < 0 \text{ for } t \geq a\}.
\end{aligned}
\tag{2.1}
$$

Next theorem establishes the continuability for all solutions of (1.1), in other words, all solutions can be extended to $[a, \infty)$.

Theorem 2.3. *Assume the following assmputions hold.*

(H2) *There exists a real number $m > 0$ and a continuous function $G(r) : \mathbb{R} \to \mathbb{R}$ such that $G(r)$ is increasing and $|g(r)| \leq |G(r)|$ for $|r| \geq m$, and $rG(r) > 0$ for $r \neq 0$;*

(H3) *There exists a real number $r_0 > 0$ such that*

$$
\int_{r_0}^{\infty} \frac{dr}{f^{-1}(z(r))} = \infty, \qquad \int_{-\infty}^{-r_0} \frac{dr}{f^{-1}(z(r))} = -\infty,
\tag{2.2}
$$

where $z(r) = G(r)/h(r)$.

Then all solutions of (1.1) *can be extended to $[a, \infty)$.*

Proof. The proof is similar to that of Theorem 2.3 [17]. We point out that as in the proof of Theorem 2.3 [17], for a class A solution x, we have

$$
\begin{aligned}
f(x'(t)) &\leq \frac{G(x(t))}{p(t)h(x(t))}\left(\frac{p(d)h(x(d))f(x'(d))}{G(x(d))} + \int_d^t q(s)ds\right), \\
\int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} &\leq M_1^2 f^{-1}(k) \int_{t_1}^{t} f^{-1}\left(\frac{1}{p(s)} \int_d^s q(\sigma)d\sigma\right)ds.
\end{aligned}
\tag{2.3}
$$

□

Remark 2.4. The function $g(r) = r + \sin r$ is not monotonic. Clearly, $|g(r)| \leq 2|r|$, so g is bounded by an increasing function $G(r) = 2r$. Therefore, the existing results which require the monotonic condition for g would not apply, but Theorem 2.3 does.

From Remark 2.2 and Theorem 2.3, all solutions of (1.1) can be classified further into four disjoint classes

$$\begin{aligned} A_b &= \left\{ x \in A : \lim_{t\to\infty} |x(t)| = \ell < \infty \right\}, \\ A_\infty &= \left\{ x \in A : \lim_{t\to\infty} |x(t)| = \infty \right\}, \\ B_b &= \left\{ x \in B : \lim_{t\to\infty} x(t) = \ell \neq 0 \right\}, \\ B_0 &= \left\{ x \in B : \lim_{t\to\infty} x(t) = 0 \right\}. \end{aligned} \tag{2.4}$$

We will show that the existence of solutions in each class and the boundedness of all solutions are fully characterized by means of convergence or divergence of the following integrals:

$$\begin{aligned} J_1 &= \int_a^\infty f^{-1}\left(\frac{1}{p(t)} \int_a^t q(s)ds \right) dt, \\ J_2 &= \int_a^\infty f^{-1}\left(-\frac{1}{p(t)} \int_a^t q(s)ds \right) dt, \\ J_3 &= \int_a^\infty f^{-1}\left(\frac{1}{p(t)} \int_t^\infty q(s)ds \right) dt, \\ J_4 &= \int_a^\infty f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds \right) dt, \\ J_5 &= \int_a^\infty f^{-1}\left(\frac{1}{p(t)} \right) dt. \end{aligned} \tag{2.5}$$

Theorem 2.5. *Let (H2) and (H3) hold. Then all positive (negative) solutions of* (1.1) *are bounded if and only if* $J_1 < \infty (J_2 > -\infty)$.

Proof. We consider positive solutions only since the case of negative solutions can be handled similarly.

Necessity. Let x be a positive bounded class A solution. Then $x(t) > 0$ and $x'(t) > 0$ for $t \geq b > a$ and $\lim_{t\to\infty} x(t) = l \in (0, \infty)$. By the Extreme Value Theorem, we have $L_1 := \min_{x(b)\leq r\leq l} g(r) > 0$. Hence

$$p(t)h(x(t))f\left(x'(t)\right) = p(b)h(x(b))f\left(x'(b)\right) + \int_b^t q(s)g(x(s))ds \geq L_1 \int_b^t q(s)ds. \tag{2.6}$$

Since x is continuous and bounded and $h(r)$ is continuous, then $h(x(t))$ is bounded. Let $h(x(t)) \le K$ for $t \in [a, \infty)$. Then

$$Kp(t)f(x'(t)) \ge p(t)h(x(t))f(x'(t)) \ge L_1 \int_b^t q(s)ds,$$
$$\frac{K}{L_1} f(x'(t)) \ge \frac{1}{p(t)} \int_b^t q(s)ds. \tag{2.7}$$

By (H1), we have

$$f^{-1}\left(\frac{1}{p(t)}\int_b^t q(s)ds\right) \le f^{-1}\left(\frac{K}{L_1} f(x'(t))\right) \le M_1 f^{-1}\left(\frac{K}{L_1}\right) x'(t). \tag{2.8}$$

Integrating from b to t and letting $t \to \infty$, we have

$$J_1 = \int_b^\infty f^{-1}\left(\frac{1}{p(t)}\int_b^t q(s)ds\right)dt \le M_1 f^{-1}\left(\frac{K}{L_1}\right)(l - x(b)) < \infty. \tag{2.9}$$

Sufficiency. We will prove by contradiction. Let x be a unbounded class A solution. Then $x(t) > 0$ and $x'(t) > 0$ on $[b, \infty)$, and there exists a real number $d \ge b$ such that $x(t) \ge m$ for $d \le t < \infty$. Similar to the proof of Theorem 2.3, we have

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k) \int_{t_1}^t f^{-1}\left(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\right)ds. \tag{2.10}$$

Letting $t \to \infty$ and noting that $x(\infty) = \infty$, we have

$$\int_{x(t_1)}^\infty \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k) \int_{t_1}^\infty f^{-1}\left(\frac{1}{p(s)}\int_b^s q(\sigma)d\sigma\right)ds \le M_1^2 f^{-1}(k) J_1 < \infty, \tag{2.11}$$

a contradiction to (H3). Therefore, x is bounded. □

Corollary 2.6. *Let (H2) and (H3) hold. If* (1.1) *has a positive (negative) bounded class A solution, then all positive (negative) solutions are bounded. On the other hand, if* (1.1) *has an unbounded positive (negative) class A solution, then all positive (negative) solutions are unbounded.*

3. Class A Solutions

In this section, we consider the existence of class A_b and class A_∞ solutions of (1.1). The necessary and sufficient conditions for the existence of class A_b solutions and the sufficient conditions for the existence of class A_∞ solutions are provided.

Theorem 3.1. *Equation* (1.1) *has both positive and negative class A solutions.*

Proof. Similar to the proof of Theorem 3.1 in [17]. □

Theorem 3.2. *Equation* (1.1) *has a positive (negative) A_b solution if and only if* $J_1 < \infty$ $(J_2 > -\infty)$.

Proof. *Necessity*. Without loss of generality, we assume that x is a positive A_b solution. In this case, there exists a $b \geq a$ such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq b$. Note that $x(\infty) := \lim_{t\to\infty} x(t) < \infty$, we have

$$m_1 := \min_{x(b)\leq r\leq x(\infty)} g(r) > 0,$$
$$c_1 := \max_{b\leq t<\infty} h(x(t)) \leq \max_{x(b)\leq r\leq x(\infty)} h(r) < \infty. \tag{3.1}$$

Then

$$p(t)h(x(t))f\left(x'(t)\right) = p(b)h(x(b))f\left(x'(b)\right) + \int_b^t q(s)g(x(s))ds \geq m_1 \int_b^t q(s)ds, \tag{3.2}$$

and hence

$$\frac{1}{p(t)}\int_b^t q(s)ds \leq \frac{1}{m_1}h(x(t))f\left(x'(t)\right). \tag{3.3}$$

Taking f^{-1} on both sides and applying (H1) imply that

$$f^{-1}\left(\frac{1}{p(t)}\int_b^t q(s)ds\right) \leq f^{-1}\left(\frac{1}{m_1}h(x(t))f\left(x'(t)\right)\right) \leq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)x'(t). \tag{3.4}$$

Therefore

$$J_1 = \int_b^\infty f^{-1}\left(\frac{1}{p(t)}\int_b^t q(s)ds\right)ds \leq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)(x(\infty)) - (x(b)) < \infty. \tag{3.5}$$

Sufficiency. Define

$$m_2 = \max_{1\leq r\leq 2} g(r) > 0, \qquad c_2 = \min_{1\leq r\leq 2} h(r) > 0. \tag{3.6}$$

Since $J_1 < \infty$, we may select a $d \geq a$ such that

$$\int_d^\infty f^{-1}\left(\frac{1}{p(t)}\int_d^t q(s)ds\right)dt \leq \frac{1}{M_1 f^{-1}(m_2/c_2)}. \tag{3.7}$$

Let $CB[d,\infty)$ be the Banach space of all bounded and continuous functions defined on $[d,\infty)$ endowed with the supremum norm, and let $X = \{x \in CB[d,\infty) : 1 \le x(t) \le 2,\ t \ge d\}$. Clearly, X is a bounded convex subset of $CB[d,\infty)$. Define a mapping $F_1 : X \to CB[d,\infty)$ by

$$(F_1x)(t) = 1 + \int_d^t f^{-1}\left(\frac{1}{p(s)h(x(s))}\int_d^s q(\sigma)g(x(\sigma))d\sigma\right)ds. \tag{3.8}$$

In order to apply Schauder's fixed-point theorem to show that F_1 has a fixed point in X, we need to prove that F_1 maps into X and is continuous, and $F_1(X)$ is precompact in $CB[d,\infty)$.

Let $x \in X$. Considering (3.7), we have

$$\begin{aligned} 1 \le (F_1x)(t) &\le 1 + M_1 f^{-1}\left(\frac{m_2}{c_2}\right)\int_d^t f^{-1}\left(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\right)ds \\ &\le 1 + M_1 f^{-1}\left(\frac{m_2}{c_2}\right)\int_d^\infty f^{-1}\left(\frac{1}{p(t)}\int_d^t q(s)ds\right)dt \le 2. \end{aligned} \tag{3.9}$$

Hence, F_1 maps X into X.

Now, we show that if $x_n, x^* \in X$ and $\|x_n - x^*\| \to 0$ as $n \to \infty$, then $\|F_1x_n - F_1x^*\| \to 0$. Indeed, for any fixed $s \in [d,\infty)$, since $x_n(s) \to x^*(s)$ as $n \to \infty$, we have

$$\begin{aligned} &\left| f^{-1}\left(\frac{1}{p(s)h(x_n(s))}\int_d^s q(\sigma)g(x_n(\sigma))d\sigma\right)\right. \\ &\quad \left. - f^{-1}\left(\frac{1}{p(s)h(x^*(s))}\int_d^s q(\sigma)g(x^*(\sigma))d\sigma\right)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned} &\left| f^{-1}\left(\frac{1}{p(s)h(x_n(s))}\int_d^s q(\sigma)g(x_n(\sigma))d\sigma\right) - f^{-1}\left(\frac{1}{p(s)h(x^*(s))}\int_d^s q(\sigma)g(x^*(\sigma))d\sigma\right)\right| \\ &\quad \le \left| f^{-1}\left(\frac{1}{p(s)h(x_n(s))}\int_d^s q(\sigma)g(x_n(\sigma))d\sigma\right)\right| \\ &\qquad + \left| f^{-1}\left(\frac{1}{p(s)h(x^*(s))}\int_d^s q(\sigma)g(x^*(\sigma))d\sigma\right)\right| \\ &\quad \le 2M_1 f^{-1}\left(\frac{m_2}{c_2}\right) f^{-1}\left(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\right) := F(s), \end{aligned} \tag{3.11}$$

and that

$$\int_d^\infty F(s)ds = \int_d^\infty 2M_1 f^{-1}\left(\frac{m_2}{c_2}\right) f^{-1}\left(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\right)ds = 2M_1 f^{-1}\left(\frac{m_2}{c_2}\right)J_1. \tag{3.12}$$

By Lebesgue's dominated convergence theorem and considering (3.11) and (3.12) we have

$$\begin{aligned}
&\|F_1x_n - F_1x^*\| \\
&\quad\le \sup_{b\le t<\infty}\int_d^t \left| f^{-1}\left(\frac{1}{p(s)h(x_n(s))}\int_d^s q(\sigma)g(x_n(\sigma))d\sigma\right)\right. \\
&\qquad\qquad \left. - f^{-1}\left(\frac{1}{p(s)h(x^*(s))}\int_d^s q(\sigma)g(x^*(\sigma))d\sigma\right)\right| ds \\
&\quad\le \int_d^\infty \left| f^{-1}\left(\frac{1}{p(s)h(x_n(s))}\int_d^s q(\sigma)g(x_n(\sigma))d\sigma\right)\right. \\
&\qquad\qquad \left. -f^{-1}\left(\frac{1}{p(s)h(x^*(s))}\int_d^s q(\sigma)g(x^*(\sigma))d\sigma\right)\right| ds \longrightarrow 0
\end{aligned} \tag{3.13}$$

as $n \to \infty$. Therefore, F_1 is continuous in X.

Finally, we show the precompactness of $F_1(X)$ in $CB[d,\infty)$, which means that for any sequence $x_n \in X$, F_1x_n has a convergent subsequence in $CB[d,\infty)$. This can be proved by showing that F_1x_n has a convergent subsequence in $C[b_1,b_2]$ for any compact subinterval $[b_1,b_2]$ of $[b,\infty)$ as well as the diagonal rule. In fact, F_1x_n is uniformly bounded on $[b_1,b_2]$. Since

$$(F_1x_n)'(t) = f^{-1}\left(\frac{1}{p(t)h(x_n(t))}\int_d^t q(s)g(x_n(s))ds\right) \le M_1 f^{-1}\left(\frac{m_2}{c_2}\right) f^{-1}\left(\frac{1}{p(t)}\int_d^t q(t)ds\right). \tag{3.14}$$

By the Mean Value Theorem, we have

$$\begin{aligned}
&|(F_1x_n)(t_1) - (F_1x_n)(t_2)| \\
&\quad = \left|(F_1x_n)'(\xi)(t_1-t_2)\right| \le M_1 f^{-1}\left(\frac{m_2}{c_2}\right)\max_{b_1\le t\le b_2} f^{-1}\left(\frac{1}{p(t)}\int_d^t q(s)ds\right)|t_1-t_2|.
\end{aligned} \tag{3.15}$$

Then F_1x_n is uniformly bounded and equicontinuous in $C[b_1,b_2]$. So F_1x_n has a convergent subsequence in $C[b_1,b_2]$ by Arzelà-Ascoli Theorem.

Now all conditions of Schauder's fixed-point theorem are satisfied, so F_1 has a fixed point $\overline{x}$ in X, that is,

$$\overline{x}(t) = 1 + \int_d^t f^{-1}\left(\frac{1}{p(s)h(\overline{x}(s))}\int_d^s q(\sigma)g(\overline{x}(\sigma))d\sigma\right)ds. \tag{3.16}$$

It is easy to verify that $[p(t)h(\overline{x}(t))f(\overline{x}'(t))]' = q(t)g(\overline{x}(t))$. Hence, $\overline{x}$ is a positive A_b solution of (1.1). The proof is complete. □

Theorem 3.3. *Let (H2) and (H3) hold. Then*

(a) $A_\infty = \emptyset$ *if and only if* $J_1 < \infty$ *and* $J_2 > -\infty$.

(b) *Equation* (1.1) *has a positive (negative)* A_∞ *solution if* $J_1 = \infty$ ($J_2 = -\infty$).

Proof. By Theorem 2.5 all solutions of (1.1) are bounded if and only if $J_1 < \infty$ and $J_2 > -\infty$, so part (a) follows.

If $J_1 = \infty$, there is no positive A_b solution of (1.1) from Theorem 3.2. Therefore, Theorem 3.1 guarantees the existence of a positive A_∞ solution of (1.1). Similarly, there exists a negative A_∞ solution of (1.1) if $J_2 = -\infty$. □

4. Class B Solutions

In this section the existence of class B, B_b, and B_0 solutions are discussed. We assume that (1.1) has a unique solution for any initial conditions $x(a) = x_0 \neq 0$ and $x'(a) = x_1$.

Theorem 4.1. *Assume the following assumptions hold.*

(H2a) *There exists a continuous function* $G(r) : \mathbb{R} \to \mathbb{R}$ *such that* G *is increasing,* $rG(r) > 0$ *for* $r \neq 0$ *and* $|g(r)| \leq |G(r)|$;

(H4) *There exists* $r_0 > 0$ *such that*

$$\int_0^{\pm r_0} \frac{dr}{f^{-1}(z(r))} = \infty. \tag{4.1}$$

Then (1.1) *has*

(a) *both positive and negative solutions in class B;*

(b) *no solution which is eventually identically equal to zero.*

Proof. (a) We prove that class B has a positive solution, the case of having a negative solution is similar. Assume $x_0 > 0$. The solution of (1.1) with initial conditions $x(a) = x_0$ and $x'(a) = c$, denoted by $x(t) := x(t, c)$, has the form

$$x(t) = x_0 + \int_a^t f^{-1}\left(\frac{p(a)h(x_0)f(c)}{p(s)h(x(s))} + \frac{1}{p(s)h(x(s))}\int_a^s q(\sigma)g(x(\sigma))d\sigma\right)ds. \tag{4.2}$$

Define two sets U and L as

$$\begin{aligned} U &= \left\{c \in \mathbb{R} : \text{there exists some } \bar{t} \geq a \text{ such that } x'\left(\bar{t}, c\right) > 0\right\}, \\ L &= \left\{c \in \mathbb{R} : \text{there exists some } \bar{t} \geq a \text{ such that } x\left(\bar{t}, c\right) < 0\right\}. \end{aligned} \tag{4.3}$$

Then $U \cap L = \emptyset$. Clearly, $U \neq \emptyset$. We claim that U is open. Indeed, if $c_0 \in U$, there exists $\bar{t} > a$ such that $x'(\bar{t}, c_0) > 0$. For any $c \in \mathbb{R}$, we have

$$\begin{aligned} & p(\bar{t})h(x(\bar{t},c_0))f(x'(\bar{t},c_0)) - p(\bar{t})h(x(\bar{t},c))f(x'(\bar{t},c)) \\ &\quad = p(a)h(x_0)f(c_0) - p(a)h(x_0)f(c) + \int_a^{\bar{t}} q(s)(g(x(s,c_0)) - g(x(s,c)))ds. \end{aligned} \tag{4.4}$$

Since (1.1) has a unique solution for any initial conditions $x(a) \neq 0$, $x'(a)$, this solution is continuously dependent on initial data. If $c \to c_0$, we have $g(x(s,c)) - g(x(s,c_0)) \to 0$ uniformly for s on $[a, \bar{t}\,]$. Hence, $x'(\bar{t}, c) > 0$ for all c that are close to c_0, this proves the openness of U.

Next we show that $L \neq \emptyset$. Define

$$M_2 := \min_{0 \le r \le x_0} h(r) > 0, \qquad M_3 := \min_{a \le t \le a+1} p(t) > 0. \tag{4.5}$$

Let

$$c < f^{-1}\left(\frac{M_2 M_3 f^{-1}(-x_0) - G(x_0)\int_a^{a+1} q(s)ds}{p(a)h(x_0)}\right) < 0. \tag{4.6}$$

If there exists $b \in (a, a+1]$ such that $x(b,c) < 0$, then $c \in L$ and $L \neq \emptyset$. Otherwise, $x(t,c) \ge 0$ on $[a, a+1]$. In this case, we claim $x'(t,c) < 0$ on $[a, a+1]$. If this is not true, since $x'(a,c) = c < 0$, there exists $t_1 \in (a, a+1]$ such that $x'(t_1, c) = 0$ and $x'(t,c) < 0$ for $t \in [a, t_1)$. Taking into account (4.6) we have

$$\begin{aligned} 0 &= p(t)h(x(t_1,c))f(x'(t_1,c)) \\ &= p(a)h(x_0)f(c) + \int_a^{t_1} q(s)g(x(s,c))ds \\ &\le p(a)h(x_0)f(c) + G(x_0)\int_a^{a+1} q(s)ds < 0. \end{aligned} \tag{4.7}$$

This is a contradiction and hence $x'(t,c) < 0$ on $[a, a+1]$. Notice that

$$\begin{aligned} x(a+1,c) &= x_0 + \int_a^{a+1} f^{-1}\left(\frac{p(a)h(x_0)f(c)}{p(t)h(x(t))} + \frac{1}{p(t)h(x(t))}\int_a^t q(s)g(x(s))ds\right)dt \\ &\le x_0 + \int_a^{a+1} f^{-1}\left(\frac{p(a)h(x_0)f(c) + G(x_0)\int_a^{a+1} q(s)ds}{M_2 M_3}\right)dt < 0, \end{aligned} \tag{4.8}$$

we know $c \in L$. Clearly, L is open, then $\mathbb{R} - (U \cup L) \neq \emptyset$. Take $c \in \mathbb{R} - (U \cup L)$, $x(t,c)$ is a nonincreasing nonnegative solution on $[a, \infty)$. We will show that $x(t,c) > 0$ on $[a, \infty)$. If not, there exists $t_0 > a$ such that $x(t_0) = 0$ and $x(t) = 0$ for $t \ge t_0$ and $x'(t_0) = 0$. Note that for $t \in [a, t_0]$ we have

$$\begin{aligned} x'(t) &= f^{-1}\left(-\frac{1}{p(t)h(x(t))}\int_t^{t_0} q(s)g(x(s))ds\right) \\ &\geq f^{-1}\left(-\frac{G(x(t))}{p(t)h(x(t))}\int_t^{t_0} q(s)ds\right) \\ &\geq M_1 f^{-1}(z(x(t)))f^{-1}\left(-\frac{1}{p(t)}\int_t^{t_0} q(s)ds\right). \end{aligned} \tag{4.9}$$

Dividing both sides by $f^{-1}(z(x(t)))$ and integrating from a to t_0, we have

$$\int_a^{t_0} \frac{x'(t)}{f^{-1}(z(x(t)))}dt \geq M_1 \int_a^{t_0} f^{-1}\left(-\frac{1}{p(t)}\int_t^{t_0} q(s)ds\right)dt. \tag{4.10}$$

That is

$$\int_0^{x_0} \frac{1}{f^{-1}(z(r))}dr \leq -M_1 \int_a^{t_0} f^{-1}\left(-\frac{1}{p(t)}\int_t^{t_0} q(s)ds\right)dt < \infty, \tag{4.11}$$

a contradiction to (H4). Therefore, $x(t) > 0$ for $t \geq a$ and $x \in B$.

The proof of part (b) follows from the end part of the proof of part (a). □

Theorem 4.2. *Equation* (1.1) *has a positive (negative) B_b solution if and only if $J_4 > -\infty$ ($J_3 < \infty$).*

Proof. Necessity. We assume that x is a positive B_b solution. The case of negative B_b solution is similar. In this case, we have $x(t) > 0$ and $x'(t) < 0$ for $t \geq a$. Let

$$m_1 = \min_{x(\infty)\leq r\leq x(a)} g(r) > 0, \qquad c_1 = \max_{x(\infty)\leq r\leq x(a)} h(r) > 0 \tag{4.12}$$

and note that $p(t)h(x(t))f(x'(t)) < 0$, $(p(t)h(x(t))f(x'(t)))' > 0$. Then

$$\lim_{t\to\infty} p(t)h(x(t))f(x'(t)) = B \leq 0. \tag{4.13}$$

Integrating both sides of (1.1) from t to ∞ implies that

$$\begin{aligned} m_1 \int_t^\infty q(s)\mathrm{d}s &\leq \int_t^\infty q(s)g(x(s))ds = B - (p(t)h(x(t))f(x'(t))) \\ &\leq -p(t)h(x(t))f(x'(t)). \end{aligned} \tag{4.14}$$

Hence,

$$f^{-1}\left(-\frac{1}{p(t)}\int_t^{\infty} q(s)ds\right) \geq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)x'(t). \tag{4.15}$$

Again, integrating both sides of the above inequality we have

$$J_4 = \int_a^{\infty} f^{-1}\left(-\frac{1}{p(t)}\int_t^{\infty} q(s)ds\right)dt \geq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)(x(\infty) - x(a)) > -\infty. \tag{4.16}$$

Sufficiency. Let

$$m_2 = \max_{1\leq r\leq 2} g(r) > 0, \qquad c_2 = \min_{1\leq r\leq 2} h(r) > 0. \tag{4.17}$$

Since $J_4 > -\infty$ we choose $d > a$ such that

$$\int_d^{\infty} f^{-1}\left(-\frac{1}{p(t)}\int_t^{\infty} q(s)ds\right)dt \geq -\frac{1}{M_1 f^{-1}(m_2/c_2)}. \tag{4.18}$$

Let X and $CB[d,\infty)$ as defined in Theorem 3.2. Define $F_2 : X \to CB[d,\infty)$ by

$$(F_2x)(t) = 1 - \int_t^{\infty} f^{-1}\left(-\frac{1}{p(s)h(x(s))}\int_s^{\infty} q(\sigma)g(x(\sigma))d\sigma\right)ds. \tag{4.19}$$

For any $x \in X$, we have

$$1 \leq (F_2x)(t) \leq 1 - \int_d^{\infty} M_1 f^{-1}\left(\frac{m_2}{c_2}\right)f^{-1}\left(-\frac{1}{p(s)}\int_s^{\infty} q(\sigma)d\sigma\right)ds \leq 2. \tag{4.20}$$

This proves that F_2 maps X into X. Similar to the proof of Theorem 3.2, we are able to show that F_2 is continuous in X, and $F_2(X)$ is precompact in $CB[d,\infty)$. Then F_2 has a fixed-point $\overline{x}(t)$ in X by Schauder's fixed-point theorem, that is,

$$\overline{x}(t) = 1 - \int_t^{\infty} f^{-1}\left(-\frac{1}{p(s)h(\overline{x}(s))}\int_s^{\infty} q(\sigma)g(\overline{x}(\sigma))d\sigma\right)ds. \tag{4.21}$$

It is easy to verify that $\overline{x}(t)$ is a positive B_b solution of (1.1). The proof is complete. □

Theorem 4.3. *Let (H2a) and (H4) hold and let* $J_5 = \infty$. *Then* (1.1) *has a positive (negative)* B_0 *solution if and only if* $J_4 = -\infty$ ($J_4 = \infty$).

Proof. We prove the assertion for positive solutions without loss of generality.

Necessity. Assume $x(t)$ is a positive B_0 solution. Then $x(t) > 0$ and $x'(t) < 0$ for $t \geq a$, $x(\infty) = 0$, and $\lim_{t\to\infty} p(t)h(x(t))f(x'(t)) = L \in (-\infty, 0]$. We claim $L = 0$. In fact, if $L < 0$, since $p(t)h(x(t))f(x'(t))$ is negative and increasing on $[a, \infty)$, then $p(t)h(x(t))f(x'(t)) \leq L$ and

$$x'(t) \leq f^{-1}\left(\frac{L}{c_1 p(t)}\right) \leq M_1 f^{-1}\left(\frac{L}{c_1}\right) f^{-1}\left(\frac{1}{p(t)}\right), \tag{4.22}$$

where $c_1 = \max_{0\leq r\leq x(a)} h(r) > 0$.

Integrating both sides from a to ∞ and noting that $x(\infty) = 0$, we have

$$x(a) \geq -M_1 f^{-1}\left(\frac{L}{c_1}\right) \int_a^\infty f^{-1}\left(\frac{1}{p(t)}\right) dt, \tag{4.23}$$

a contradiction to $J_5 = \infty$ and hence $L = 0$.

Integrating both sides of (1.1) from t to ∞ we have

$$p(t)h(x(t))f(x'(t)) = -\int_t^\infty q(s)g(x(s))ds. \tag{4.24}$$

Then

$$\begin{aligned} x'(t) &= f^{-1}\left(-\frac{1}{p(t)h(x(t))} \int_t^\infty q(s)g(x(s))ds\right) \\ &\geq M_1 f^{-1}(z(x(t))) f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right). \end{aligned} \tag{4.25}$$

Hence,

$$\frac{x'(t)}{f^{-1}(z(x(t)))} \geq M_1 f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right). \tag{4.26}$$

Integrating both sides of the above inequality from a to ∞ implies that

$$\int_0^{x(a)} \frac{dr}{f^{-1}(z(r))} \leq -M_1 \int_a^\infty f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right) dt. \tag{4.27}$$

Therefore, $J_4 = -\infty$ from (H4).

Sufficiency. By Theorem 4.1 (1.1) has a positive class B solution x, either $x \in B_b$ or $x \in B_0$. Note that $J_4 = -\infty$ implies that $x \notin B_b$ from Theorem 4.2. So $x \in B_0$. The proof is complete. □

5. Remarks

In this section, we present several remarks about comparison of our results with the existing ones in the literature.

Theorems 2.3 and 2.5 improve [10, Theorem 1] since (H3) reduces to (iii) of [10] if $f(r) = r$ and the differentiability of $p(\cdot)$ and $h(\cdot)$ is not required. Theorems 2.3, 2.5, and 4.2 complement and generalize [2, Theorem 8]. Moreover, under (H2), Theorems 2.3, 2.5, and 4.2 improve [2, Theorem 8] since (H3) improves (22) of [2]; see the discussion in [16]. Theorem 2.5 generalizes [13, Theorem 3.9]. Theorems 2.3, 3.1, and 4.2 generalize [16, Theorem 1]. Theorem 3.2 generalizes [2, Theorem 3], [16, Theorem 3], and [18, Theorem 2.1]. Theorem 3.3 generalizes [18, Theorem 2.2]. Theorem 4.1 generalizes [13, Theorem 2.1] and improves [3, Theorem 6] under (H2a) since (hp) in [3] is replaced by a weaker condition (H4). Theorem 4.2 generalizes [2, Theorem 1], [16, Theorem 5], and [18, Theorem 3.1]. Theorem 4.3 generalize [16, Theorem 6] and [18, Theorem 3.2].

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