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Research Article

Normal Criterion Concerning Shared Values

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We study normal criterion of meromorphic functions shared values, we obtain the following. Let F be a family of meromorphic functions in a domain D, such that function $f \in F$ has zeros of multiplicity at least 2, there exists nonzero complex numbers b_f, c_f depending on f satisfying (i) b_f/c_f is a constant; (ii) $\min\{\sigma(0,b_f),\sigma(0,c_f),\sigma(b_f,c_f) \geq m\}$ for some $m > 0$; (iii) $(1/c_f^{k-1})(f')^k(z) + f(z) \neq b_f^k/c_f^{k-1}$ or $(1/c_f^{k-1})(f')^k(z) + f(z) = b_f^k/c_f^{k-1} \Rightarrow f(z) = b_f$, then F is normal. These results improve some earlier previous results.

1. Introduction and Main Results

We use C to denote the open complex plane, $\widehat{C}(= C \cup \{\infty\})$ to denote the extended complex plane and D to denote a domain in C. A family F of meromorphic functions defined in $D \subset C$ is said to be normal, if for any sequence $\{f_n\} \subset F$ contains a subsequence which converges spherically, and locally, uniformly in D to a meromorphic function or ∞. Clearly F is said to be normal in D if and only if it is normal at every point of D see [1].

Let D be a domain in C. For f meromorphic on C and $a \in C$, set

$$\overline{E_f}(a) = f^{-1}(\{a\}) \cap D = \{z \in D : f(z) = a\}. \tag{1.1}$$

Two meromorphic functions f and g on D are said to share the value a if $\overline{E_f}(a) = \overline{E_g}(a)$. Let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Longrightarrow g(z) = b. \tag{1.2}$$

If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Longleftrightarrow g(z) = b. \tag{1.3}$$

According to Bloch's principle [2], every condition which reduces a meromorphic function in the plane C to a constant forces a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [3]), many authors proved normality criterion for families of meromorphic functions by starting from Liouville-Picard type theorem (see [4]). It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [5] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values have emerged [6–9]. In recent years, this subject has attracted the attention of many researchers worldwide.

In this paper, we use $\sigma(x, y)$ to denote the spherical distance between x and y and the definition of the spherical distance can be found in [10].

In 2008, Fang and Zalcman [11] proved the following results.

Theorem 1.1 (see [11]). *Let f be a transcendental function. Let $a(\neq 0)$ and b be complex numbers, and let $n(\geq 2)$, k be positive integers, then $f + a(f')^n$ assumes every value $b \in C$ infinitely often.*

Theorem 1.2 (see [11]). *Let F be a transcendental function. Let $a(\neq 0)$ and b be complex numbers, and let $n(\geq 2)$, k be positive integers. If for every $f \in F$ has multiple zeros, and $f + a(f')^n \neq b$, then F is normal in D.*

In 2009, Xu et al. [12] proved the following results.

Theorem 1.3 (see [12]). *Let f be a transcendental function. Let $a(\neq 0)$ and let b be complex numbers, and n, k be positive integers, which satisfy $n \geq k + 1$, then $f + a(f^{(k)})^n$ assumes each value $b \in C$ infinitely often.*

Theorem 1.4 (see [12]). *Let f be a transcendental function. Let $a(\neq 0)$ and b be complex numbers, and let n, k be positive integers, which satisfy $n \geq k + 1$. If for every $f \in F$ has only zeros of multiplicity at least $k + 1$, and satisfies $f + a(f^{(k)})^n \neq b$, then F is normal in D.*

In Theorems 1.2 and 1.4, the constants are the same for each $f \in F$. Now we will prove the condition for the constants be the same can be relaxed to some extent.

Theorem A. *Let F be a family of meromorphic functions in the unit disc Δ, and k be a positive integer and $k \geq 3$. For every $f \in F$, such that all zeros of f have multiplicity at least 2, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying that*

(i) *b_f / c_f is a constant;*

(ii) *$\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f) \geq m\}$ for some $m > 0$;*

(iii) *$(1/c_f^{k-1})(f')^k(z) + f(z) \neq b_f^k / c_f^{k-1}$.*

Then F is normal in Δ.

Theorem B. *Let F be a family of meromorphic functions in the unit disc Δ, and $k(\geq 3)$ be a positive integer. For every $f \in F$, such that all zeros of f have multiplicity at least 2, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying that*

(i) b_f/c_f *is a constant;*

(ii) $\min\{\sigma(0,b_f),\sigma(0,c_f),\sigma(b_f,c_f)\geq m\}$ *for some $m > 0$;*

(iii) $(1/c_f^{k-1})(f')^k(z) + f(z) = b_f^k/c_f^{k-1} \Rightarrow f(z) = b_f$.

Then F is normal in Δ.

2. Some Lemmas

In order to prove our theorems, we require the following results.

Lemma 2.1 (see [7]). *Let F be a family of meromorphic functions in a domain D, and k be a positive integer, such that each function $f \in F$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in F$. If F is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0^+$, and a subsequence of functions $f_n \in F$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\varsigma)}{\rho_n^{\alpha}} \longrightarrow g(\zeta) \tag{2.1}$$

locally uniformly with respect to the spherical metric in C, where g is a nonconstant meromorphic function, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA+1$. Morever, g has order at most 2.

Here as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

Lemma 2.2 (see [10]). *Let m be any positive number. Then, Möbius transformation g satisfies $\sigma(g(a), g(b)) \geq m$, $\sigma(g(b), g(c)) \geq m$, $\sigma(g(c), g(a)) \geq m$, for some constants a, b, and c also satisfy the uniform Lipschitz condition*

$$\sigma(g(z), g(w)) \leq k_m\sigma(z,w), \tag{2.2}$$

where k_m is a constant depending on m.

3. Proof of Theorems

Proof of Theorem A. Let $M = b_f/c_f$. We can find nonzero constants b and c satisfying $M = b/c$. For each $f \in F$, define a Möbius map g_f by $g_f = c_f z/c$, thus $g_f^{-1} = cz/c_f$.

Next we will show $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal in Δ. Suppose to the contrary, G is not normal in Δ. Then by Lemma 2.1. We can find $g_n \in G$, $z_n \in \Delta$, and $\rho_n \to 0^+$, such that $T_n(\zeta) = g_n(z_n + \rho_n\zeta)/\rho_n^{1/(k+1)}$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\zeta)$ whose zeros of multiplicity at least 2 and spherical derivative is limited and T has order at most 2.

We now consider three cases.

Case 1. If $(1/c^{k-1})(T')^k(\zeta) \equiv b^k/c^{k-1}$, then $T(\zeta)$ is a polynomial with degree at most 1, a contradiction.

Case 2. If there exists ζ_0 such that $(1/c^{k-1})(T')^k(\zeta_0) = b^k/c^{k-1}$. Noting that $\rho_n T_n(\zeta) + (1/c^{k-1})(T_n')^k(\zeta) - (b^k/c^{k-1}) \rightarrow (1/c^{k-1})(T')^k(\zeta) - (b^k/c^{k-1})$. By Hurwitz's theorem, there exist a sequence of points $\zeta_n \rightarrow \zeta_0$ such that (for large enough n)

$$\begin{aligned} 0 &= \rho_n T_n(\zeta_n) + \frac{1}{c^{k-1}}(T_n')^k(\zeta_n) - \frac{b^k}{c^{k-1}} \\ &= g_n(z_n + \rho_n\zeta_n) + \frac{1}{c^{k-1}}(g_n')^k(z_n + \zeta_n) - \frac{b^k}{c^{k-1}} \\ &= \frac{c}{c_f} f_n(z_n + \rho_n\zeta_n) + \frac{1}{c^{k-1}}\frac{c^k}{c_f^k}(f_n')^k(z_n + \zeta_n) - \frac{b^k}{c^{k-1}}. \end{aligned} \tag{3.1}$$

Hence $f_n(z_n + \rho_n\zeta_n) + (1/c_f^{k-1})(f_n')^k(z_n + \zeta_n) = b_f^k/c_f^{k-1}$. This contradicts with the suppose of Theorem A.

Case 3. If $(1/c^{k-1})(T')^k(\zeta) \neq b^k/c^{k-1}$. Let $c_1, c_2, \ldots, c_k$ be the solution of the equation $w^k = c^k$, then $T'(\zeta) \neq c_i$ $(i = 1, 2, \ldots, k)$. When $T(\zeta)$ is a rational function, then $T'(\zeta)$ is also a rational function. By Picard Theorem we can deduce that $T'(\zeta)$ is a constant $(k \geq 3)$. Hence $T(\zeta)$ is a polynomial with degree at most 1. This contradicts with $T(\zeta)$ has zeros of multiplicity at least 2. When $T(\zeta)$ is a transcendental function, combining with the second main theorem, we have

$$\begin{aligned} T(r, T') &\leq \overline{N}(r, T') + \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{T'} - c_i\right) + s(r, T') \\ &\leq \overline{N}(r, T') + s(r, T') \leq \frac{1}{2}N(r, T') + s(r, T') \leq \frac{1}{2}T(r, T') + s(r, T'). \end{aligned} \tag{3.2}$$

Hence, $T(r, T') \leq s(r, T')$, a contradiction.

Hence $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal and equicontinuous in Δ. There given $(\varepsilon/k_m > 0)$, where k_m is the constant of Lemma 2.2, there exists $\delta > 0$ such that for the spherical distance $\sigma(x, y) < \delta$,

$$\sigma\left(\left(g_f^{-1} \circ f\right)(x), \left(g_f^{-1}\right)(y)\right) < \frac{\varepsilon}{k_m} \tag{3.3}$$

for each $f \in F$. Hence by Lemma 2.2.

$$\begin{aligned} \sigma(f(x), f(y)) &= \sigma\left(\left(g_f \circ g_f^{-1} \circ f\right)(x), \left(g_f \circ g_f^{-1} \circ f\right)(y)\right) \\ &= k_m \sigma\left(\left(g_f^{-1} \circ f\right)(x), \left(g_f^{-1} \circ f\right)(y)\right) < \varepsilon. \end{aligned} \tag{3.4}$$

Therefore, the family is equicontinuous in Δ. This completes the proof of Theorem A. □

Proof of Theorem B. Let $M = b_f/c_f$. We can find nonzero constants b and c satisfying $M = b/c$. For each $f \in F$, define a Möbius map g_f by $g_f = c_f z/c$, thus $g_f^{-1} = cz/c_f$.

Next we will show $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal in Δ. Suppose to the contrary, G is not normal in Δ. Then by Lemma 2.1. We can find $g_n \in G,\ z_n \in \Delta$, and $\rho_n \to 0^+$, such that $T_n(\zeta) = g_n(z_n + \rho_n\zeta)/\rho_n^{1/(k+1)}$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\zeta)$ whose spherical derivate is limited and T has order at most 2.

We will also consider three cases.

Case 1. If $(1/c^{k-1})(T')^k(\zeta) \equiv b^k/c^{k-1}$, then $T(\zeta)$ is a polynomial with degree at most 1, a contradiction.

Case 2. If there exists ζ_0 such that $(1/c^{k-1})(T')^k(\zeta_0) = b^k/c^{k-1}$. Noting that $\rho_n T_n(\zeta) + (1/c^{k-1})(T_n')^k(\zeta) - (b^k/c^{k-1}) \to (1/c^{k-1})(T')^k(\zeta) - (b^k/c^{k-1})$. By Hurwitz's theorem, there exist a sequence of points $\zeta_n \to \zeta_0$ such that (for large enough n)

$$\begin{aligned}
0 &= \rho_n T_n(\zeta_n) + \frac{1}{c^{k-1}}(T_n')^k(\zeta_n) - \frac{b^k}{c^{k-1}} \\
&= g_n(z_n + \rho_n\zeta_n) + \frac{1}{c^{k-1}}(g_n')^k(z_n + \zeta_n) - \frac{b^k}{c^{k-1}} \\
&= \frac{c}{c_f} f_n(z_n + \rho_n\zeta_n) + \frac{1}{c^{k-1}}\frac{c^k}{c_f^k}(f_n')^k(z_n + \zeta_n) - \frac{b^k}{c^{k-1}}.
\end{aligned} \tag{3.5}$$

Hence $f_n(z_n + \rho_n\zeta_n) + (1/c_f^{k-1})(f_n')^k(z_n + \zeta_n) = b_f^k/c_f^{k-1}$, then we have $f_n(z_n + \rho_n\zeta_n) = b_f$ by the condition (iii) $(1/c_f^{k-1})(f')^k(z) + f(z) = b_f^k/c_f^{k-1} \Rightarrow f(z) = b_f$.

Thus

$$T(\zeta_0) = \lim_{n\to\infty} \frac{g_n(z_n + \rho_n\zeta_n)}{\rho_n} = \lim_{n\to\infty} \frac{cf(z_n + \rho_n\zeta_n)}{c_f\rho_n} = \lim_{n\to\infty} \frac{b}{\rho_n} = \infty. \tag{3.6}$$

This is a contradiction.

Case 3. If $(1/c^{k-1})(T')^k(\zeta) \neq b^k/c^{k-1}$. Let $c_1, c_2, \ldots, c_k$ be the solution of the equation $w^k = c^k$, then $T'(\zeta) \neq c_i$ $(i = 1, 2, \ldots, k)$. When $T(\zeta)$ is a rational function, then $T'(\zeta)$ is also a rational function. By Picard theorem we can deduce that $T'(\zeta)$ is a constant $(k \geq 3)$. Hence $T(\zeta)$ is a polynomial with degree at most 1. This contradicts with $T(\zeta)$ has zeros of multiplicity at least 2. When $T(\zeta)$ is a transcendental function, combining with the second main theorem, we have

$$\begin{aligned}
T(r, T') &\leq \overline{N}(r, T') + \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{T'} - c_i\right) + s(r, T') \\
&\leq \overline{N}(r, T') + s(r, T') \leq \frac{1}{2}N(r, T') + s(r, T') \leq \frac{1}{2}T(r, T') + s(r, T').
\end{aligned} \tag{3.7}$$

Hence, $T(r, T') \leq s(r, T')$, a contradiction.

Hence $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal and equicontinuous in Δ. There given $(\varepsilon/k_m > 0)$, where k_m is the constant of Lemma 2.2, there exists $\delta > 0$ such that for the spherical distance $\sigma(x, y) < \delta$,

$$\sigma\left(\left(g_f^{-1} \circ f\right)(x), \left(g_f^{-1}\right)(y)\right) < \frac{\varepsilon}{k_m} \tag{3.8}$$

for each $f \in F$. Hence by Lemma 2.2.

$$\begin{aligned} \sigma(f(x), f(y)) &= \sigma\left(\left(g_f \circ g_f^{-1} \circ f\right)(x), \left(g_f \circ g_f^{-1} \circ f\right)(y)\right) \\ &= k_m \sigma\left(\left(g_f^{-1} \circ f\right)(x), \left(g_f^{-1} \circ f\right)(y)\right) < \varepsilon. \end{aligned} \tag{3.9}$$

Therefore, the family is equicontinuous in Δ. This completes the proof of Theorem B. □

Remark 3.1. Using the similar argument, if the condition (iii) $f(z) = b_f$ when $(1/c_f^{k-1})(f')^k(z) + f(z) = b_f^k/c_f^{k-1}$ is replaced by (iii) $|f(z)| \geq |b_f|$ when $(1/c_f^{k-1})(f')^k(z) + f(z) = b_f^k/c_f^{k-1}$, then F is normal too.

Authors' Contribution

W. Chen performed the proof and drafted the paper. All authors read and approved the final paper.

Conflict of Interests

The authors declare that they have no conflict of interests.

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